Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–6.
ISSN-0037-8712 in press
doi:10.5269/bspm.51700

Certain Types of Ladder Prime Graphicable Algebras *

R. Anantha Lakshmi and K. Jayalakshmi

abstract: This investigation comprises the continuation of creative research in Discrete Mathematics presented in previous papers on algebras in general, regarding the utilization of graphs to contemplate the specific instance of graphicable algebras, which form a subset of evolution algebras. Evolution algebras are especially fascinating since they are intrinsically connected with other Mathematical fields, for example, group theory, stochastics processes and dynamical systems. Depiction on primeness of particular type of graphicable and subgraphicable algebras is described in view of the newline of research initiated previously by some of the authors.

Key Words: Evolution algebra, ladder graphicable algebra, graphicable algebra, graphs.

Contents

1. Introduction

In this short note we give a simple complete characterization of graph labelings on evolution algebra which was introduced by Tian in last decade. The concept of evolution algebra lies between algebras and dynamical system. Algebraically, evolution algebras are non-associative Banach algebras, dynamically, they represent discrete dynamical system. Evolution algebras have many connections with various branches of mathematics, such as graph theory, group theory, stochastic processes, mathematical physics etc. Since evolution algebra are not defined by identities, they cannot belong to any well-known classes of non-associative algebras, as Lie, alternative and Jordan algebras. These algebras are commutative and flexible, but not power associative. Also the direct sum of evolution algebra is an evolution algebra.

Research in graph labelings is mainly due to Rosa [8] who introduced β-valuation in 1967.A number of graph labelings can be found in x-ray crystallography, coding theory, radar, circuit design astronomy and communication design. Evolution algebras and graphs together have attained the attention of many researchers in recent years. Juan Nunez et.al. [7] have introduced some particular families of graphicable algebras which are subsets of evolution algebras viz. complete bipartite graph, star graph, n-partite graph $A(K_{a_1,a_2,\ldots,a_n})$, friendship graph, wheel graph, snark graph, Tietze graph, generalized Petersen graphs, Durer graph, Mobius - Kantor graph.

Tian gave the best way to connect a graphicable algebra with a directed graph. He took the set of generators of algebra as the set of vertices and as the set of edges those connecting the vertex e_i with the vertices corresponding to generators appearing in the expression of e_i^2, for each generator e_i.

Tian himself, although by using a different notation to that followed here (he used x_i for generators, instead of the notation that we use, e_i) introduced the graphicable algebras associated with cyclic graphs,

* The project is fully supported by JNTUA (16PH0911)
2010 *Mathematics Subject Classification:* 17D99, 05C20, 05C50.
Submitted March 19, 2020. Published April 29, 2021

paths, and complete graphs, referred to as cyclic algebras, path algebras and complete graphicable algebras, respectively. Later, Nunez, et. al. following the study by Tian, obtained in [7] graphicable algebras associated with cubic graphs Q_6 and Q_8 and with the Petersen and Heawood graphs. They also associated a graphicable algebra with complete bipartite graphs and wheel graphs. Following the lines initiated by Tian, in this paper we introduce certain types of ladder graphicable algebras which are prime and investigate the edge deletion case in these algebras.

2. Preliminaries

This section is devoted to recalling some basic notions on the concepts dealt with in the paper. For a more general overview on evolution algebras and on Graph Theory, the reader can consult, for instance, [2,4,5], respectively.

Definition 2.1. *Let $A = (A, \cdot)$ be an algebra over a field K equipped with multiplication and let $S := \{e_1, e_2, ..., e_n, ...\}$ be a basis of A. We say that A is an evolution algebra if*

$$\begin{cases} e_i \cdot e_i = \Sigma c_{ik} e_k, for\ any\ i\ and\ e_i \cdot e_j = 0,\ if\ i \neq j \end{cases} \tag{2.1}$$

The scalars $c_{ik} \in K$ are called the structure constants of A relative to S.

A basis S satisfying (2.1) is called natural basis of A, A is real if $K = R$, and it is non negative, if it is real and the structure constants c_{ik} are non negative. In addition, if $0 \leq c_{ik} \leq 1$, and $\Sigma_{k=1}^{n} c_{ik} = 1$, for any i, k, then A is called Markov evolution algebra.

In his book [5] Tian introduced the concept of graphicable algebra as follows:

Definition 2.2. *A commutative nonassociative algebra $G = (V, E)$, where V is the set of vertices and E the set of edges of G is called graphicable, if it has the set of generators $V = \{e_1, e_2, e_3, ..., e_r\}$ with the two defining relations*

$$R = \left\{ \begin{array}{c} e_i^2 = \Sigma_{e_{k \in N(e_i)}} e_k; \\ e_i . e_j = 0 \ if\ i \neq j, i, j = 1, 2, ..., r. \end{array} \right\}.$$

It is immediate to see that any graphicable algebra $A(G)$ is an evolution algebra, although the converse is not true in general. In the particular case where $e_{ij} = e_{ji}$ for $i \neq j$ and $e_{ii} = 0$ for every $i \in A(G)$, we call this algebra as a S-graphicable algebra.

Definition 2.3. *If the gcd of $(c_{ik}, k \in N(i)) = 1$ for $1 \leq i \leq n$, then the graphicable algebra $A(G)$ is a prime graphicable algebra.*

The accompanying Theorem is particularly useful in two ways. From one perspective, it provides a characterization of a graphicable algebra based as far as its related graph. Then again, it likewise characterizes a graph in terms of its associated evolution algebra. The sufficient condition is defined in [5]. A more direct proof than that presented by Tian was given by Nunez et. al. [6].

Theorem 2.4 (6 Theorem 10)**.** *Let $A(G_1)$ and $A(G_2)$ be two graphicable algebras associated by graphs G_1 and G_2. Then $A(G_1)$ and $A(G_2)$ are isomorphic if and only if the graphs G_1 and G_2 are isomorphic.*

3. Ladder graphicable algebra

In this section we show our main results obtained following a procedure similar to that proposed by the last mentioned authors.

Let us recall that the Cartesian product of a complete graph K_2 and a path P_n with $n \geq 1$ vertices is a Ladder graph L_n with $2n$ vertices and $3n - 2$ edges, that is $L_n = K_2 \times P_n$.

Our objective is to associate this family of graphs with a family of graphicable algebras and to show that the all types of ladder graphicable algebras are prime. To do this, as the simplest case $n = 2$ is trivial, we will begin with the ladder which has two steps and then try to generalize to the case of n steps by induction.

According to our procedure, the graphicable algebra associated to this graph has the generators $\{e_1, e_2, e_3, e_4\}$ starting from $n = 2$ and satisfies the law $e_1^2 = 2e_2 + e_3$, $e_2^2 = 2e_1 + 3e_4$, $e_3^2 = e_1 + 4e_4$ and $e_4^2 = 3e_2 + 4e_3$ and same could be done with L_3 and so on. So, as a consequence we can give the following.

Definition 3.1. *The Ladder graphicable algebra $A(L_n)$ is associated to a Ladder graph L_n if it satisfies the following equalities:*

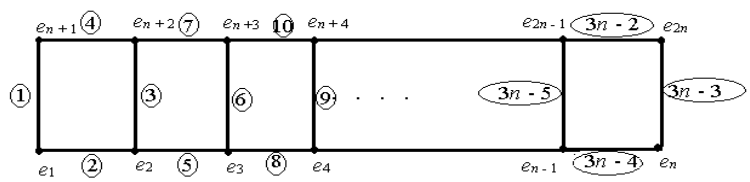
$$e_1^2 = e_{n+1} + e_2;$$
$$e_{n+1}^2 = e_1 + e_{n+2};$$
$$e_n^2 = e_{n-1} + e_{2n};$$
$$e_{2n}^2 = e_n + e_{2n-1};$$
$$e_i^2 = e_{i+1} + e_{n+i} + e_{i-1};\ \text{for } 2 \le i \le n-1$$
$$e_{n+i}^2 = e_i + e_{i+n-1} + e_{i+n+1};\ \text{for } 2 \le i \le n-1$$

Now by assigning the weights $\{1, 2, 3, ..., 3n-2\}$ to the generators of $A(L_n)$, the following relations are obtained

$$e_1^2 = e_{n+1} + 2e_2;$$
$$e_{n+1}^2 = e_1 + 4e_{n+2};$$
$$e_n^2 = (3n-4)e_{n-1} + (3n-3)e_{2n};$$
$$e_{2n}^2 = (3n-3)e_n + (3n-2)e_{2n-1};$$
$$e_i^2 = (3i-1)e_{i+1} + (3i-3)e_{n+i} + (3i-4)e_{i-1}\ ;\ \text{for } 2 \le i \le n-1$$
$$e_{n+i}^2 = (3i-3)e_i + (3i-2)e_{i+n-1} + (3i+1)e_{i+n+1};\ \text{for } 2 \le i \le n-1$$

Thus, it is easy to check that the greatest common divisor of the coefficients of each term in the above result i.e., the gcd of $(1, 2), (1, 4), (3n-4, 3n-3), (3i-3, 3i-2), (3i-4, 3i-3, 3i-1), (3i-3, 3i-2, 3i+1)$ are all equal to 1. Hence, the Ladder graphicable algebra is prime.

The figure below is Ladder graphicable algebra $A(L_n)$.



Ladder graphicable algebra A(Ln)

4. Circular ladder graphicable algebra

Recall that the Cartesian product of a complete graph K_2 and the cycle C_n is a Circular ladder graph CL_n with $2n$-vertices and $3n$-edges, that is $CL_n = K_2 \times C_n$.

By using the same reasoning as before, the graphicable algebra associated to this graph has the generators $\{e_1, e_2, e_3, e_4 e_5, e_6\}$ starting from $n = 3$ and satisfies the law $e_1^2 = e_2 + 3e_3 + 4e_4$, $e_2^2 = e_1 + 2e_3 + 5e_5$, $e_3^2 = 2e_2 + 3e_1 + 6e_6$, $e_4^2 = 4e_1 + 7e_5 + 9e_6$, $e_5^2 = 7e_4 + 5e_2 + 8e_6$ and $e_6^2 = 8e_5 + 6e_3 + 9e_4$, and same could be done with CL_4 and so on. So, as a consequence we can give the following.

Definition 4.1. *The graphicable algebra $A(CL_n)$ is associated to the circular ladder graph CL_n, if it satisfies the following equalities.*

$$e_i^2 = e_{i-1} + e_{i+1} + e_{n+i}\ \text{for } 2 \le i \le n-1;$$
$$e_i^2 = e_{i-1} + e_{i+1} + e_{i-n}\ \text{for } n+2 \le i \le 2n-1;$$
$$e_1^2 = e_2 + e_n + e_{n+1};$$

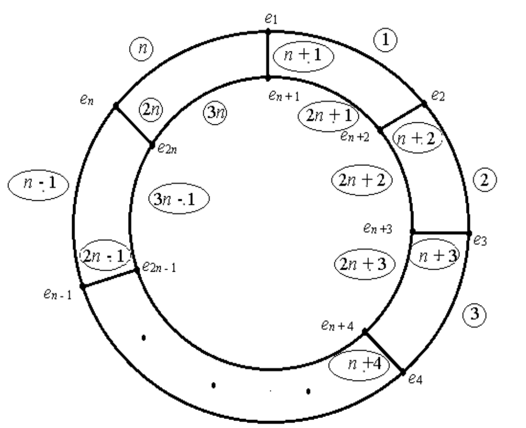
$$e_n^2 = e_1 + e_{n-1} + e_{2n};$$
$$e_{n+1}^2 = e_1 + e_{n+2} + e_{2n};$$
$$e_{2n}^2 = e_{n+1} + e_{2n-1} + e_n.$$

Now by assigning the weights $\{1, 2, 3, ..., 3n\}$ for $n \geq 3$ to the generators of $A(CL_n)$, the following relations are obtained:

$$e_i^2 = (i-1)e_{i-1} + ie_{i+1} + (n+i)e_{n+i}, \text{ for } 2 \leq i \leq n-1;$$
$$e_i^2 = (n+i-1)e_{i-1} + (n+i)e_{i+1} + ie_{i-n}, \text{ for } n+2 \leq i \leq 2n-1;$$
$$e_1^2 = e_2 + ne_n + (n+1)e_{n+1};$$
$$e_n^2 = ne_1 + (n-1)e_{n-1} + 2ne_{2n};$$
$$e_{n+1}^2 = (n+1)e_1 + (2n+1)e_{n+2} + 3ne_{2n};$$
$$e_{2n}^2 = 3ne_{n+1} + (3n-1)e_{2n-1} + 2ne_n.$$

Likewise, one can easily check that the greatest common divisor of the coefficients of each term in the above result i.e., the gcd of $(i-1, i, i+n), (n+i, n+i-1, i), (1, n, n+1), (n, n-1, 2n), (n+1, 2n+1, 3n), (3n-1, 3n, 2n)$ are all equal to 1. Thus, the Circular ladder graphicable algebra is prime.

The figure below is a Circular ladder graphicable algebra $A(CL_n)$.



Circular Ladder graphicable algebra A(CLn)

Recall that the subgraphicable algebra $A(H)$ is the graphicable algebra associated to the subgraph H of G. This subgraphicable algebra is prime, if the prime relations on $A(G)$ is also prime on the set of weights of generators of $A(H)$. We now proceed to the edge deletion case of subgraphicable algebra by defining the edge deleted graphicable algebra as follows

Definition 4.2. *Let $A(G)$ be the graphicable algebra that is associated to a graph $G = (V, E)$. Then for any edge e of G, $A(G-e)$ is the edge deleted subgraphicable algebra of $A(G)$ with vertex set V, and edge set $E - e$.*

Translating the above fact of edge deletion into the language of graphicable algebras, it allows us to set the following Theorem.

Theorem 4.3. *Let any single edge of the form $e = v_i v_{n+i}$, for $2 \leq i \leq n$ be deleted. Then the subgraphicable algebras so obtained are $n-1$ edge deleted prime subgraphicable algebras $A(CL_n - e)$ of the Circular graphicable algebras $A(CL_n)$.*

Corollary 4.4. *Let all edges of the form $v_i v_{n+i}$ for $2 \leq i \leq n$ be deleted at a time.Then the prime subgrphicable algebra is obtained with $2n$ vertices and $2n+1$ edges.*

All the results stated in the previous two sections allows us to set the relation between the Ladder and Circular ladder graphs into the language of graphicable algebra in the form of the following theorem.

Theorem 4.5. *In the expressions of e_1^2 and e_{n+1}^2 for Circular ladder graphicable algebra $A(CL_n)$, additional terms viz., e_n and e_{2n} respectively and vice-versa will appear a term greater than the corresponding ladder graphicable algebra $A(L_n)$.*

5. Mobius Ladder Graphicable Algebra

The Mobius Ladder graph M_n is a cubic circular graph obtained from a cycle which has even number of vertices and by adding edges between the opposite pair of vertices. It has n vertices and $\frac{3n}{2}$ edges, where n is even number. Its corresponding associated graphicable algebra is defined as follows.

Using the same arguments, the graphicable algebra associated to this graph has the generators $\{e_1, e_2, e_3, e_4\}$ starting from $n = 4$ where n takes only even vertices and satisfies the law $e_1^2 = e_2 + 4e_4 + 5e_3$, $e_2^2 = e_1 + 2e_3 + 6e_4$, $e_3^2 = 2e_2 + 3e_4 + 5e_1$ and $e_4^2 = 3e_3 + 4e_1 + 6e_2$

and same could be done with M_6 and so on. So, as a consequence we can give the following.

Definition 5.1. *The Mobius graphicable algebra $A(M_n)$ for even $n \geq 4$ is accociated to the Mobius ladder graph M_n, if it satisfies the following equalities*

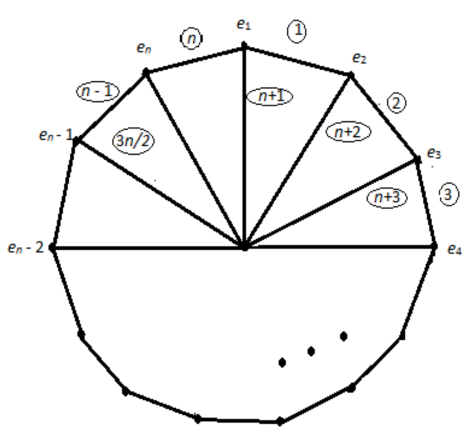
$$e_i^2 = e_{i-1} + e_{i+1} + e_{n/2+i}, \text{ for } 2 \leq i \leq \tfrac{n}{2};$$
$$e_i^2 = e_{i-1} + e_{i+1} + e_{in/2}, \text{ for } \tfrac{n+2}{2} \leq i \leq n-1;$$
$$e_1^2 = e_2 + e_n + e_{n/2+1}$$
$$\text{and}$$
$$e_n^2 = e_1 + e_{n-1} + e_{n/2}.$$

As in the case of Ladder graphicable algebras let $\{1, 2, 3, ..., |E|\}$ be the weights of the generators of $A(M_n)$. Then with respect to the Mobius ladder graphicable algebra we obtain the following results:

$$e_i^2 = (i-1)e_{i-1} + ie_{i+1} + (n+i)e_{n/2+i} \text{ for } 2 \leq i \leq \tfrac{n}{2};$$
$$e_i^2 = (i-1)e_{i-1} + ie_{i+1} + (\tfrac{n}{2} + i)e_{i-n/2} \text{ for } \tfrac{n+2}{2} \leq i \leq n-1;$$
$$e_1^2 = e_2 + ne_n + (n+1)e_{n/2+1};$$
$$e_n^2 = ne_1 + (n-1)e_{n-1} + \tfrac{3n}{2}e_{n/2}\ .$$

Thus, one can note that the greatest common divisors in above relations of any two successive natural numbers is 1. Therefore with the above weights, the Mobius graphicable algebra is prime.

Figure below is a Mobius ladder graphicable algebra $A(M_n)$.



Mobius Ladder graphicable algebra $A(M_n)$

Remark 5.2. *The ladder and Mobius ladder graphicable algebras are subalgebras of circular ladder graphicable algebras.*

6. Conclusion

In this study of vertex prime labelings of Ladder graphs, we see that the ladder graphicable algebra, circular ladder graphicable algebra and Mobius ladder graphicable algebras are prime.

References

1. Cadavid P., Rodino Montoya M. L. and Rodríguez P. M., *On the isomorphisms between evolution algebras of graphs and random walks*. (eprint arXiv:1710.10516).
2. Clark, J., Holton. D. A., *A first look at graph theory*, World scientific, (1991).
3. Camacho, L. M., Gomez J. R., Omirov. B. A., and Turdibaev, R. M.,*Some properties of evolution algebras*, Bull. Korean. Math. Soc., 50, No. 5, 1481-1494, (2013).
4. Tian, J. P. and Vojtechovsky. P., *Mathematical concepts of evolution algebra in non-mendelian genetics*, Quasi groups and related systems, 14, No. 1, 111-122, (2006).
5. Tian, J. P. and Vojtechovsky. P., *Evolution algebras and their applications*, (Lecture notes in Mathematics), No. 1921, Springer-Verlag, Berlin, (2008).
6. Nunez. J., Silverio. M. and Trinidad Villar. M., *Graph theory : a tool to study evolution algebras*, (Preprint), (2012).
7. Nunez. J., Rodriguez Arevalo. M. L. and Trinidad Villar. M., *Certain perticular families of graphicable algebras*, Appl. Math. Comput., 246, 416-425, (2014).
8. Rosa. A., *On certain valuations of the vertices of a graph*, Theory of graphs, Int. Symp. Rome July, 349-355, (1966).

R. Anantha Lakshmi and K.Jayalakshmi,
Department of Mathematics,
JNTUA College Of Engineering (Ananthapuramu),
India.
E-mail address: anantha.reddem@gmail.com, kjay.maths@jntua.ac.in