Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–11.
ISSN-0037-8712 in press
doi:10.5269/bspm.63371

Notions of β-Closure Compatible Topology with an Ideal

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ABSTRACT: In this paper, we have defined β-local closure function. Its properties and characterizations are analyzed. The set operator is defined and its properties are discussed. The notions of β-closure compatible topology with an ideal $\mathcal{I}$ are introduced and investigated. Moreover, -dense set and β_{**}-codense ideal are defined and explored.

Key Words: Ideal, β-local closure function, β-open set, β_{**}-codense ideal.

Contents

1. Introduction

Ahmad Al-Omari and Takashi Noiri [2] characterized local closure function and investigate its properties. In this paper, we have introduced and studied the notions of β-closure compatible topology with an ideal $\mathcal{I}$. Also, we define an operator $\beta_{**}\ (A)(\mathcal{I}, \tau)$ called β-local closure function of A with respect to $\mathcal{I}$ and τ as follows: An ideal on a nonempty set X is a collection of subsets of X which satisfies the following properties: (i) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$ (heredity) (ii) $A\in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$(finite additivity). An ideal topological space is a topological space $(X, \tau\ )$ with an ideal $\mathcal{I}$ on X, and is denoted by $(X, \tau, \mathcal{I})$. Given a topological space $(X, \tau\ )$ with an ideal $\mathcal{I}$ on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^*: P(X) \to P(X)$, called a *local function* [16] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X\colon U \cap A \notin \mathcal{I}$ for every $U \in \tau\ (x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$, when there is no chance for confusion $A^*(\mathcal{I}, \tau)$ is denoted by A^*. For every ideal topological space $(X, \tau, \mathcal{I})$ there exists a topology τ^* finer than τ, generated by the base $\beta(\mathcal{I}, \tau) = \{U - K : U \in \tau$ and $K \in \mathcal{I}\}$. In general, $\beta(\mathcal{I}, \tau)$ is not a topology.

A Kuratowski closure operator $cl\ ^*(.)$ for the topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$ *-topology* finer than τ is defined by $cl\ ^*(A){=}A \cup A^*(\mathcal{I}, \tau)$ [16]. In an ideal space $\mathcal{I}$ is said to be *codense* [6] if $\tau \cap \mathcal{I} = \{\phi\}$. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X, τ) and $cl^*(A)$ and $int^*(A)$ will respectively denote the closure and interior of A in (X, τ^*). A subset A of a space (X, τ) is *β-open* or *semi-pre-open* [3] set if $A \subset cl(int(cl(A)))]$. The complement of *$\beta$-open* or *semi-pre-open* set is *β-closed* or *semi-pre-closed*. The *semi pre-closure* of a subset A of X, denoted by $spcl(A)$ or $\beta cl(A)$, is defined to be the intersection of all semi-pre-closed sets containing A. The *semi pre-interior* of a subset A of X, denoted by $spint(A)$ or $\beta int(A)$, is defined to be the union of all semi-pre-open sets contained in A We have introduced in [10], a set operator $(.)^{**}: P(X) \to P(X)$, called a *semi-pre local function* or *β-local function* of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A_{**}\left(\mathcal{I},\ \tau\right) = \{x \in X\ : U \cap A \notin \mathcal{I} for every U \in \beta O\left(x\right)\}$ where the family of semi-preopen sets

2020 *Mathematics Subject Classification:* 54A05, 54F65.
Submitted April 26, 2022. Published December 11, 2022

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$style.

$\beta O(x) = \{U \in \beta O(X) : x \in U\}$, when there is no ambiguity, we will write simply A_{**} for $A_{**}(\mathfrak{I}, \tau)$. A set A is said to be θ-open [17] if every point of A has an open neighborhood whose closure is contained in A. In [13], Newcomb defines $A = B$ [*mod* $\mathfrak{I}$] if $(A \backslash B) \cup (B \backslash A) \in \mathfrak{I}$ and observed that "= [mod $\mathfrak{I}$]" is an equivalence relation and denoted the symmetric difference" $(A \backslash B) \cup (B \backslash A)$ by $A \triangle B$. In [14], the topology τ is compatible with the ideal $\mathfrak{I}$, denoted $\tau \sim \mathfrak{I}$, if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists a $U \in N(x)$ such that $U \cap A \in \mathfrak{I}$, then $A \in \mathfrak{I}$, where $N(x)$ denotes the open neighborhood system at x.

2. βlocal closure function in ideal topological space

Definition 2.1. A set A is said to be θ^{β}-open if every point of A has a β-open neighborhood whose closure is contained in A. The θ^{β}-interior of A in X is the union of all θ^{β}-open sets contained in A and is denoted by $int_{\theta^{\beta}}(A)$. Naturally, the complement of a θ^{β}-open set is said to be θ^{β}-closed. The θ^{β}-closure of A in X is the intersection of all θ^{β}-closed sets containing A and is denoted by $cl_{\theta^{\beta}}(A)$. Equivalently, $cl_{\theta^{\beta}}(A) = \{x \in \mathrm{X} : cl_{\theta^{\beta}}(U) \cap A \neq \phi$ for every $U \in \tau(x)\}$ and the set $A$is θ^{β}-closed if and only if $A = cl_{\theta^{\beta}}(A)$. Note that all θ^{β}-open sets form a topology on X which is coarser than τ, and is denoted by $\tau_{\theta^{\beta}}$ and that a space (X, τ) is β-regular if and only if $\tau = \tau_{\theta^{\beta}}$. Also, the θ^{β}-closure of a given set need not be a θ^{β}-closed set.

Example 2.2. Let $X = \{a,b,c,d\}$ with the topology $\tau = \{\phi, X, \{a\},\{b\},\{a,b\}\}$ and an ideal $\mathfrak{I} = \{\phi, \{b\},\{c\},\{b,c\}\}$. Here $A = \{c, d\} = cl_{\theta^{\beta}}(A)$, $B = \{a, c, d\} = cl_{\theta^{\beta}}(B)$, $C = \{b, c, d\} = cl_{\theta^{\beta}}(C)$ hence A, B, C are θ^{β}-closed and $\tau_{\theta^{\beta}} = \{\{c, d\}, \{a, c, d\}, \{b, c, d\}, X, \phi\}$.

Definition 2.3. Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. For a subset A of X, we define the following operator: $\beta_{**}(A)(\mathfrak{I}, \tau) = \{x \in X : A \cap cl_{\beta}(U) \notin \mathfrak{I}$ for every $U \in \beta O(x)\}$, where $\beta O(x) = \{U \in \beta O(X) : x \in U\}$ and in case there is no confusion $\beta_{**}(A)(\mathfrak{I}, \tau)$ is briefly denoted by $\beta_{**}(A)$ and is called the β-local closure function of A with respect to $\mathfrak{I}$ and τ.

For example:

1. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}\}$ and an ideal $\mathfrak{I} = \{\phi, \{b\}\}$. Here, $\beta O(X) = \{\{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X, \phi\}$. Also, $A = \{a\}$, $\beta_{**}(A) = \{a\}$, $B = \{b\}$, $\beta_{**}(B) = \phi$, $C = \{c\}, \beta_{**}(C) = \{c\}$, $D = \{a, b\}, \beta_{**}(D) = \{a\}$, $E = \{a,c\}, \beta_{**}(E) = \{a,c\}$, $F = \{b,c\}, \beta_{**}(F) = \{c\}$, $G = \phi, \beta_{**}(G) = \phi$, $H = X$, $\beta_{**}(H) = \{a,c\}$.
2. Consider $\mathbb{R}$ with the usual topology τ_u and the ideal $\mathfrak{I} = \{\phi\}$. For the set $A = [0,1]$, $int(A) = (0,1), cl(A) = \mathbb{R}$. Therefore, $A = [0,1] \in \beta O(X)$ and $cl_{\beta}(A) = \mathbb{R}$. Hence, $\beta_{**}(A) = \mathbb{R}$.
3. Consider $\mathbb{R}$ with the topology and any ideal $\mathfrak{I}$. For the set $A = \mathbb{Z}^{+}$, $int(A) = \phi, cl(A) = \mathbb{R}$.Therefore, $A = \mathbb{Z}^{+} \in \beta O(X)$ and $cl_{\beta}(A) = \mathbb{R}$. Hence, $\beta_{**}(A) = \mathbb{R}$.

Lemma 2.4. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. Then $A_{**} \subseteq \beta_{**}(A)$ for every subset A of X.*
Proof: Let $x \in A_{**}$. Then, $A \cap U \notin \mathfrak{I}$ for every $U \in \beta O(x)$. Since $A \cap U \subseteq A \cap cl_{\beta}(U)$, $A \cap cl_{\beta}(U) \notin \mathfrak{I}$. Thus, $x \in \beta_{**}(A)$.

Example 2.5. Let $X = \{a,b,c,d\}$ with the topology $\tau = \{\phi, X, \{a\},\{b\},\{a,b\}\}$ and an ideal $\mathfrak{I} = \{\phi, \{b\},\{c\},\{b,c\}\}$. Here $\beta O(X) = \{\{a\}, \{b\},\{a, b\},\{a, c\},\{a, d\},\{b, c\},\{b, d\},\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $A = \{a\}$ then $\beta_{**}(A) = \{a\}$.

Lemma 2.6. *Let (X, τ) be a topological space and A be a subset of X. Then*

1. *If A is closed, then $cl(A) = cl_{\theta^{\beta}}(A)$,*
2. *If A is open, then $int(A) = int_{\theta^{\beta}}(A)$*

Theorem 2.7. *Let (X, τ) be a topological space, $\mathfrak{I}_1$, $\mathfrak{I}_2$ be two ideals on X, and let A and B be subsets of X. Then the following properties hold:*

1. *If $A \subseteq B$, then $\beta_{**}(A) \subseteq \beta_{**}(B)$*

2. *If* $\mathcal{I}_1 \subseteq \mathcal{I}_2$, *then* $\beta_{**}(A)(\mathcal{I}_1) \supseteq \beta_{**}(A)(\mathcal{I}_2)$.

3. *If* $A \subseteq X$, *then* $\beta_{**}(A) = cl(\beta_{**}(A)) \subseteq cl_{\theta^\beta}(A)$ *and* $\beta_{**}(A)$ *is closed.*

4. *If* $A \subseteq \beta_{**}(A)$ *and* $\beta_{**}(A)$ *is open, then* $\beta_{**}(A) = cl_{\theta^\beta}(A)$.

5. *If* $A \in \mathcal{I}$, *then* $\beta_{**}(A) = \phi$.

Proof:

1. Suppose that $x \notin \beta_{**}(B)$. Then there exists $U \in \beta O(x)$ such that $B \cap cl_\beta(U) \in \mathcal{I}$. Since $A \cap cl_\beta(U) \subseteq B \cap cl_\beta(U)$ implies $A \cap cl_\beta(U) \in \mathcal{I}$. Hence $x \notin \beta_{**}(A)$. Thus $X - \beta_{**}(B) \subseteq X - \beta_{**}(A)$ that is $\beta_{**}(A) \subseteq \beta_{**}(B)$.

2. Suppose that $x \notin \beta_{**}(A)(\mathcal{I}_1)$. There exists $U \in \beta O(X))$ such that $A \cap cl_\beta(U) \in \mathcal{I}_1$. Since $\mathcal{I}_1 \subseteq \mathcal{I}_2$, $A \cap cl_\beta(U) \in \mathcal{I}_2$ and $x \notin \beta_{**}(A)(\mathcal{I}_2)$. Hence $\beta_{**}(A)(\mathcal{I}_2) \subseteq \beta_{**}(A)(\mathcal{I}_1)$.

3. We have $\beta_{**}(A) \subseteq cl(\beta_{**}(A))$. Suppose that $x \in cl(\beta_{**}(A))$. Then $\beta_{**}(A) \cap U \neq \phi$ for every $U \in \beta O(X)$. Therefore, there exists some $y \in \beta_{**}(A) \cap U$ and $U \in \beta O(y)$. Since $y \in \beta_{**}(A)$, $A \cap cl_\beta(U) \notin \mathcal{I}$ and hence $x \in \beta_{**}(A)$. Therefore, we have $cl(\beta_{**}(A)) \subseteq \beta_{**}(A)$ and hence $\beta_{**}(A) = cl(\beta_{**}(A))$. Now, let $x \in cl_\beta(\beta_{**}(A)) = \beta_{**}(A)$, then $A \cap cl_\beta(U) \notin \mathcal{I}$ for every $U \in \beta O(x)$ and it implies that $A \cap cl_\beta(U) \neq \phi$ for every $U \in \beta$O(x)). Therefore, $x \in cl_{\theta^\beta}(A)$. Thus, $\beta_{**}(A) = cl(\beta_{**}(A)) \subseteq cl_{\theta^\beta}(A)$.

4. For any subset A of X, by (3) we have $\beta_{**}(A) = cl(\beta_{**}(A)) \subseteq cl_{\theta^\beta}(A)$. Since $A \subseteq \beta_{**}(A)$ and $\beta_{**}(A)$ is open, then $cl_{\theta^\beta}(A) \subseteq cl_{\theta^\beta}(\beta_{**}(A)) = cl(\beta_{**}(A)) = \beta_{**}(A) \subseteq cl_{\theta^\beta}(A)$, by Lemma 2.6, and hence $\beta_{**}(A) = cl_{\theta^\beta}(A)$.

5. Suppose that $x \in \beta_{**}(A)$. Then for any $U \in \beta O(x)$, $A \cap cl_\beta(U) \notin \mathcal{I}$. Since $A \in \mathcal{I}$, then $A \cap cl_\beta(U) \in \mathcal{I}$ for every $U \in \beta O(x)$. This is contradiction. Hence $\beta_{**}(A) = \phi$.

Lemma 2.8. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* $U \in \tau_{\theta^\beta}$, *then* $U \cap \beta_{**}(A) = U \cap \beta_{**}(U \cap A) \subseteq \beta_{**}(U \cap A)$ *for any subset* A *of* X.

Proof: Suppose that $U \in \tau_{\theta^\beta}$ and $x \in U \cap \beta_{**}(A)$. Then $x \in U$ and $x \in \beta_{**}(A)$. Since $U \in \tau_{\theta^\beta}$, then there exists $G \in \beta O(x)$ such that $x \in G \subseteq cl(G) \subseteq U$. Let H be any β-open set containing x. Then $H \cap G \in \beta O(x)$ and $cl_\beta(H \cap G) \cap A \notin \mathcal{I}$ and hence $cl_\beta(H) \cap (U \cap A) \notin \mathcal{I}$. This implies that $x \in \beta_{**}(U \cap A)$ and hence we obtain $U \cap \beta_{**}(A) \subseteq \beta_{**}(U \cap A)$. Also, $U \cap \beta_{**}(A) \subseteq U \cap \beta_{**}(U \cap A)$ and by Theorem 2.7, $\beta_{**}(U \cap A) \subseteq \beta_{**}(A)$ and $U \cap \beta_{**}(U \cap A) \subseteq U \cap \beta_{**}(A)$. Thus, $U \cap \beta_{**}(A) = U \cap \beta_{**}(U \cap A)$.

Theorem 2.9. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* A, B *any subsets of* X. *Then the following properties hold:*

1. $\beta_{**}(\phi) = \phi$

2. $\beta_{**}(A) \cup \beta_{**}(B) = \beta_{**}(A \cup B)$.

Proof:

1. The proof is obvious.

2. It follows from Theorem 2.7 that $\beta_{**}(A \cup B) \supseteq \beta_{**}(A) \cup \beta_{**}(B)$. Let $x \notin \beta_{**}(A) \cup \beta_{**}(B)$. Then x belongs neither to $\beta_{**}(A)$ nor to $\beta_{**}(B)$ and there exist $U_x, V_x \in \beta O(x)$ such that $A \cap cl_\beta(U_x) \in \mathcal{I}$ and $B \cap cl_\beta(V_x) \in \mathcal{I}$. Since $\mathcal{I}$ is additive, $[A \cap cl_\beta(U_x)] \cup [B \cap cl_\beta(V_x)] \in \mathcal{I}$. Also, by hereditary $[A \cap cl_\beta(U_x)] \cup [B \cap cl_\beta(V_x)] = [(A \cap cl_\beta(U_x)) \cup B] \cap [(A \cap cl_\beta(U_x)) \cup cl_\beta(V_x)] = [(A \cup B) \cap (cl_\beta(U_x) \cup B)] \cap [(A \cup cl_\beta(V_x)) \cap (cl_\beta(U_x) \cup cl_\beta(V_x))] \supseteq cl_\beta(U_x \cap V_x) \cap (A \cup B)$ and $cl_\beta(U_x \cap V_x) \cap (A \cup B) \in \mathcal{I}$. Since $U_x \cap V_x \in \beta O(x)$, $x \notin \beta_{**}(A \cup B)$. Hence $(X - \beta_{**}(A)) \cap (X - \beta_{**}(B)) \subseteq X - \beta_{**}(A \cup B)$ or $\beta_{**}(A \cup B) \subseteq \beta_{**}(A) \cup \beta_{**}(B)$. Thus, $\beta_{**}(A) \cup \beta_{**}(B) = \beta_{**}(A \cup B)$.

Lemma 2.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B any subsets of X. Then $\beta_{**}(A) - \beta_{**}(B) = \beta_{**}(A - B) - \beta_{**}(B)$.*

Proof: Let $A, B \subseteq X$ then by Theorem 2.9, $\beta_{**}(A) = \beta_{**}[(A - B) \cup (A \cap B)] = \beta_{**}(A - B) \cup \beta_{**}(A \cap B) \subseteq \beta_{**}(A - B) \cup \beta_{**}(B)$. Thus, $\beta_{**}(A) - \beta_{**}(B) \subseteq \beta_{**}(A - B) - \beta_{**}(B)$. By Theorem 2.7, $\beta_{**}(A - B) \subseteq \beta_{**}(A)$ and hence $\beta_{**}(A - B) - \beta_{**}(B) \subseteq \beta_{**}(A) - \beta_{**}(B)$. Hence, $\beta_{**}(A) - \beta_{**}(B) = \beta_{**}(A - B) - \beta_{**}(B)$.

Corollary 2.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be any subsets of X with $B \in \mathcal{I}$. Then $\beta_{**}(A \cup B) = \beta_{**}(A) = \beta_{**}(A - B)$.*

Proof: Since $B \in \mathcal{I}$, by Theorem 2.7, $\beta_{**}(B) = \phi$. By Lemma 2.10, $\beta_{**}(A) = \beta_{**}(A - B)$ and by Theorem 2.9, $\beta_{**}(A \cup B) = \beta_{**}(A) \cup \beta_{**}(B) = \beta_{**}(A)$.

Theorem 2.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\beta_{**}(A) \supseteq A - \cup\{U \subseteq X : U \in \mathcal{I}\}$ for all $A \subseteq X$.*

Proof: Let $B = \cup\{U \subseteq X : U \in \mathcal{I}\}$ and $x \in A - B$. Then $x \notin B$ implies that $x \notin U$ for all $U \in \mathcal{I}$, so that $\{x\} = \{x\} \cap A \notin \mathcal{I}$, because $x \in A$. For every $V \in \beta O(x)$, we have $\{x\} \cap A \subseteq cl_\beta(V) \cap A \notin \mathcal{I}$, by heredity and hence $x \in \beta_{**}(A)$.

Theorem 2.13. *Let (X, τ) be a topological space with ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on X and $A \subseteq X$, then $\beta_{**}(A)(\mathcal{I}_1 \cap \mathcal{I}_2) \subseteq \beta_{**}(A)(\mathcal{I}_1) \cap \beta_{**}(A)(\mathcal{I}_2)$.Moreover, $\beta_{**}(A)(\mathcal{I}_1 \cap \mathcal{I}_2) \subseteq \beta_{**}(A)(\mathcal{I}_1) \cup \beta_{**}(A)(\mathcal{I}_2)$*

Proof: Let $x \in \beta_{**}(A)(\mathcal{I}_1 \cap \mathcal{I}_2)$. Then $cl_\beta(U) \cap A \notin \mathcal{I}_1$ and $cl_\beta(U) \cap A \notin \mathcal{I}_2$ for every $U \in \beta O(x)$. Hence, $x \in \beta_{**}(A)(\mathcal{I}_1)$ and $x \in \beta_{**}(A)(\mathcal{I}_2)$ implies $x \in \beta_{**}(A)(\mathcal{I}_1) \cap \beta_{**}(A)(\mathcal{I}_2)$. Thus $x \in \beta_{**}(A)(\mathcal{I}_1) \cup \beta_{**}(A)(\mathcal{I}_2)$.

3. $\Psi_{\beta_{**}}$ operator in ideal topological space

Definition 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. An operator $\Psi_{\beta_{**}} : P(X) \to \tau$ is defined as follows: for every $A \subseteq X$, $\Psi_{\beta_{**}}(A) = \{x \in X :$ there exists $U \in \beta O(x)$ such that $cl_\beta(U) - A \in \mathcal{I}\}$ and observe that $\Psi_{\beta_{**}}(A) = X - \beta_{**}(X - A)$.

For example: Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}\}$ and an ideal $\mathcal{I} = \{\phi, \{b\}\}$. Here $\beta O(X) = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X, \phi\}$. For the set $A = \{a\}$, $\beta_{**}(A) = \{a\}$ and $\beta_{**}(X - A) = \beta_{**}(\{b, c\}) = X$. Then, $\Psi_{\beta_{**}}(A) = X - \beta_{**}(X - A) = X - X = \phi$.

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties hold:*

1. *If $A \subseteq X$, then $\Psi_{\beta_{**}}(A)$ is β-open.*
2. *If $A \subseteq B$, then $\Psi_{\beta_{**}}(A) \subseteq \Psi_{\beta_{**}}(B)$*
3. *If A, $B \in P(X)$, then $\Psi_{\beta_{**}}(A \cap B) = \Psi_{\beta_{**}}(A) \cap \Psi_{\beta_{**}}(B)$.*
4. *If $A \subseteq X$, then $\Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}(\Psi_{\beta_{**}}(A))$ if and only if $\beta_{**}(X - A) = \beta_{**}(\beta_{**}(X - A))$.*
5. *If $A \in \mathcal{I}$, then $\Psi_{\beta_{**}}(A) = X - \beta_{**}(X)$.*
6. *If $A \subseteq X$, $I \in \mathcal{I}$, then $\Psi_{\beta_{**}}(A - I) = \Psi_{\beta_{**}}(A)$.*
7. *If $A \subseteq X$, $I \in \mathcal{I}$, then $\Psi_{\beta_{**}}(A \cup I) = \Psi_{\beta_{**}}(A)$.*
8. *If $(A - B) \cup (B - A) \in \mathcal{I}$,then $\Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}(B)$.*

Proof:

1. Obvious by Definition.
2. Let $A \subseteq B$ and $x \in \Psi_{\beta_{**}}(A)$ then there exists $U \in \beta O(x)$ such that $cl_\beta(U) - A \in \mathcal{I}$. Since $A \subseteq B$ then $cl_\beta(U) - B \in \mathcal{I}$, for $U \in \beta O(x)$. Hence $x \in \Psi_{\beta_{**}}(B)$.

3. Let $A,\ B \in P(X)$. $\Psi_{\beta_{**}}(A \cap B) = X - \beta_{**}\ (X - (A \cap B)) = X - \beta_{**}\ [(X - A) \cup (X - B)] = [X - \beta_{**}\ (X - A)] \cap [X - \beta_{**}\ (X - B)] = \Psi_{\beta_{**}}(A) \cap \Psi_{\beta_{**}}(B)$.

4. Let $A \subseteq X$ and $\beta_{**}\ (X - A) = \beta_{**}\ (\beta_{**}\ (X - A))$. Then $\Psi_{\beta_{**}}(A) = X - \beta_{**}\ (X - A)$. Now, $\Psi_{\beta_{**}}\left(\Psi_{\beta_{**}}(A)\right) = X - \beta_{**}\ [(X - (X - \beta_{**}\ (X - A))] = X - \beta_{**}(\beta_{**}(X - A)) = X - \beta_{**}(X - A) = \Psi_{\beta_{**}}(A)$. Conversely, assume that $\Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}\left(\Psi_{\beta_{**}}(A)\right)$. Then $X - \beta_{**}\ (X - A) = X - \beta_{**}\ [(X - (X - \beta_{**}\ (X - A))]$. Hence, $\beta_{**}(X - X) = \beta_{**}(\beta_{**}(X - A))$.

5. If$A \in \mathcal{I}$, then $\Psi_{\beta_{**}}(A) = X - \beta_{**}(X - A) = X - \beta_{**}\ (X)$ by Corollary 2.11.

6. If $A \subseteq X$ and $I \in \mathcal{I}$, then $\Psi_{\beta_{**}}(A - I) = X - \beta_{**}[X - (A - I)] = X - \beta_{**}[(X - A) \cup I] = X - [\beta_{**}(X - A) \cup \beta_{**}\ (I)] = X - [\beta_{**}(X - A) \cup \phi] = X - \beta_{**}(X - A) = \Psi_{\beta_{**}}(A)$ by Theorem 2.7 (5).

7. If $A \subseteq X$, $I \in \mathcal{I}$, then $\Psi_{\beta_{**}}(A \cup I) = X - \beta_{**}\ [X - (A \cup I)] = X - \beta_{**}[(X - A) - I] = X - \beta_{**}\ (X - A) = \Psi_{\beta_{**}}(A)$, by Corollary 2.11.

8. Assume that $(A - B) \cup (B - A) \in \mathcal{I}$. Let $A - B = I$ and $B - A = J$. Observe that $I,\ J \in \mathcal{I}$ by heredity. Also, $B = (A - I) \cup J$. Thus $\Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}(A - I) = \Psi_{\beta_{**}}[(A - I) \cup J] = \Psi_{\beta_{**}}(B)$, by Corollary 2.11.

Corollary 3.3. *Let $(X,\ \tau,\ \mathcal{I})$ be an ideal topological space. Then $A \subseteq \Psi_{\beta_{**}}(A)$ for every θ^{β}-open set $A \subseteq X$.*
Proof: We know that $\Psi_{\beta_{**}}(A) = X - \beta_{**}\ (X - A)$. Now, $\beta_{**}\ (X - A) \subseteq cl_{\theta^{\beta}}\ (X - A) = X - A$, since $X - A$ is θ^{β}-closed. Therefore, $A = X - (X - A) \subseteq X - \beta_{**}\ (X - A) = \Psi_{\beta_{**}}(A)$.

The converse of this statement is not true by the following example.

Example 3.4. Let $X = \{a,b,c,d\}$ with the topology $\tau = \{\phi,\ X,\ \{a\},\ \{b\},\ \{a,b\}\}$ and the ideal $\mathcal{I} = \{\phi,\ \{b\}, \{c\},\{b,c\}\}$. Here $\beta O(X) = \{\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},X,\phi\}$. Let $A = \{a\}$. Then $\Psi_{\beta_{**}}(A) = X - \beta_{**}\ (X - A) = X - \beta_{**}\ (X - \{a\}) = X - \beta_{**}(\{b,c,d\}) = X - \{d\} = \{a,b,c\}$, since $\beta_{**}(\{b,c,d\}) = \{d\}$. Hence, $A \subseteq \Psi_{\beta_{**}}(A)$ but A is not θ^{β}-open.
Theorem 3.5. *Let $(X,\ \tau,\ \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following properties hold:*

1. $\Psi_{\beta_{**}}(A) = \bigcup\{U \in \beta O(X) : cl_{\beta}\,(U) - A \in \mathcal{I}\}$

2. $\Psi_{\beta_{**}}(A) \supseteq \bigcup\{U \in \beta O(X) : (cl_{\beta}\,(U) - A) \cup (A - cl_{\beta}\,(U)) \in \mathcal{I}\}$.

Proof: (a) This follows immediately from the definition of $\Psi_{\beta_{**}}$-operator.
(b) Since $\mathcal{I}$ is heredity, it is obvious that $\bigcup\{U \in \beta O(X) : (cl_{\beta}\,(U) - A) \cup (A - cl_{\beta}\,(U)) \in \mathcal{I}\} \subseteq \bigcup\{U \in \beta O(X) : cl_{\beta}\,(U) - A \in \mathcal{I}\} = \Psi_{\beta_{**}}(A)$ for every $A \subseteq X$.

Theorem 3.6. *Let $(X,\ \tau,\ \mathcal{I})$ be an ideal topological space. If $\mu = \{A \subseteq X : A \subseteq \Psi_{\beta_{**}}(A)\}$. Then μ is a topology for X.*
Proof: Let $\mu = \{A \subseteq X : A \subseteq \Psi_{\beta_{**}}(A)\}$. Since $\phi \in \mathcal{I}$, by Theorem 2.6 (5), $\beta_{**}(\phi) = \phi$ and $\Psi_{\beta_{**}}(X) = X - \beta_{**}(X - X) = X - \beta_{**}(\phi) = X$. Also, $\Psi_{\beta_{**}}(\phi) = X - \beta_{**}\ (X - \phi) = X - X = \phi$. Therefore, we obtain that $\phi \subseteq \Psi_{\beta_{**}}(\phi)$ and $X \subseteq \Psi_{\beta_{**}}(X) = X$ implies ϕ and $X \in \mu$. Now if $A,\ B \in \mu$, then by Theorem 3.2, $A \cap B \subseteq \Psi_{\beta_{**}}(A) \cap \Psi_{\beta_{**}}(B) = \Psi_{\beta_{**}}(A \cap B)$ which implies that $A \cap B \in \mu$. If $\{A_{\alpha} : \alpha \in \Delta\} \subseteq \mu$, then $A_{\alpha} \subseteq \Psi_{\beta_{**}}(A_{\alpha}) \subseteq \Psi_{\beta_{**}}(\cup A_{\alpha})$ for every α and hence $\cup A_{\alpha} \subseteq \Psi_{\beta_{**}}(\cup A_{\alpha})$. This shows that μ is a topology.

Lemma 3.7. *If either $A \in \beta O(X)$ or $B \in \beta O(X)$, then $int(cl(A \cap B)) = int(cl(A)) \cap int(cl(B))$.*
Theorem 3.8. *Let $\mu_0 = \{A \subseteq X : A \subseteq int(cl(\Psi_{\beta_{**}}(A)))\}$, then μ_0 is a topology for X.*
Proof: For any subset A of X, $\Psi_{\beta_{**}}$(A) is open and $\mu \subset \mu_0$ by Theorem 3.2. Hence $\phi,\ X \in \mu_0$. Let $A,\ B \in \mu_0$. Then $A \cap B \subset int(cl(\Psi_{\beta_{**}}(A))) \cap int(cl(\Psi_{\beta_{**}}(B))) = int(cl(\Psi_{\beta_{**}}(A) \cap \Psi_{\beta_{**}}(B))) = int(cl(\Psi_{\beta_{**}}(A \cap B)))$, by Lemma 3.7 and Theorem 3.2. Hence $A \cap B \in \mu_0$. Let $A_{\alpha} \in \mu_0$ for each $\alpha \in J$. By Theorem 3.2, for each $\alpha \in J$, $A_{\alpha} \subseteq int(cl(\Psi_{\beta_{**}}(A_{\alpha}))) \subseteq int(cl(\Psi_{\beta_{**}}(\cup A_{\alpha})))$ implies $\cup A_{\alpha} \subset int(cl(\Psi_{\beta_{**}}(\cup A_{\alpha})))$. Hence $\cup A_{\alpha} \in \mu_0$. Thus, μ_0 is a topology for X. The above discussion is shown clearly in the set implications: Figure 1.

$$\begin{tikzcd} \theta^{\beta}\text{-open} \arrow[r] \arrow[d] & \beta\text{-open} \\ \mu\text{-open} \arrow[r] & \mu_0\text{-open} \end{tikzcd}$$

Figure 1: Sets implications

4. βclosure compatible topology with an ideal

Definition 4.1. Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. We say the τ is β-closure compatible with the ideal $\mathfrak{I}$, denoted $\tau \sim_{\beta_{**}} \mathfrak{I}$ if the following holds for every $A \subseteq X$, if for every $x \in A$ there exists $U \in \beta O(x)$ such that $A \cap cl_{\beta}(U) \in \mathfrak{I}$, then $A \in \mathfrak{I}$.

For example: Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi,\ X,\{a\},\{b\},\{a,b\}\}$ and an ideal $\mathfrak{I} = \{\phi, \{b\}\}$. Here $\beta O\,(X) = \{\{a\}\,,\{b\}\,,\{a,b\}\,,\ \{a,c\}\,,\{b,c\}\,,X,\phi\}$. For the set $A_1 = \{a\}$ there exists $U = \{a,c\} \in \beta O\,(X,a)$ such that $cl_{\beta}\,(U) \cap A_1 = \{a\} \notin \mathfrak{I}$ and $A_1 \notin \mathfrak{I}$, for the set $A_2 = \{$b$\}$ there exists $U = \{b,c\} \in \beta O\,(X,b)$ such that $U \cap A_2 = \{b\} \in \mathfrak{I}$ and $A_2 \in \mathfrak{I}$, for the set $A_3 = \{c\}$ there exists $U = \{b,c\} \in \beta O\,(X,c)$ such that $U \cap A_3 = \{c\} \notin \mathfrak{I}$ and $A_3 \notin \mathfrak{I}$, and so on. Hence $\tau \sim_{\beta_{**}} \mathfrak{I}$.

Theorem 4.2. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space, the following properties are equivalent:*

1. $\tau \sim_{\beta_{**}} \mathfrak{I}$
2. *If a subset A of X has a cover of open sets each of whose β-closure intersection with A is in $\mathfrak{I}$, then $A \in \mathfrak{I}$.*
3. *For every $A \subseteq X$, $A \cap \beta_{**}(A) = \phi$ implies that $A \in \mathfrak{I}$.*
4. *For every $A \subseteq X$, $A - \beta_{**}(A) \in \mathfrak{I}$.*
5. *For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq \beta_{**}(B)$, then $A \in \mathfrak{I}$.*

Proof: (1) $\Rightarrow$ (2): The proof is obvious by Definitions.
(2) $\Rightarrow$ (3): Let $A \subseteq X$ and $x \in A$. Then $x \notin \beta_{**}(A)$ and there exists $V_x \in \beta O(x)$ such that $A \cap cl_{\beta}\,(V_x) \in \mathfrak{I}$. Therefore, we have $A \subseteq \bigcup\{\,V_x : x \in A\}$ and $V_x \in \beta O(x)$ and hence by $A \in \mathfrak{I}$.
(3) $\Rightarrow$ (4): For any $A \subseteq X$, $A - \beta_{**}(A) \subseteq A$ and $(A - \beta_{**}(A)) \cap \beta_{**}(A - \beta_{**}(A)) \subseteq (A - \beta_{**}(A)) \cap \beta_{**}(A) = \phi$. Hence $A - \beta_{**}(A) \in \mathfrak{I}$, by (3).
(4) $\Rightarrow$ (5): For every $A \subseteq X$, $A - \beta_{**}(A) \in \mathfrak{I}$, by (4). Let $A - \beta_{**}(A) = J \in \mathfrak{I}$, then $A = J \cup (A \cap \beta_{**}(A))$ and by Theorem 2.9 (2) and Theorem 2.6 (5), $\beta_{**}(A) = \beta_{**}\,(J) \cup \beta_{**}(A \cap \beta_{**}(A)) = \beta_{**}(A \cap \beta_{**}(A))$. Therefore, we have $A \cap \beta_{**}(A) = A \cap \beta_{**}(A \cap \beta_{**}(A)) \subseteq \beta_{**}(A \cap \beta_{**}(A))$ and $A \cap \beta_{**}(A) \subseteq A$. By hypothesis, $A \cap \beta_{**}(A) = \phi$ and hence $A - \beta_{**}\,(A) \in \mathfrak{I}$.
(5) $\Rightarrow$ (1): Let $A \subseteq X$ and assume that for every $x \in A$, there exists $U \in \beta O(x)$ such that $A \cap cl(U) \in \mathfrak{I}$. Then $A \cap \beta_{**}(A) \subseteq A$. Hence, $A \cap \beta_{**}(A) = \phi$, by (5). Suppose that A contains B such that $B \subseteq \beta_{**}(B)$. Then $B = B \cap \beta_{**}(B) \subseteq A \cap \beta_{**}\,(A) = \phi$. Hence, A contains no nonempty subset B with $B \subseteq \beta_{**}(B)$. Thus, $A \in \mathfrak{I}$.

Theorem 4.3. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. If $\tau \sim_{\beta_{**}} \mathfrak{I}$, then the following are equivalent.*

1. *For every $A \subseteq X$, $A \cap \beta_{**}(A) = \phi$ implies that $\beta_{**}(A) = \phi$*
2. *For every $A \subseteq X$, $\beta_{**}(A - \beta_{**}(A)) = \phi$*
3. *For every $A \subseteq X$, $\beta_{**}(A \cap \beta_{**}(A)) = \beta_{**}(A)$.*

Proof: First we show that (1) holds if $\tau \sim_{\beta_{**}} \mathfrak{I}$. For every $A \subseteq X$, and $A \cap \beta_{**}(A) = \phi$, by Theorem 4.2 (3), $A \in \mathfrak{I}$ and by Theorem 2.7 (5), $\beta_{**}(A) = \phi$.
(1) $\Rightarrow$ (2): Let $B = A - \beta_{**}(A)$, then $B \cap \beta_{**}(B) = (A - \beta_{**}(A)) \cap \beta_{**}(A - \beta_{**}(A)) = [A \cap (X -$

β_{**} $(A))] \cap \beta_{**}(A \cap (X - \beta_{**}(A))) \subseteq [A \cap (X - \beta_{**}(A))] \cap [\beta_{**}(A) \cap \beta_{**}(X - \beta_{**}(A))] = \phi$. This implies $\beta_{**}(B) = \phi$, by hypothesis. Hence, $\beta_{**}(A - \beta_{**}(A)) = \phi$.
(2) $\Rightarrow$ (3): Assume that for every $A \subseteq X$, $\beta_{**}(A - \beta_{**}$ $(A)) = \phi$. Now, $A = (A - \beta_{**}(A)) \cup (A \cap \beta_{**}(A))$ implies β_{**} $(A) = \beta_{**}[(A - \beta_{**}(A)) \cup (A \cap \beta_{**}(A))] = \beta_{**}(A - \beta_{**}(A)) \cup \beta_{**}(A \cap \beta_{**}(A))$ $= \beta_{**}(A \cap \beta_{**}(A))$.
(3) $\Rightarrow$ (1): Assume that for every $A \subseteq X$, $A \cap \beta_{**}(A) = \phi$ and $\beta_{**}(A \cap \beta_{**}$ $(A)) = \beta_{**}$ (A). This implies that $\phi = \beta_{**}(\phi) = \beta_{**}(A)$.

Theorem 4.4. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space, and $\tau{\sim}_{\beta_{**}}\mathfrak{I}$. Then for every $G \in \tau_{\theta^\beta}$ and any subset A of X, $cl(\beta_{**}(G \cap A)) = \beta_{**}(G \cap A) \subseteq \beta_{**}(G \cap \beta_{**}(A)) \subseteq cl_{\theta^\beta}$ $(G \cap \beta_{**}(A))$.*
Proof: By Theorem 4.3 (3) and Theorem 2.7, we have β_{**} $(G \cap A) = \beta_{**}((G \cap A) \cap \beta_{**}(G \cap A))$ $\subseteq \beta_{**}(G \cap \beta_{**}(A))$. Also, $cl(\beta_{**}(G \cap A)) = \beta_{**}(G \cap A) \subseteq \beta_{**}(G \cap \beta_{**}(A)) \subseteq cl_{\theta^\beta}$ $(G \cap \beta_{**}(A))$, by Theorem 2.6.

Theorem 4.5. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. Then $\tau{\sim}_{\beta_{**}}\mathfrak{I}$ if and only if $\Psi_{\beta_{**}}(A) - A \in \mathfrak{I}$ for every $A \subseteq X$.*
Proof: *Necessity:* Let $A \subseteq X$. Assume that $\tau \sim_{\beta_{**}}\mathfrak{I}$. Observe that $x \in \Psi_{\beta_{**}}$ $(A) - A$ if and only if $x \notin A$ and $x \notin \beta_{**}(X - A)$ if and only if $x \notin A$ and there exists $U_x \in \beta O(x)$ such that $cl_\beta(U_x) - A \in \mathfrak{I}$ if and only if there exists $U_x \in \beta O(x)$ such that $x \in cl_\beta(U_x) - A \in \mathfrak{I}$. Now, for each $x \in \Psi_{\beta_{**}}(A) - A$ and $U_x \in \beta O(x)$, $cl_\beta(U_x) \cap (\Psi_{\beta_{**}}(A) - A) \in \mathfrak{I}$, by heredity and hence $\Psi_{\beta_{**}}(A) - A \in \mathfrak{I}$, by assumption that $\tau \sim_{\beta_{**}}\mathfrak{I}$.
Sufficiency: Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in \beta O(x)$ such that $cl_\beta(U_x) \cap A \in \mathfrak{I}$. Observe that $\Psi_{\beta_{**}}(X - A) - (X - A) = A - \beta_{**}(A) = \{x :$ there exists $U_x \in \beta O(x)$ such that $x \in cl_\beta(U_x) \cap A \in \mathfrak{I}\}$. Hence $A \subseteq \Psi_{\beta_{**}}(X - A) - (X - A) \in \mathfrak{I}$ and implies $A \in \mathfrak{I}$, by heredity of $\mathfrak{I}$.

Theorem 4.6. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space with $\tau \sim_{\beta_{**}}\mathfrak{I}$, $A \subseteq X$. If U is a nonempty open subset of $\beta_{**}(A) \cap \Psi_{\beta_{**}}(A)$, then $U - A \in \mathfrak{I}$ and $cl_\beta(U) \cap A \notin \mathfrak{I}$.*
Proof: If $U \subseteq \beta_{**}(A) \cap \Psi_{\beta_{**}}(A)$, then $U - A \subseteq \Psi_{\beta_{**}}(A) - A \in \mathfrak{I}$, by Theorem 4.5. Hence, $U - A \in \mathfrak{I}$, by heredity. Since $U \in \beta O(X) - \{\phi\}$ and $U \subseteq \beta_{**}(A)$, we have $cl_\beta(U) \cap A \notin \mathfrak{I}$ by the definition of $\beta_{**}(A)$.

5. β_{**}codense ideal

Definition 5.1. Let $(X, \tau, \mathfrak{I})$ be an ideal topological space, then an ideal $\mathfrak{I}$ is said to be β_{**} -codense if $\mathcal{C}\beta O(X) \cap \mathfrak{I} = \phi$ where $\mathcal{C}\beta O(X) = \{cl_\beta(U) : U \in \beta O(X)\}$.

For example: Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a,c\}\}$ and the ideal $\mathfrak{I} = \{\phi, \{a\}\}$. Here $\beta O(X) = \{\{b\}, \{c\}, \{b,c\}, X, \phi\}$. Then, $\mathcal{C}\beta O(X) = \{\{b\}, \{c\}, \{b,c\}, X, \phi\}$. Therefore, $\mathcal{C}\beta O(X)$ $\cap\, \mathfrak{I} = \phi$. Hence our ideal $\mathfrak{I}$ is a β_{**}-codense ideal.
Theorem 5.2. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. Then the following are equivalent:*

1. *$\mathfrak{I}$ is β_{**}-codense.*
2. *If $I \in \mathfrak{I}$, then $int_{\theta^\beta}(I) = \phi$*
3. *For every clopen G, $G \subseteq \beta_{**}$(G)*
4. *$X = \beta_{**}(X)$.*

Proof: (1) $\Rightarrow$ (2): Let $\mathcal{C}\beta O(X) \cap \mathfrak{I} = \phi$ and $I \in \mathfrak{I}$. Suppose that $x \in int_{\theta^\beta}(I)$. Then there exists U $\in \tau(x)$ such that $x \in U \subseteq cl(U) \subseteq I$. Since $I \in \mathfrak{I}$ and then $\phi \neq \{x\} \subseteq cl(U) \in \mathcal{C}\beta O(X) \cap \mathfrak{I}$. This is contradiction to $\mathcal{C}\beta O(X) \cap \mathfrak{I} = \phi$. Hence $int_{\theta^\beta}(I) = \phi$.
(2) $\Rightarrow$ (3): Let $x \in G$. Assume that $x \notin \beta_{**}(G)$, then there exists $U_x \in \beta O(x)$ such that $G \cap cl_\beta(U_x) \in \mathfrak{I}$ and hence $G \cap U_x \in \mathfrak{I}$. Since G is clopen, by (2) and Lemma 2.6, $x \in G \cap U_x = int(G \cap U_x) \subseteq int G$ $\cap\, cl_\beta(U_x) \subseteq int(G \cap cl(U_x)) = int_{\theta^\beta}(G \cap cl_\beta(U_x) = \phi$. It is a contradiction. Hence $x \in \beta_{**}(G)$. Thus, $G \subseteq \beta_{**}(G)$.

(3) $\Rightarrow$ (4): Since X is clopen, then $X = \beta_{**}(X)$.
(4) $\Rightarrow$ (1): $X = \beta_{**}(X) = \{x \in X : X \cap cl_\beta(U) = cl_\beta(U) \notin \mathcal{I}$ for every $U \in \beta O(x)\}$. Hence $\mathcal{C}\beta O(X) \cap \mathcal{I} = \phi$.

Definition 5.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a *Baire-* β_{**}*-set* with respect to τ and $\mathcal{I}$, denoted by $A \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I})$, if there exists a θ^β-open set U such that $A = U$ [mod $\mathcal{I}$]. Also, $\mathcal{U}_{\beta_r}(X, \tau, \mathcal{I})$, denote $\{A \subseteq X$: there exists $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Lemma 5.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_{\beta_{**}} \mathcal{I}$. If $U, V \in \tau_{\theta^\beta}$ and $\Psi_{\beta_{**}}(U) = \Psi_{\beta_{**}}(V)$, then $U = V$ [mod $\mathcal{I}$].*

Proof: Since $U \in \tau_{\theta^\beta}$, then $U \subseteq \Psi_{\beta_{**}}(U)$, by Corollary 3.3, and hence $U - V \subseteq \Psi_{\beta_{**}}(U) - V \in \mathcal{I}$, by Theorem 4.5. Therefore, $U - V \in \mathcal{I}$. Similarly, $V - U \in \mathcal{I}$. Now, $(U - V) \cup (V - U) \in \mathcal{I}$, by additivity. Hence $U = V$ [mod $\mathcal{I}$].

Theorem 5.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_{\beta_{**}} \mathcal{I}$. If $A, B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I})$ and $\Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}(B)$, then $A = B$ [mod $\mathcal{I}$].*

Proof: Let $U, V \in \tau_{\theta^\beta}$ be such that $A = U$ [mod $\mathcal{I}$] and $B = V$ [mod $\mathcal{I}$]. Now, $\Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}(U)$ and $\Psi_{\beta_{**}}(B) = \Psi_{\beta_{**}}(V)$, by Theorem 3.2 (8). Since $\Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}(B)$, then $\Psi_{\beta_{**}}(U) = \Psi_{\beta_{**}}(V)$ and $U = V$ [mod $\mathcal{I}$], by Lemma 5.4. Hence $A = B$ [mod $\mathcal{I}$], by transitivity.

Proposition 5.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space.*

1. *If $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$, then there exists $A \in \tau_{\theta^\beta} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$].*

2. *Let $\mathcal{C}\beta O(X) \cap \mathcal{I} = \phi$, then $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$ if and only if there exists $A \in \tau_{\theta^\beta} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$].*

Proof: (1) Assume that $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$ then $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I})$. Then there exists $A \in \tau_{\theta^\beta}$ such that $B = A$ [mod $\mathcal{I}$]. If $A = \phi$, then we have $B = \phi$ [mod $\mathcal{I}$]. This implies that $B \in \mathcal{I}$. It is contradiction.
(2) Assume that there exists $A \in \tau_{\theta^\beta} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$]. Hence by Definition 4.11, $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I})$. So, $A = (B - J) \cup I$, where $J = B - A$, $I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$, then $A \in \mathcal{I}$, by heredity and additivity. Since $A \in \tau_{\theta^\beta} - \{\phi\}$, $A \neq \phi$ and there exists $U \in \beta O(X)$ such that $\phi \neq U \subseteq cl_\beta(U) \subseteq$ A. Since $A \in \mathcal{I}$, $cl_\beta(U) \in \mathcal{I}$. Therefore, $cl_\beta(U) \in \mathcal{C}\beta O(X) \cap \mathcal{I}$. This contradicts that $\mathcal{C}\beta O(X) \cap \mathcal{I} = \phi$.

Proposition 5.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be β_{**}-codense. If $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$, then $\Psi_{\beta_{**}}(B) \cap int_{\theta^\beta}(\beta_{**}(B)) \neq \phi$.*

Proof: Assume that $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$, then by Proposition 5.6 (1), there exists $A \in \tau_{\theta^\beta} - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$]. By Theorem 5.2 and Lemma 2.8, $A = A \cap X = A \cap \beta_{**}(X) \subseteq \beta_{**}(A \cap X) = \beta_{**}(A)$. This implies that $\phi \neq A \subseteq \beta_{**}(A) = \beta_{**}((B - J) \cup I\ ) = \beta_{**}(B)$, where $J = B - A$, $I = A - B \in \mathcal{I}$, by Corollary 2.11. Since $A \in \tau_{\theta^\beta}$, $A \subseteq int_{\theta^\beta}(\beta_{**}(B))$. Also, $\phi \neq A \subseteq \Psi_{\beta_{**}}(A) = \Psi_{\beta_{**}}(B)$, by Corollary 3.3 and Theorem 3.2 (8). Consequently, we obtain $A \subseteq \Psi_{\beta_{**}}(B) \cap int_{\theta^\beta}(\beta_{**}(B))$.

Proposition 5.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be β_{**}-codense. If $\tau = \tau_{\theta^\beta}$, then the following statements are equivalent:*

1. A $\in \mathcal{U}_{\beta_r}$(X, τ, $\mathcal{I}$)

2. $\Psi_{\beta_{**}}(A) \cap int_{\theta^\beta}(\beta_{**}(A)) \neq \phi$

3. $\Psi_{\beta_{**}}(A) \cap \beta_{**}(A) \neq \phi$

4. $\Psi_{\beta_{**}}(A) \neq \phi$

5. $int_{**}(A) \neq \phi$

6. *There exists $G \in \beta O(X) - \{\phi\}$ such that $G - A \in \mathcal{I}$ and $G \cap A \notin \mathcal{I}$.*

Proof: (1) $\Rightarrow$ (2): Let $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A$. Then $int_{\theta^\beta}(\beta_{**}(B)) \subseteq int_{\theta^\beta}(\beta_{**}(A))$ and $\Psi_{\beta_{**}}(B) \subseteq \Psi_{\beta_{**}}(A)$ and hence $int_{\theta^\beta}(\beta_{**}(B)) \cap \Psi_{\beta_{**}}(B) \subseteq int_{\theta^\beta}(\beta_{**}(A)) \cap \Psi_{\beta_{**}}(A)$. By Proposition 5.7, we have $\Psi_{\beta_{**}}(A) \cap int_{\theta^\beta}(\beta_{**}(A)) \neq \phi$.
(2) $\Rightarrow$ (3): The proof is obvious.
(3) $\Rightarrow$ (4): The proof is obvious.
(4) $\Rightarrow$ (5): If $\Psi_{\beta_{**}}(A) \neq \phi$, then there exists $U \in \beta O(X) - \{\phi\}$ such that $U - A \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - A) \cup (U \cap A)$, we have $U \cap A \notin \mathcal{I}$. By Theorem 3.2, $\phi \neq (U \cap A) \subseteq \Psi_{\beta_{**}}(U) \cap A = \Psi_{\beta_{**}}((U - A) \cap (U \cap A)) \cap A = \Psi_{\beta_{**}}(U \cap A) \cap A \subseteq \Psi_{\beta_{**}}(A) \cap A = int_{**}(A)$. Hence, $int_{**}(A) \neq \phi$.
(5) $\Rightarrow$ (6): If $int_{**}(A) \neq \phi$, then there exists $G \in \beta O(X) - \{\phi\}$ and $I \in \mathcal{I}$ such that $\phi \neq G - I \subseteq A$. Since every open set is β-open, we have $G - A \in \mathcal{I}$, $G = (G - A) \cup (G \cap A)$ and $G \notin \mathcal{I}$. This implies that $G \cap A \notin \mathcal{I}$.
(6) $\Rightarrow$ (1): Let $B = G \cap A \notin \mathcal{I}$ with $G \in \tau_{\theta^\beta} - \{\phi\}$ and $G - A \in \mathcal{I}$. Then $B \in \mathcal{B}_{\beta_r}(X, \tau, \mathcal{I}) - \mathcal{I}$, since $B \notin \mathcal{I}$ and $(B - G) \cup (G - B) = G - A \in \mathcal{I}$.

Theorem 5.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_{\beta_{**}} \mathcal{I}$ and $\mathcal{I}$ be β_{**}-codense. Then for $A \subseteq X$, $\Psi_{\beta_{**}}(A) \subseteq \beta_{**}(A)$.*
Proof: Suppose that $x \in \Psi_{\beta_{**}}(A)$ and $x \notin \beta_{**}(A)$. Then there exists a nonempty neighborhood $U_x \in \beta O(x)$ such that $cl_\beta(U_x) \cap A \in \mathcal{I}$. Since $x \in \Psi_{\beta_{**}}(A)$, by Theorem 3.5, $x \in \cup\{U \in \beta O(X)\colon cl_\beta(U) - A \in \mathcal{I}\}$ and there exists $V \in \beta O(x)$ and $cl_\beta(V) - A \in \mathcal{I}$. Now, we have $U_x \cap V \in \beta O(x)$, $cl_\beta(U_x \cap V) \cap A \in \mathcal{I}$ and $cl_\beta(U_x \cap V) - A \in \mathcal{I}$, by heredity. Hence, by finite additivity, we have $(cl_\beta(U_x \cap V) \cap A) \cup (cl_\beta(U_x \cap V) - A) = cl_\beta(U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in \beta O(x)$, this is contrary to $\mathcal{C}\beta O(X) \cap \mathcal{I} = \phi$. Therefore, $x \in \beta_{**}(A)$. This implies that $\Psi_{\beta_{**}}(A) \subseteq \beta_{**}(A)$.

Theorem 5.10. *Let* (X, τ, $\mathcal{I}$) *be an ideal topological space with $\tau \sim_{\beta_{**}} \mathcal{I}$ and $\mathcal{I}$ be β_{**}-codense. Then $\Psi_{\beta_{**}}(A) \cap \Psi_{\beta_{**}}(X - A) = \phi$ for every subset A of X.*
Proof: Let $x \in \Psi_{\beta_{**}}(A) \cap \Psi_{\beta_{**}}(X - A)$ for some $x \in X$, then there exist β-open sets G, F containing x such that $cl_\beta(G) - A \in \mathcal{I}$ and $cl_\beta(F) \cap A \in \mathcal{I}$ respectively. Hence, $cl_\beta(G \cap F) \cap A \in \mathcal{I}$ and $cl_\beta(G \cap F) - A \in \mathcal{I}$, by heredity, so $cl_\beta(G \cap F) \in \mathcal{I}$. Since $G \cap F \in \beta O(x)$, this is contrary to $\mathcal{C}\beta O(X) \cap \mathcal{I} = \phi$. Thus, $\Psi_{\beta_{**}}(A) \cap \Psi_{\beta_{**}}(X - A) = \phi$.

Corollary 5.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_{\beta_{**}} \mathcal{I}$ and $\mathcal{I}$ be β_{**} -codense. Then* β_{**} (A) $\cup$ β_{**} (X $-$ A) $= X$ *for every subset A of X.*
Theorem 5.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:*

1. $\mathcal{I}$ is β_{**}-codense.
2. $\Psi_{\beta_{**}}(\phi) = \phi$
3. *If $A \subseteq X$ is β-closed, then $\Psi_{\beta_{**}}(A) - A = \phi$*
4. *If $J \in \mathcal{I}$, then $\Psi_{\beta_{**}}(J) = \phi$.*

Proof: (1) $\Rightarrow$ (2): Since $\mathcal{C}\beta O(X) \cap \mathcal{I} = \phi$, then $\Psi_{\beta_{**}}(\phi) = \bigcup \{U \in \beta O(X)$ such that $cl_\beta(U) - \phi = cl_\beta(U) \in \mathcal{I}\} = \phi$, by Theorem 3.5.
(2) $\Rightarrow$ (3): Assume that $x \in \Psi_{\beta_{**}}(A) - A$, then there exists a $U_x \in \beta O(x)$ such that $x \in cl_\beta(U_x) - A \in \mathcal{I}$ and $cl_\beta(U_x) - A \in \beta O(x)$. But $cl_\beta(U_x) - A \in \{U \in \beta O(x) : cl_\beta(U) \in \mathcal{I}\} = \Psi_{\beta_{**}}(\phi)$ implies that $\Psi_{\beta_{**}}(\phi) \neq \phi$. It is a contradiction. Hence, $\Psi_{\beta_{**}}(A) - A = \phi$.
(3) $\Rightarrow$ (4): Let $J \in \mathcal{I}$ and by hypothesis ϕ is β-closed, then $\Psi_{\beta_{**}}(J) = \Psi_{\beta_{**}}(J \cup \phi) = \Psi_{\beta_{**}}(\phi) = \phi$.
(4) $\Rightarrow$ (1): Assume that $A \in \mathcal{C}\beta O(X) \cap \mathcal{I}$, then $A \in \mathcal{I}$ and by (4), $\Psi_{\beta_{**}}(A) = \phi$. Since $A \in \mathcal{C}\beta O(X)$, then $A \subseteq \Psi_{\beta_{**}}(A) = \phi$ by Corollary 3.3. Hence $\mathcal{C}\beta O(X) \cap \mathcal{I} = \phi$.

Definition 5.13. A subset A in an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_{\beta_{**}}$-dense set if $\beta_{**}(A) = X$. The collection of all $\mathcal{I}_{\beta_{**}}$-dense sets in $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}_{\beta_{**}}D(X, \tau)$. The collection of all dense sets in (X, τ) is denoted by $D(X, \tau)$.

For example: Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a,c\}\}$ and the ideal $\mathcal{I} = \{\phi,$

$\{a\},\{b\},\{a,b\}\}$. Here, $\beta O(X)=\{\{a\},\{a,b\},\{a,c\},X,\phi\}$. Then, $A=\{a\}$, $\beta_{**}(A)=\phi$, $B=\{b\}$, $\beta_{**}(B)=\phi$, $C=\{c\}, \beta_{**}(C)=X$, $D=\{a,b\}, \beta_{**}(D)=\phi$, $E=\{a,c\}, \beta_{**}(E)=X$, $F=\{b,c\}, \beta_{**}(F)=X$, $G=\phi, \beta_{**}(G)=\phi$, $H=X$, $\beta_{**}(H)=X$. Hence, $\mathfrak{I}_{\beta_{**}}D(X,\tau)=\{\{c\},\{a,c\},\{b,c\},X\}$.

Lemma 5.14. *$\tau_{**}(\mathfrak{I})$ is a topology on X, generated by the sub basis $\{cl_\beta(U)-E: U\in\beta O(X)$ and $E\in\mathfrak{I}\}$ or equivalently $\tau_{**}(\mathfrak{I})=\{U\subset X: cl_\beta(X-U)=\beta_{**}(X-U)\}$.*

Remark 5.15. The closure operator cl_{**} for a topology $\tau_{**}(\mathfrak{I})$ defined as follows: for $A\subseteq X$, $cl_{**}(A)=A\cup\beta_{**}(A)(\mathfrak{I},\tau)$ and $int_{**}(A)$ denote the interior of the set A in $(X,\tau_{**},\mathfrak{I})$. It is known that $\tau\subseteq\tau_{**}(\mathfrak{I})$. Clearly if $A\subset\beta_{**}(A)$, then $\beta_{**}(A)=cl_{**}(A)$. Also, $cl_{**}\ (X-A)=X-int_{**}(\text{A})$.

Theorem 5.16. *Let $(X,\tau,\mathfrak{I})$ be an ideal topological space. If $\mathfrak{I}$ is β_{**}-codense, then $\mathfrak{I}_{\beta_{**}}D(X,\tau)=D(X,\tau_{**})$, where $D(X,\tau_{**})$ is the collection of all dense sets in $(X,\tau_{**},\mathfrak{I})$.*

Proof: Let $A\in\mathfrak{I}_{\beta_{**}}D(X,\tau)$. Then $cl_{**}(A)=A\cup\beta_{**}(A)=X$ implies $A\in D(X,\tau_{**})$. Hence $\mathfrak{I}_{\beta_{**}}D(X,\tau)\subseteq D(X,\tau_{**})$. Now, let $A\in D(X,\tau_{**})$. Then, $cl_{**}(A)=A\cup\beta_{**}(A)=X$. We prove that $\beta_{**}(A)=X$. Let $x\in X$ such that $x\notin A_{**}$. Then there exists $\phi\neq U\in\beta O(X)$ such that $cl_\beta(U)\cap A\in\mathfrak{I}$. Since $cl_\beta(U)\notin\mathfrak{I}$, $cl_\beta(U)\cap(X-A)\notin\mathfrak{I}$. So, $cl_\beta(U)\cap(X-A)\neq\phi$. Let $x_0\in cl_\beta(U)\cap(X-A)$. Then $x_0\notin A$ and also $x_0\notin A_{**}$. Because $x_0\in A_{**}$ implies that $cl_\beta(U)\cap A\notin\mathfrak{I}$ which is contrary to $cl_\beta(U)\cap A\in\mathfrak{I}$. Hence, $x_0\notin A\cup\beta_{**}(A)=cl_{**}(A)=X$. This is contradiction, and it implies that $A\in\mathfrak{I}_{\beta_{**}}D(X,\tau)$. Therefore, $D(X,\tau_{**})\subseteq\mathfrak{I}_{\beta_{**}}D(X,\tau)$. Thus, $\mathfrak{I}_{\beta_{**}}D(X,\tau)=D(X,\tau_{**})$.

Theorem 5.17. *Let $(X,\tau,\mathfrak{I})$be an ideal topological space. Then for $x\in X$, $X-\{x\}$ is $\mathfrak{I}_{\beta_{**}}$-dense if and only if $\Psi_{\beta_{**}}(\{x\})=\phi$.*

Proof: Follows from the definition of $\mathfrak{I}_{\beta_{**}}$-dense set, since $\Psi_{\beta_{**}}(\{x\})=X-\beta_{**}(X-\{x\})=\phi$ if and only if $X=\beta_{**}(X-\{x\})$.

Theorem 5.18. *Let $(X,\tau,\mathfrak{I})$ be an ideal topological space and $\mathfrak{I}$ be β_{**}-codense. Then $\Psi_{\beta_{**}}(A)\neq\phi$ if and only if A contains the nonempty τ_{**}-interior.*

Proof: Suppose that $\Psi_{\beta_{**}}(A)\neq\phi$ then $\Psi_{\beta_{**}}(A)=\bigcup\{U\in\beta O(X)$ such that $cl_\beta(U)-A\in\mathfrak{I}\}$ by Theorem 3.5 (1), and there exists a nonempty set $U\in\beta O(X)$ such that $cl_\beta(U)-A\in\mathfrak{I}$. Let $cl_\beta(U)-A=J$, where $J\in\mathfrak{I}$. Hence, $cl_\beta(U)-J\subseteq A$ implies $cl_\beta(U)-J\in\tau_{**}$ and A contains the nonempty τ_{**}-interior by Lemma 5.14. Conversely, suppose that A contains the nonempty τ_{**}-interior. Then there exists $U\in\beta O(X)$ and $J\in\mathfrak{I}$ such that $cl_\beta(U)-J\subseteq A$. Hence, $cl_\beta(\text{U})-A\subseteq J$. Let $G=cl_\beta(U)-A\subseteq J$, then $G\in\mathfrak{I}$. Thus, $\bigcup\{U\in\beta O(X): cl_\beta(U)-A\in\mathfrak{I}\}=\Psi_{\beta_{**}}(A)\neq\phi$.

6. Conclusion

In this paper, we have introduced and discussed the salient features of β-local closure function, a set operator $\Psi_{\beta_{**}}$, β_{**}-codense ideals and $\mathfrak{I}_{\beta_{**}}$-dense sets. The notions of the β-closure compatible topology τ with the ideal $\mathfrak{I}$ are explored in the ideal topological space $(X,\tau,\mathfrak{I})$. Furthermore, we have shown that the heredity nature of ideals is imported under β-local closure function. These results are some modes of future development in ideal topological spaces.

Conflicts of interest

The authors declare that there is no conflict of interests.

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