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First Module Cohomology Group of Induced Semigroup Algebras

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abstract: Let S be a discrete semigroup and T be a left multiplier on S. A new product on S defined by T creates a new induced semigroup S_T. In this paper, we show that if T is bijective, then the first module cohomology groups $\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ and $\mathcal{H}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T))$ are equal, where E and E_T are sets of idempotent elements in S and S_T, respectively. Which in particular means that $\ell^1(S)$ is weak $\ell^1(E)$-module amenable if and only if $\ell^1(S_T)$ is weak $\ell^1(E_T)$-module amenable. Finally, by giving an example, we show that the bijectivity of T, is necessary.

Key Words: Semigroup, induced semigroup, module cohomology group, weak module amenability.

Contents

1. Introduction

The difference between amenability and weak amenability is important for group algebras and semigroup algebras. Amini in [1] and Amini along with Bagha in [2] by introducing the concepts of module amenability and weak module amenability, tried to make these differences clearer.

Amini in [1] showed that, inverse semigroup S with subsemigroup E of idempotent elements is amenable if and only if semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module amenable, when $\ell^1(E)$ acts on $\ell^1(S)$ by multiplication from right and trivially from left.

After that, Amini and Bagha in [2], showed that, for every commutative inverse semigroup S with idempodent set E, semigroup algebra $\ell^1(S)$ is always weakly $\ell^1(E)$-module amenable, where module actions $\ell^1(E)$ on $\ell^1(S)$ is

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_{es} \qquad (e \in E, s \in S). \tag{1.1}$$

Then this sentence has been expanded by second author of the current paper along with Pourabbas. They in [7] and [8], by introducing the concept of module cohomology group for Banach algebras extended this result and showed that the first and second $\ell^1(E)$-module cohomology groups of $\ell^1(S)$ with coefficients in $\ell^1(S)^{(2n-1)}$ $(n \in \mathbb{N})$, are zero and Banach space, respectively, when $\ell^1(S)$ is a Banach $\ell^1(E)$-bimodule with actions (1.1).

Let S be a semigroup and S_T be induced semigroup by a left multiplier $T : S \to S$, and E and E_T are sets of idempotent elements in S and S_T, respectively. These two semigroups are sometims different and we try to find conditions on S and T such that the semigroups $\ell^1(S)$ and $\ell^1(S_T)$ have the same module cohomological properties.

It is worth mention that, this idea has started by Birtel in [3] and continued by Larsen in [6]. Also the relation between weak amenability (not weak module amenability) Banach algebra A and the induced Banach algebra A_T studied by Laali in [6], where T is left multiplier on Banach algebra A for more details see, [3], [5], and [9].

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In this paper, we show that if T is bijective, then the first $\ell^1(E)$-module cohomology group $\ell^1(S)$ with coefficents in $\ell^\infty(S)$ is eqvalence with the first $\ell^1(E_T)$-module cohomology group $\ell^1(S_T)$ with coefficents in $\ell^\infty(S_T)$. Indeed, we prove

$$\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{H}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)).$$

Which in particular means that $\ell^1(S)$ is weak $\ell^1(E)$-module amenable if and only if $\ell^1(S_T)$ is weak $\ell^1(E_T)$-module amenable. Finally, by giving an example, we show that the condition of bijectivity for T, is necessary.

We begin recalling some terminology.

Let $\mathfrak{A}$ and A be Banach algebras such that A is a $\mathfrak{A}$-bimodule with compatible actions. Also, let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions then we say that X is a Banach A-$\mathfrak{A}$-module (For more details see [1], [2] and [7]). Moreover, X is called commutative (bi-commutative) Banach A-$\mathfrak{A}$-module, if

$$\alpha \cdot x = x \cdot \alpha \quad (a \cdot x = x \cdot a) \quad (\alpha \in \mathfrak{A}, a \in A, x \in X).$$

If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is dual space X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined by

$$(\alpha \cdot f)(x) = f(x \cdot \alpha), \quad (a \cdot f)(x) = f(x \cdot a), \quad (\alpha \in \mathfrak{A}, a \in A, x \in X, f \in X^*),$$

and the same for the other side actions. In particular, if A is a commutative Banach $\mathfrak{A}$-module, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space A^* is also a commutative Banach A-$\mathfrak{A}$-module.

A bounded map $D : A \longrightarrow X$ is called a $\mathfrak{A}$-module derivation if for $\alpha \in \mathfrak{A}$ and $a, b \in A$, we have

$$D(a \pm b) = D(a) \pm D(b), \quad D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha,$$

and

$$D(ab) = D(a) \cdot b + a \cdot D(b).$$

Note that, $D : A \longrightarrow X$ is bounded if there exist $M > 0$ such that $\|D(a)\| \leq M \|a\|$, for each $a \in A$. Although D is not necessarily $\mathbb{C}$ linear, but still its boundedness implies its norm continuity (since D preserve subtraction). Each $x \in \mathrm{Cent}_{\mathfrak{A}} X = \{x \in X;\ x \cdot \alpha = \alpha \cdot x, \text{ for each } \alpha \in \mathfrak{A}\}$, defines an $\mathfrak{A}$-module derivation

$$D(a) = \mathbf{ad}_x(a) = a \cdot x - x \cdot a \quad (a \in A)$$

which is called inner $\mathfrak{A}$-module derivation. Note that, $\mathrm{Cent}_{\mathfrak{A}} X = X$, if X is commutative Banach A-$\mathfrak{A}$-module. Also if X is a bi-commutative Banach A-$\mathfrak{A}$-module, then the inner derivations are zero.

Definition 1.1. *The Banach algebra A is called $\mathfrak{A}$-module amenable, if for every Banach A-$\mathfrak{A}$-module X, every $\mathfrak{A}$-module derivation $D : A \longrightarrow X^*$ is inner.*

Definition 1.2. *The $\mathfrak{A}$-module Banach algebra A is called weak $\mathfrak{A}$-module amenable, if every $\mathfrak{A}$-module derivation $D : A \longrightarrow A^*$ is inner.*

We use the notation $\mathcal{Z}^1_{\mathfrak{A}}(A, X)$ for the set of all $\mathfrak{A}$-module derivations $D : A \longrightarrow X$ and $\mathcal{B}^1_{\mathfrak{A}}(A, X)$, for those which are inner. The first $\mathfrak{A}$-module cohomology group with coefficent in X is denoted by $\mathcal{H}^1_{\mathfrak{A}}(A, X)$ which is the quotient $\mathcal{Z}^1_{\mathfrak{A}}(A, X)/B^1_{\mathfrak{A}}(A, X)$. Hence, A is $\mathfrak{A}$-module amenable if and only if $\mathcal{H}^1_{\mathfrak{A}}(A, X^*) = 0$, for each Banach A-$\mathfrak{A}$-module X. Similarly, if A is a Banach A-$\mathfrak{A}$-module, then the Banach algebra A is weak $\mathfrak{A}$-module amenable if $\mathcal{H}^1_{\mathfrak{A}}(A, A^*) = 0$. Note that, Banach algebra A is called amenable, (resp. weak amenable) if it is $\mathbb{C}$-module amenable (resp. weak $\mathbb{C}$-module amenable).

2. Induced Semigroup S_T with The Left Multiplier Map T

Let S be a semigroup, the elemant $s \in S$ is called idempotent if $s \cdot s = s$. The set of all idempotent elements of S is denoted by $E(S) = E$. If also S is a Hausdorff topological space and the binary operation on S is jointly continuous, then S is called a topological semigroup.

A map $T : S \longrightarrow S$ is called a left (right) multiplier if $T(st) = T(s)t$ ($T(st) = sT(t)$), for all $s, t \in S$. The class of left (right) multiplier map on S is denoted by $\mathrm{Mul}_l(S)$ ($\mathrm{Mul}_r(S)$). The map T is called multiplier if $T \in \mathrm{Mul}_l(S) \cap \mathrm{Mul}_r(S)$. The space of all multiplier maps on S is denoted by $\mathrm{Mul}(S)$.

Let $T \in \mathrm{Mul}_l(S)$, we define a new operation "$\circ$" on S as follow $s \circ t := sT(t)$ for every s and t in S. It's easy to check that $(S, \circ)$ is a semigroup which is called **induced semigroup** dependent on left multiplier T and is denoted by S_T. Let E and E_T are sets of idempotent elements in S and S_T, respectively.

Lemma 2.1. *Let S be a semigroup and $T : S \to S$ be a bijective map, then*
(i) $T \in \mathrm{Mul}_l(S)$ if and only if $T^{-1} \in \mathrm{Mul}_l(S)$.
(ii) If $T \in \mathrm{Mul}_l(S)$, then $T(E_T) = E$ and $T^{-1}(E) = E_T$.
(iii) If $T \in \mathrm{Mul}(S)$, then $s \circ T(t) = T(s) \circ t$ and $s \circ T^{-1}(t) = T^{-1}(s) \circ t$ for every $s, t \in S$.

Proof. (i) Let $s, t \in S$ and $T^{-1}(s) = z$, we have

$$T^{-1}(st) = T^{-1}(T(z)t) = T^{-1}(T(zt)) = zt = T^{-1}(s)t.$$

The other side of proof is proven, since $T = (T^{-1})^{-1}$.

(ii) Let $p \in E_T$, we have

$$T(p) = T(p \circ p) = T(pT(p)) = T(p)T(p).$$

This shows that $T(p) \in E$ and so $T(E_T) \subseteq E$. For the other side let $e \in E$ and $e = T(p)$. We have to show $p \in E_T$. Since $ee = e$ and $T^{-1} \in \mathrm{Mul}_l(S)$ by (i), we have

$$p \circ p = pT(p) = T^{-1}(e)e = T^{-1}(ee) = T^{-1}(e) = p.$$

Therefore $T(E_T) = E$. The proof of $T^{-1}(E) = E_T$ is similar.

(iii) Since $T \in \mathrm{Mul}_l(S) \cap \mathrm{Mul}_r(S)$, so $aT(b) = T(ab) = T(a)b$, for every $a, b \in S$. Let $s, t \in S$,

$$s \circ T(t) = sT(T(t)) = T(s)T(t) = T(s) \circ t.$$

Finally, since $T \in \mathrm{Mul}(S)$ is equal to $T^{-1} \in \mathrm{Mul}(S)$ by (i), similarly we can show that $s \circ T^{-1}(t) = T^{-1}(s) \circ t$, for every $s, t \in S$. □

The next examples show that, when T is not bijective or not multiplier, previous lemma not necessarily true, therefore, bijective and multiplier conditions are neccessary for T.

Example 2.2. *Let $S = \left\{\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}, x, y \in \mathbb{R}\right\}$. S with matrix product is a semigroup and one can verify that, its idempotent set is $E = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ y & 0 \end{bmatrix}, y \in \mathbb{R}\right\}$. Now let $T : S \longrightarrow S$ be left multiplier L_a, where $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Indeed, $T\left(\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$. Clearly T is not right multiplier and bijective. It easy to show that $T(E_T) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \neq E$. Now for every $s = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}, t = \begin{bmatrix} z & 0 \\ p & 0 \end{bmatrix} \in S$, where $y, z \neq 0$, a simple computation shows that $s \circ T(t) \neq T(s) \circ t$.*

Example 2.3. *Let* $S = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, x, y, z \in \mathbb{R}\right\}$*. S with matrix product is a semigroup which its idempotent set is* $E = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} y \in \mathbb{R}\right\}$*. Now let* $T : S \longrightarrow S$ *be left multiplier* L_a*, where* $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$*. Indeed,*

$$T\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y+z \\ 0 & z \end{bmatrix}.$$

We know that T *is bijective but not right multiplier. It is easy to show that*

$$E_T = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} y \in \mathbb{R}\right\}$$

and $T(E_T) = E$*, also* $T^{-1} : S \longrightarrow S$ *is left multiplier* $T^{-1} = L_b$*, where* $b = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ *and* $T^{-1}(E) = E_T$*. Now for every* $s = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, t = \begin{bmatrix} m & n \\ 0 & q \end{bmatrix} \in S$*, where* $x \neq z$*, a simple computation shows that* $s \circ T(t) \neq T(s) \circ t$*. Similarly can be shown* $s \circ T^{-1}(t) \neq T^{-1}(s) \circ t$*.*

3. First Module Cohomology Group and Weak Module Amenability of Induced Semigroup Algebras

Throughout this paper, unless otherwise indicated, we will assume that S is a discrete semigroup, $T \in \mathrm{Mul}(S)$ and T is bijective. We know that the set of point masses $\{\delta_s; s \in S\}$ is dense in $\ell^1(S)$. Since module actions and module derivations are continuous, we consider point masses as representing elements of semigroup algebras $(\ell^1(S), *)$ and $(\ell^1(S_T), \circledast)$, where $*$ is the convolution on $\ell^1(S)$ and $\circledast$ is convolution on $\ell^1(S_T)$, defined as follow

$$\delta_s \circledast \delta_t = \delta_{s\circ t} = \delta_s * \delta_{T(t)} = \delta_{sT(t)} \qquad (s, t \in S). \tag{3.1}$$

Lemma 3.1. *Let* S*,* S_T *and* T *be as above. Then*

$$\mathrm{Cent}_{\ell^1(E)}\, \ell^\infty(S) = \mathrm{Cent}_{\ell^1(E_T)}\, \ell^\infty(S_T).$$

Proof. Let $\phi \in \mathrm{Cent}_{\ell^1(E)}\, \ell^\infty(S)$ and $p \in E_T$. Since $T(p) \in E$ by Lemma 2.1, we have

$$\begin{aligned}
[\phi \circledast \delta_p](\delta_z) &= \phi(\delta_p \circledast \delta_z) = \phi(\delta_{pT(z)}) = \phi(\delta_{T(p)z}) \\
&= \phi(\delta_{T(p)} * \delta_z) = [\phi * \delta_{T(p)}](\delta_z) = [\delta_{T(p)} * \phi](\delta_z) \\
&= \phi(\delta_z * \delta_{T(p)}) = \phi(\delta_{z\circ p}) = \phi(\delta_z \circledast \delta_p) \\
&= [\delta_p \circledast \phi](\delta_z).
\end{aligned}$$

This shows that, $\phi \in \mathrm{Cent}_{\ell^1(E_T)}\, \ell^\infty(S_T)$. □

Lemma 3.2. *Let* S*,* S_T *and* T *be as above. Then* $D : \ell^1(S) \to \ell^\infty(S)$ *is* $\ell^1(E)$*-module derivation if and only if* $\widetilde{D} : \ell^1(S_T) \longrightarrow \ell^\infty(S_T)$ *defined as* $\widetilde{D}(f) := D(f \circ T^{-1})$ *is* $\ell^1(E_T)$*-module derivation. Furthermore,* D *is inner if and only if* $\widetilde{D}$ *is inner.*

Proof. Note that $\widetilde{D}(\delta_x) = D(\delta_{T(x)})$ for every $x \in S$. In the first, we show that $\widetilde{D}$ is derivation, when D

is derivation. Clearly $\widetilde{D}$ is additive. Let $x, y, z \in S_T$ be arbitrary elements, we have

$$\begin{aligned}
[\widetilde{D}(\delta_x) \circledast \delta_y + \delta_x \circledast \widetilde{D}(\delta_y)](\delta_z) &= [\widetilde{D}(\delta_x) \circledast \delta_y](\delta_z) + [\delta_x \circledast \widetilde{D}(\delta_y)](\delta_z) \\
&= \widetilde{D}(\delta_x)(\delta_y \circledast \delta_z) + \widetilde{D}(\delta_y)(\delta_z \circledast \delta_x) \\
&= D(\delta_{T(x)})(\delta_{yT(z)}) + D(\delta_{T(y)})(\delta_{zT(x)}) \\
&= D(\delta_{T(x)})(\delta_{T(y)z}) + D(\delta_{T(y)})(\delta_{zT(x)}) \\
&= D(\delta_{T(x)})(\delta_{T(y)} * \delta_z) + D(\delta_{T(y)})(\delta_z * \delta_{T(x)}) \\
&= [D(\delta_{T(x)}) * \delta_{T(y)}](\delta_z) + [\delta_{T(x)} * (D(\delta_{T(y)})](\delta_z) \\
&= [D(\delta_{T(x)}) * \delta_{T(y)} + \delta_{T(x)} * D(\delta_{T(y)})](\delta_z) \\
&= D(\delta_{T(x)} * \delta_{T(y)})(\delta_z) = D(\delta_{T(x)T(y)})(\delta_z) \\
&= D(\delta_{T(xT(y))})(\delta_z) = \widetilde{D}(\delta_{xT(y)})(\delta_z) \\
&= \widetilde{D}(\delta_x \circledast \delta_y)(\delta_z).
\end{aligned}$$

Hence, it follows that $\widetilde{D}$ is derivation.

Now we show that $\widetilde{D}$ is $\ell^1(E_T)$-module map, when D is $\ell^1(E)$-module map. For this, let $p \in E_T$ and $x, y \in S_T$, since $T \in \mathrm{Mul}_l(S)$ and $T(p) \in E$ by Lemma 2.1 (ii), we have

$$\begin{aligned}
[\delta_p \circledast \widetilde{D}(\delta_x)](\delta_y) &= \widetilde{D}(\delta_x)(\delta_y \circledast \delta_p) = \widetilde{D}(\delta_x)(\delta_{y\circ p}) \\
&= D(\delta_{T(x)})(\delta_{yT(p)}) = D(\delta_{T(x)})(\delta_y * \delta_{T(p)}) \\
&= [\delta_{T(p)} * D(\delta_{T(x)})](\delta_y) = [D(\delta_{T(p)} * \delta_{T(x)})](\delta_y) \\
&= D(\delta_{T(p)T(x)})(\delta_y) = D(\delta_{T(pT(x))})(\delta_y) \\
&= \widetilde{D}(\delta_{pT(x)})(\delta_y) = \widetilde{D}(\delta_p \circledast \delta_x)(\delta_y).
\end{aligned}$$

That shows $\delta_p \circledast \widetilde{D}(\delta_x) = \widetilde{D}(\delta_p \circledast \delta_x)$. For the other side, since $T \in \mathrm{Mul}_r(S)$, we have

$$\begin{aligned}
[\widetilde{D}(\delta_x) \circledast \delta_p](\delta_y) &= \widetilde{D}(\delta_x)(\delta_p \circledast \delta_y) = \widetilde{D}(\delta_x)(\delta_{p\circ y}) \\
&= D(\delta_{T(x)})(\delta_{pT(y)}) = D(\delta_{T(x)})(\delta_{T(py)}) \\
&= D(\delta_{T(x)})(\delta_{T(p)y}) = D(\delta_{T(x)})(\delta_{T(p)} * \delta_y) \\
&= [(D(\delta_{T(x)}) * \delta_{T(p)}](\delta_y) = D(\delta_{T(x)} * \delta_{T(p)})(\delta_y) \\
&= D(\delta_{T(x)T(p)})(\delta_y) = D(\delta_{T(xT(p))})(\delta_y) \\
&= \widetilde{D}(\delta_{xT(p)})(\delta_y) = \widetilde{D}(\delta_x \circledast \delta_p)(\delta_y).
\end{aligned}$$

That shows $\widetilde{D}(\delta_x) \circledast \delta_p = \widetilde{D}(\delta_x \circledast \delta_p)$. Therefore $\widetilde{D}$ is $\ell^1(E_T)$-module derivation.

Conversely, let $\widetilde{D}$ be $\ell^1(E_T)$-module derivation. Since $T^{-1} \in \mathrm{Mul}(S)$, similarly we can show that D is $\ell^1(E)$-module derivation.

Finally, let D be inner. Then there exists $\psi \in \mathrm{Cent}_{\ell^1(E)}\ell^\infty(S)$, such that $D(f) = f * \psi - \psi * f$, for every $f \in \ell^1(S)$. Let $x, y \in S_T$, be arbitrary elements, we have

$$\begin{aligned}
[\widetilde{D}(\delta_x)](\delta_y) &= [D(\delta_{T(x)}](\delta_y) = [\delta_{T(x)} * \psi - \psi * \delta_{T(x)}](\delta_y) \\
&= \psi(\delta_y * \delta_{T(x)} - \delta_{T(x)} * \delta_y) = \psi(\delta_{yT(x)} - \delta_{T(x)y}) \\
&= \psi(\delta_{yT(x)} - \delta_{xT(y)}) = \psi(\delta_{y\circ x} - \delta_{x\circ y}) \\
&= [\delta_x \circledast \psi - \psi \circledast \delta_x](\delta_y).
\end{aligned}$$

Hence, by Lemma 3.1, it follows that $\widetilde{D}$ is inner.

Conversely, let $\widetilde{D}$ be inner. So there exists $\psi \in \mathrm{Cent}_{\ell^1(E_T)}\ell^\infty(S_T)$, such that $\widetilde{D}(f) = f \circledast \psi - \psi \circledast f$, for every $f \in \ell^1(S_T)$. Let $x, y \in S_T$ be arbitrary elements, we have

$$\begin{aligned}
[D(\delta_x)](\delta_y) &= [\widetilde{D}(\delta_{T^{-1}(x)}](\delta_y) \\
&= [\delta_{T^{-1}(x)} \circledast \psi - \psi \circledast \delta_{T^{-1}(x)}](\delta_y) \\
&= \psi(\delta_y \circledast \delta_{T^{-1}(x)} - \delta_{T^{-1}(x)} \circledast \delta_y) \\
&= \psi(\delta_{y\circ T^{-1}(x)} - \delta_{T^{-1}(x)\circ y}) \\
&= \psi(\delta_{y\circ T^{-1}(x)} - \delta_{x\circ T^{-1}(y)}) \\
&= \psi(\delta_{yx} - \delta_{xy}) \\
&= [\delta_x * \psi - \psi * \delta_x](\delta_y).
\end{aligned}$$

Hence, D is inner by Lemma 3.1 and the proof is complete. □

Theorem 3.3. *Let S, T and S_T be as above. Then*

$$\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{H}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)).$$

Proof. Consider the map:

$$\begin{aligned}
\Gamma : \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) &\longrightarrow \mathcal{H}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)) \\
D &\mapsto \widetilde{D} + \mathcal{B}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)).
\end{aligned}$$

Clearly Γ is linear and Lemma 3.2 shows that Γ is well-define.

For surjectivity, let $P \in \mathcal{Z}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)$ and $D : \ell^1(S) \longrightarrow \ell^\infty(S)$ defined by $D(f) := P(f \circ T)$. Clearly $\Gamma(D) = \widetilde{D} = P$. But $D \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)$ by Lemma 3.2. That shows, Γ is surjective. On the other hand, Lemma 3.2, also shows that $\ker\Gamma = \mathcal{B}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$. But

$$\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = \frac{\mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))}{\ker\Gamma} \simeq \mathrm{Im}\,\Gamma = \mathcal{H}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)).$$

□

Corollary 3.4. *$\ell^1(S)$ is weakly $\ell^1(E)$-module amenable if and only if $\ell^1(S_T)$ is weakly $\ell^1(E_T)$-module amenable.*

Now by Corollary 3.4 and [7, Thorem 2.4], the following result is obtained:

Corollary 3.5. *Let S be a commutative inverse semigroup and T be a bijective and left multiplier (or right multiplier) on S. Then $\ell^1(S_T)$ is weak $\ell^1(E_T)$-module amenable.*

Finally, we will show in the next example, if T is not bijective, then weak $\ell^1(E)$-module amenability (resp. amenability, weak amenability and $\ell^1(E)$-module amenability) $\ell^1(S)$, may not be equivalent to weak $\ell^1(E_T)$-module amenability (resp. amenability, weak amenability and $\ell^1(E_T)$-module amenability) $\ell^1(S_T)$.

Example 3.6. *Put $S = \{0, 1, 2, 3\}$ with the operation $s \cdot t = \mathrm{Max}\{s, t\}$ $(s, t \in S)$. Then S is a finite commutative idempotent semigroup and so is amenable by (0.18) of [9]. S is a unital semigroup and has a zero, indeed $1_S = 0$ and $0_S = 3$. Also S is regular, since each $s \in S$ is idempotent. Therefore the semigroup algebra $\ell^1(S)$ is amenable (and so weak amenable) by Colorally 5.3 of [4]. Now let $T : S \to S$ be not bijective left multiplier map L_2 (i.e. $T(s) = L_2(s) = \mathrm{Max}\{2, s\}$). We know that S_T is amenable and not regular, since for $0 \in S_T$ no exists $x \in S_T$ that $0 = 0 \circ x \circ 0$. This shows that $\ell^1(S_T)$ is not amenable by Corollary 5.3 of [4]. Beyond that, $A = \ell^1(S_T)$ is not weak amenable, since $\overline{A^2} \neq A$. Therefore $\ell^1(S)$ is amenable (weak amenable) while $\ell^1(S_T)$ is not amenable (weak amenable).*

Now let $A = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$. $\mathfrak{A}$ has a bounded approximate identity for $\ell^1(S)$ and so by Proposition 2.1 of [1], $\ell^1(S)$ is $\ell^1(E)$-module amenable. But $\ell^1(E_T)$ has not a bounded approximate identity for $\ell^1(S_T)$, so $\ell^1(S_T)$ is not $\ell^1(E_T)$-module amenable by Proposition 2.2 of in [1]. On the other hands, Proposition 2.3 of [2], shows that $\ell^1(S)$ is weak $\ell^1(E)$-module amenable, while $\ell^1(S_T)$ is not weak $\ell^1(E_T)$-module amenable by Proposition 2.4 of [2], because $\overline{\ell^1(S_T)\ell^1(E_T)\ell^1(S_T)} \neq \ell^1(S_T)$.

Regarding our results, we need to make the following remark.

Remark 3.7. *(i) Let $S = \{z \in \mathbb{C} : |z| \leq 1\}$ is a compact topological semigroup, under complex multiplication with idempotent elements $E = \{0, 1\}$, Put $T = L_i$ where $L_i(x) = ix, (i = \sqrt{-1})$, clear that $T \in \mathrm{Mul}_l(S)$ and $S_T = (S, \circ)$ is a compact topological semigroup with idempotent elements $E_T = \{0, -i\}$. Indeed, $\ell^1(E)$-module derivations on $\ell^1(S)$ are $\mathbb{R}$-linear, while $\ell^1(E_T)$-module derivations on $\ell^1(S_T)$ are $\mathbb{C}$-linear. This shows that, about weak module amenability of some semigroup algebras, working with module derivatives of one module is easier than working with module derivatives of other module.*

(ii) Our assumptions are not additional assumptions to achieve the results, which means that they are exactly sufficient assumptions. Let S be a semigroup and $T : S \to S$ be only bijective map. Suppose that weak $\ell^1(E)$-module amenability of $\ell^1(S)$ is equal to weak $\ell^1(E_T)$-module amenability of $\ell^1(S_T)$. In this case, the module derivation $\boldsymbol{ad}_\psi$ is $\ell^1(E)$-module inner derivation if and only if it is $\ell^1(E_T)$-module inner derivation for every $\psi \in \ell^\infty(S) \simeq \ell^\infty(S_T)$. Now for every $x, y \in S$ we have

$$\begin{aligned}
\psi(\delta_{xT(y)}) &= \psi(\delta_{xT(y)}) - \psi(\delta_{yT(x)}) + \psi(\delta_{yT(x)}) \\
&= -[\delta_x \circledast \psi - \psi \circledast \delta_x](\delta_y) + \psi(\delta_{yT(x)}) \\
&= -[\delta_x \circledast \psi - \psi \circledast \delta_x](\delta_y) + \psi(\delta_{yT(x)}) \\
&= -[\widetilde{\boldsymbol{ad}_\psi}(\delta_x)](\delta_y) + \psi(\delta_{yT(x)}) \\
&= -[\boldsymbol{ad}_\psi(\delta_{T(x)})](\delta_y) + \psi(\delta_{yT(x)}) \\
&= -[\delta_{T(x)} * \psi - \psi * \delta_{T(x)}](\delta_y) + \psi(\delta_{yT(x)}) \\
&= \psi(\delta_{T(x)y}) - \psi(\delta_{yT(x)}) + \psi(\delta_{yT(x)}) \\
&= \psi(\delta_{T(x)y}).
\end{aligned}$$

This shows that, $xT(y) = T(x)y$ and so T is left multiplier if and only if it is right multiplier.

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