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Existence and Stability Results of the Solution for Nonlinear Fractional Differential Problem

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abstract: The problem of existence and stability results for fractional problem is considered. Based on the Krasnoselskii's fixed point theorem, we prove our main results. Then we give an examples to illustrate our main results.

Key Words: Fractional derivative, fractional differential equation, existence, stability, initial value problem, fixed point theory.

Contents

1. introduction and Preliminaries

It is known that fractional order equations serve as the basis for mathematical modeling of processes occurring in fractal media. When constructing mathematical models of geophysical processes, the introduction of the concept of the effective rate of change of certain physical quantities characterizing the modeled processes leads to differential equations containing a composition of fractional differentiation operators with different principles [4], [10].
In this paper we consider the following IVP of fractional differential equation

$$\begin{cases} {}^{C}D^{p}_{0^+}x(t) = g(t,x(t)) + {}^{C}D^{p-1}_{0^+}f(t,x(t)), \quad t \in [0,+\infty) \\ \\ x(0) = x_0, \quad x'(0) = x_1, \end{cases} \tag{1.1}$$

where $1 < p < 2, (x_0, x_1) \in \mathbb{R}^2, f, g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions with $f(t, 0) = g(t, 0) \equiv 0$ and ${}^{C}D^{p}$ is the standard Caputo fractional derivative of order p.
The nonlinear fractional differential problem have been studied less intensively. Unlike the classical differentiation operator of integer order, the action of the fractional differentiation operator on the product of two functions does not appear to be a finite sum, but an infinite series (The so-called generalized Leibniz rule). A particular case of which is also equation (1.1). Nevertheless, there are a number of general mathematical approaches that make it possible to construct solutions and to treat the stability, one of which is Krasnoselskii's fixed point theory (Please see [1], [2], [3], [6], [7]).
Here, we present some notations, definitions and auxiliary Lemmas concerning fractional calculus and fixed point theorems. Some preliminary concepts of fractional calculus [11].

Definition 1.1. *Let $q > 0$ and $\zeta : \mathbb{R}_+ \longrightarrow \mathbb{R}$. The Riemann-Liouville fractional integral of order q of a function ζ is defined by*

$$I^{q}_{0^+}\zeta(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\zeta(s)ds, \quad t \in \mathbb{R}_+.$$

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Definition 1.2. *[11] Let $q>0$, the Caputo fractional derivative of order q of a function $\zeta:\mathbb{R}_+\longrightarrow\mathbb{R}$ is defined by*

$$\begin{aligned} {}^{C}D^{q}_{0+}\zeta(t) &= \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}\zeta^{(n)}(s)ds \\ &= I^{n-q}_{0+}\zeta^{(n)}(t), \quad t\in\mathbb{R}_+, \end{aligned}$$

where $n=[q]+1$, provided the right side is pointwise defined on $\mathbb{R}_+$.

Lemma 1.3. *[11] For real numbers $q>0$ and appropriate function $\zeta(t)\in C^{n-1}[0,\infty)$ and $\zeta(t)$ exists almost everywhere on any bounded interval of $\mathbb{R}_+$.*

$$(I^{q}_{0+}\,{}^{C}D^{q}_{0+}\zeta)(t)=\zeta(t)-\sum_{k=0}^{n-1}\frac{\zeta^{(k)}(0)}{k!}t^k.$$

Lemma 1.4. *[8] Let be Λ be a subset of the Banach space X. Then, Λ is relatively compact in X if the following assumptions hold*
(A_1) $\left\{\frac{x(t)}{h(t)}:\ x(t)\in\Lambda\right\}$ is uniformly bounded.
(A_2) $\left\{\frac{x(t)}{h(t)}:\ x(t)\in\Lambda\right\}$ is equicontinuous on any compact interval of $\mathbb{R}_+$.
(A_3) $\left\{\frac{x(t)}{h(t)}:\ x(t)\in\Lambda\right\}$ is equiconvergent at infinity i.e. for each given $\epsilon>0$, there exists $T_0>0$ such that for any $x\in\Lambda$ and $t_1,t_2>T_0$, we have

$$\left|\frac{x(t_2)}{h(t_2)}-\frac{x(t_1)}{h(t_1)}\right|<\epsilon.$$

Lemma 1.5. *[9](Krasnoselskii's fixed point Theorem) Let E be bounded, closed and convex subset in a Banach space X. If $T_1,T_2:E\longrightarrow E$ are two applications satisfying the following conditions*
1) $T_1x+T_2y\in E$, for every $x,y\in E$.
2) T_1 is a contraction.
3) T_2 is compact and continuous.
then, there exists $z\in E$ such that $T_1z+T_2z=z$.

Let Ω be the set of all strictly increasing functions $h:\mathbb{R}_+\longrightarrow[1,+\infty)$ satisfying the following assumptions
(H_1) $h(0)=1$.
(H_2) $\lim\limits_{t\to\infty}h(t)=+\infty$.
(H_3) $h(t)\geq h(t-s)h(s)$ for all $0\leq s\leq t\leq\infty$.

Remark 1.6. *Note that Ω is a non-empty set, because the functions $h_1(t)=e^t$ and $h_2(t)=e^{t^2}$ belong to Ω.*

We denote

$$E=\Big\{x(t)\in C[0,+\infty),\quad \sup_{t\geq0}\frac{|x(t)|}{h(t)}<\infty\Big\},$$

with the norm

$$\|x\|=\sup_{t\geq0}\frac{|x(t)|}{h(t)}.$$

Then $(E,\|\cdot\|)$ is a Banach space. For more details of this Banach space we refer to [5,8].
In addition, we define $\|\phi\|_t=\max\{|\phi(s)|,0\leq s\leq t\}$ for all $t\geq0$,
all given function $\phi\in C(\mathbb{R}_+)$, and let $\mathfrak{B}(\epsilon)=\{x:x\in E,\|x\|\leq\epsilon\}$
be a non-empty closed convex subset of E, for each $\epsilon>0$.

Lemma 1.7. *Let* $g(t,x(t)) \in C[0,+\infty)$ *and* $f(t,x(t)) \in C^1[0,+\infty)$.
Then $x(t) \in C[0,+\infty)$ *is a solution of (1.1) if and only if* $x(t)$ *is a solution of the following Cauchy system:*

$$\begin{cases} x'(t) = I_{0^+}^{p-1}\Big(g(t,x(t)) + {}^C D_{0^+}^{p-1} f(t,x(t))\Big) + x_1, \quad t \geq 0 \\ \\ x(0) = x_0. \end{cases} \tag{1.2}$$

Proof. To begin the proof, note that for any $0 < \alpha < 0$, if $\varphi \in C[0,+\infty)$, then $(I_{0^+}^{\alpha}\varphi)(0) = 0$. Indeed

$$\begin{aligned} |I_{0^+}^{\alpha}\varphi(t)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}\varphi(s)ds\right| \\ &\leq \frac{\|\varphi\|_t}{\Gamma(\alpha+1)}t^{\alpha} \longrightarrow 0, \ \textbf{as } t \longrightarrow 0. \end{aligned} \tag{1.3}$$

To simplify calculations, we use the notation

$$m(t) = g(t,x(t)) + {}^C D_{0^+}^{p-1} f(t,x(t)).$$

(1) let $x(t) \in C[0,+\infty)$ be a solution of (1.1). By the definition 1.2, we get

$${}^C D_{0^+}^{p} x(t) = ({}^C D_{0^+}^{p-1}{}^C D_{0^+}^{1} x)(t) = ({}^C D_{0^+}^{p-1} x')(t) = m(t),$$

then from Lemma 1.3 we obtain

$$x'(t) = I_{0^+}^{p-1}{}^C D_{0^+}^{p-1} x'(t) = x'(0) + I_{0^+}^{p-1} m(t) = I_{0^+}^{p-1} m(t) + x_1,$$

therefore $x(t)$ is a solution of (1.2).
(2) Conversely, let $x(t)$ be a solution of the problem (1.2). Then we have

$${}^C D_{0^+}^{p} x(t) = {}^C D_{0^+}^{p-1} x'(t) = ({}^C D_{0^+}^{p-1} I_{0^+}^{p-1} m)(t) + {}^C D_{0^+}^{p-1} x_1 = m(t).$$

Since $m(t) \in C(\mathbb{R}_+)$, then we find $(I_{0^+}^{p-1} m)(0) = 0$, this implies

$$x'(0) = (I_{0^+}^{p-1} m)(0) + x_1 = x_1.$$

Thus, $x(t)$ is a solution of the problem (1.1). □

Lemma 1.8. *The problem (1.2) is equivalent to the problem*

$$\begin{cases} x'(t) = -\rho x(t) + G(t,x(t)) + \dfrac{d}{dt}\displaystyle\int_0^t \psi(t-s)x(s)ds, \\ \\ x(0) = x_0, \end{cases} \tag{1.4}$$

where: $\psi(t-s) = \frac{(t-s)^{p-1}}{\Gamma(p)} + \rho, \quad \forall \rho \in \mathbb{R}, \quad 0 \leq s \leq t < +\infty,$
and: $G(t,x(t)) = I_{0^+}^{p-1}\Big(g(t,x(t)) - x(t)\Big) + f(t,x(t)) - f(0,x_0) + x_1.$

Proof.

$$\begin{aligned}
x'(t) &= I_{0^+}^{p-1}\Big(g(t,x(t)) + {}^C D_{0^+}^{p-1} f(t,x(t))\Big) + x_1.\\
&= I_{0^+}^{p-1}\Big(g(t,x(t)) + {}^C D_{0^+}^{p-1} f(t,x(t)) - x(t)\Big) + I_{0^+}^{p-1}x(t) + x_1.\\
&= I_{0^+}^{p-1}\Big(g(t,x(t)) - x(t)\Big) + I_{0^+}^{p-1C} D_{0^+}^{p-1} f(t,x(t)) + I_{0^+}^{p-1}x(t) + x_1.\\
&= I_{0^+}^{p-1}\Big(g(t,x(t)) - x(t)\Big) + f(t,x(t)) - f(0,x_0) + I_{0^+}^{p-1}x(t) + x_1.\\
&= G(t,x(t)) + I_{0^+}^{p-1}x(t).\\
&= G(t,x(t)) + \int_0^t \frac{(t-s)^{p-2}}{\Gamma(p-1)} x(s)ds.\\
&= G(t,x(t)) + \frac{d}{dt}\int_0^t \psi(t-s)x(s)ds - \rho x(t).
\end{aligned}$$

□

Lemma 1.9. *$x(t)$ is a solution of the problem (1.4) if and only if $x(t)$ satisfies the following integral equation:*

$$\begin{aligned}
x(t) &= e^{-\rho t}x_0 + \frac{(1-e^{-\rho t})}{\rho}\Big(x_1 - f(0,x_0)\Big) + \rho\int_0^t e^{-\rho(t-u)}x(u)du \qquad (1.5)\\
&+ \int_0^t e^{-\rho(t-s)} f(s,x(s))ds + \int_0^t\int_u^t \frac{e^{-\rho(t-s)}}{\Gamma(p-1)}(s-u)^{p-2}ds \;\; g(u,x(u))du.
\end{aligned}$$

Proof. Using the variation of constants method to the first order nonlinear equation in (1.4) with integration by parts, we find:

$$\begin{aligned}
x(t) =& e^{-\rho t}x_0 + \int_0^t e^{-\rho(t-s)}\left[G(s,x(s)) + \frac{d}{ds}\int_0^s \psi(s-u)x(u)du\right]ds.\\
=& e^{-\rho t}x_0 + \int_0^t e^{-\rho(t-s)}\left[\frac{d}{ds}\int_0^s \psi(s-u)x(u)du\right]ds + \int_0^t e^{-\rho(t-s)}G(s,x(s))ds.\\
=& e^{-\rho t}x_0 + \left[e^{-\rho(t-s)}\int_0^s \psi(s-u)x(u)du\right]_{s=0}^{s=t} - \rho\int_0^t e^{-\rho(t-s)}\int_0^s \psi(s-u)x(u)duds\\
&+ \int_0^t e^{-\rho(t-s)}\left[I_{0^+}^{p-1}g(s,x(s)) - I_{0^+}^{p-1}x(s) + f(s,x(s)) - f(0,x_0) + x_1\right]ds.\\
=& e^{-\rho t}x_0 + \int_0^t \psi(t-s)x(s)ds - \rho\int_0^t\int_u^t e^{-\rho(t-s)}\psi(s-u)dsx(u)du\\
&+ \int_0^t e^{-\rho(t-s)}\int_0^s \frac{(s-u)^{p-2}}{\Gamma(p-1)}g(u,x(u))duds - \int_0^t e^{-\rho(t-s)}\int_0^s \frac{(s-u)^{p-2}}{\Gamma(p-1)}x(u)duds\\
&+ \int_0^t e^{-\rho(t-s)}f(s,x(s))ds - \int_0^t e^{-\rho(t-s)}f(0,x_0)ds + \int_0^t e^{-\rho(t-s)}x_1 ds.\\
=& e^{-\rho t}x_0 + \int_0^t \psi(t-s)x(s)ds - \rho\int_0^t\int_u^t e^{-\rho(t-s)}\psi(s-u)dsx(u)du\\
&+ \frac{1}{\Gamma(p-1)}\int_0^t\int_u^t e^{-\rho(t-s)}(s-u)^{p-2}ds \;\; g(u,x(u))du - \frac{1-e^{-\rho t}}{\rho}\Big(x_1 - f(0,x_0)\Big)\\
&+ \int_0^t e^{-\rho(t-s)}f(s,x(s))ds - \int_0^t\int_u^t e^{-\rho(t-s)}\frac{\partial\psi(s-u)}{\partial s}ds \;\; x(s)du.
\end{aligned}$$

$$\begin{aligned}
=&e^{-\rho t}x_0+\int_0^t\psi(t-s)x(s)ds-\rho\int_0^t\int_u^t e^{-\rho(t-s)}\psi(s-u)dsx(u)du\\
&+\frac{1}{\Gamma(p-1)}\int_0^t\int_u^t e^{-\rho(t-s)}(s-u)^{p-2}ds\ \ g(u,x(u))du\\
&+\frac{1-e^{-\rho t}}{\rho}\Big(x_1-f(0,x_0)\Big)+\int_0^t e^{-\rho(t-s)}f(s,x(s))ds\\
&+\int_0^t\left\{\Big[e^{-\rho(t-s)}\psi(s-u)\Big]_{s=u}^{s=t}-\rho\int_u^t e^{-\rho(t-s)}\psi(s-u)ds\right\}x(u)du.\\
=&e^{-\rho t}x_0+\frac{(1-e^{-\rho t})}{\rho}\Big(x_1-f(0,x_0)\Big)+\rho\int_0^t e^{-\rho(t-u)}x(u)du\\
&+\int_0^t e^{-\rho(t-s)}f(s,x(s))ds+\int_0^t\int_u^t\frac{e^{-\rho(t-s)}}{\Gamma(p-1)}(s-u)^{p-2}ds\ \ g(u,x(u))du.
\end{aligned}$$

Conversely, suppose that (1.5) is satisfied, then we have $x(0)=x_0$ and :

$$\begin{aligned}
\Big(e^{\rho t}x(t)\Big)' &= \rho e^{\rho t}x(t)+e^{\rho t}x'(t).\\
&= \Bigg[x_0+\frac{e^{\rho t}-1}{\rho}\Big(x_1-f(0,x_0)\Big)+\rho\int_0^t e^{\rho u}x(u)du\\
&\quad+\int_0^t e^{-\rho s}f(s,x(s))ds+\int_0^t e^{-\rho u}I_{0+}^{p-1}g(u,x(u))du\Bigg]'.\\
&= e^{\rho t}\Big[x_1-f(0,x_0)+\rho x(t)+f(t,x(t))+I_{0+}^{p-1}g(t,x(t))\Big].\\
&= e^{\rho t}\Big[I_{0+}^{p-1}g(t,x(t))+I_{0+}^{p-1\,C}D_{0+}^{p-1}f(t,x(t))+x_1\Big]+\rho e^{\rho t}x(t).
\end{aligned}$$

Thus,

$$x'(t)=I_{0+}^{p-1}g(t,x(t))+I_{0+}^{p-1\,C}D_{0+}^{p-1}f(t,x(t))+x_1.$$

□

Based on Lemma 1.7, Lemma 1.8 and Lemma 1.9, we conclude that the problem (1.1) is equivalent to the integral equation (1.5).
Section 2, provide the proofs of the existence of solution to the problem (1.1) in Banach space. Finally, a stability result and an illustrative example is presented in Section 3.

2. Existence result

In order to prove the existence of the solution for the problem (1.1) in E. We transform the problem (1.1) into fixed point problem $Px=x$ Where P is an operator defined on $\mathfrak{B}(\epsilon)$ by

$$\begin{aligned}
Px(t) &= e^{-\rho t}x_0+\frac{(1-e^{-\rho t})}{\rho}\Big(x_1-f(0,x_0)\Big)+\rho\int_0^t e^{-\rho(t-u)}x(u)du \qquad (2.1)\\
&+ \int_0^t e^{-\rho(t-s)}f(s,x(s))ds+\frac{1}{\Gamma(p-1)}\int_0^t\int_u^t e^{-\rho(t-s)}(s-u)^{p-2}ds\ \ g(u,x(u))du.
\end{aligned}$$

We decompose the operator P into two operators P_1 and P_2 (i.e. $P=P_1+P_2$) defined on $\mathfrak{B}(\epsilon)$, as follows:

$$P_1x(t)=e^{-\rho t}x_0+\frac{(1-e^{-\rho t})}{\rho}\Big(x_1-f(0,x_0)\Big)+\rho\int_0^t e^{-\rho(t-u)}x(u)du.$$

$$\begin{aligned} P_2x(t) &= \int_0^t e^{-\rho(t-s)} f(s,x(s))ds + \int_0^t \int_u^t \frac{e^{-\rho(t-s)}(s-u)^{p-2}}{\Gamma(p-1)} ds \;\; g(u,x(u))du. \\ &= \int_0^t e^{-\rho(t-s)} f(s,x(s))ds + \int_0^t K(t-u) g(u,x(u))du, \end{aligned}$$

$$\text{where :} \quad k(t-u) = \begin{cases} \displaystyle\int_u^t \frac{e^{-\rho(t-s)}(s-u)^{p-2}}{\Gamma(p-1)} ds, & t-u \geq 0 \\ 0, & t-u < 0, \end{cases}$$

Theorem 2.1. *Suppose that there are strictly positive constants $\varphi, \delta, c_1, c_2, c_3$ where $c_1 + c_2 + c_3 < 1$, $|x_0| + |x_1| + |f(0, x_0)| \leq \delta$ and the functions $\overline{f}, \overline{g} : \mathbb{R}_+ \times (0, \varphi] \longrightarrow \mathbb{R}_+$ are continuous and nondecreasing in r for fixed t with $\overline{f}, \overline{g} \in L^1[0, +\infty)$ in t for fixed r, such that*

$$\frac{|g(t, \nu h(t))|}{h(t)} \leq \overline{g}(t, |\nu|), \qquad \frac{|f(t, \nu h(t))|}{h(t)} \leq \overline{f}(t, |\nu|), \tag{2.2}$$

hold for all $t \geq 0$, $0 < |\nu| \leq \varphi$ and

$$\sup_{t\geq 0} \int_0^t \frac{k(t-u)}{h(t-u)} \frac{\overline{g}(u,r)}{r} du \leq c_2 < 1 - c_1 - c_3 \tag{2.3}$$

$$\sup_{t\geq 0} \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)} \frac{\overline{f}(s,r)}{r} ds \leq c_3, \tag{2.4}$$

hold for every $0 < r \leq \varphi$. Then there exists at least one fixed point of the operator P in $\mathfrak{B}(\epsilon)$.

Proof. Suppose that there exists constant $c_4 > 0$ such that $\dfrac{e^{-\rho t}}{h(t)} \leq c_4$ and

$$\frac{e^{-\rho t}}{h(t)} \in BC[0, +\infty) \cap L^1[0, +\infty), \quad |\rho| \int_0^{+\infty} \frac{e^{-\rho s}}{h(s)} ds \leq c_1. \tag{2.5}$$

Let

$$0 < \delta \leq \frac{\big[1 - (c_1 + c_2 + c_3)\big] |\rho|}{c_4 |\rho| + 1 + c_4} \epsilon. \tag{2.6}$$

Firstly, we will show that: $P_1 \mathfrak{B}(\epsilon) \subseteq E$, $P_2 \mathfrak{B}(\epsilon) \subseteq E$, and P_1 is a contraction mapping.
It is clear that for $x \in \mathfrak{B}(\epsilon)$, P_1 and P_2 are continuous functions on $\mathbb{R}_+$. Moreover, for all $x \in \mathfrak{B}(\epsilon)$ and each $t \geq 0$, we have

$$\begin{aligned} \frac{|P_1 x(t)|}{h(t)} &= \frac{1}{h(t)} \left| e^{-\rho t} x_0 + \frac{(1 - e^{-\rho t})}{\rho} \Big(x_1 - f(0, x_0) \Big) + \rho \int_0^t e^{-\rho(t-u)} x(u) du \right|. \\ &\leq \frac{e^{-\rho t}}{h(t)} |x_0| + \frac{(1 - e^{-\rho t})}{\rho h(t)} \Big(|x_1| + |f(0, x_0)| \Big) + \rho \int_0^t \frac{e^{-\rho(t-u)}}{h(t-u)} \frac{x(u)}{h(u)} du. \\ &\leq c_4 |x_0| + \frac{1 + c_4}{|\rho|} \Big(|x_1| + |f(0, x_0)| \Big) + \frac{|\rho|}{\rho} \epsilon. \\ &< +\infty, \end{aligned}$$

which means that $P_1 \mathfrak{B}(\epsilon) \subseteq E$.

Similarly, for any $x \in \mathfrak{B}(\epsilon)$, we have

$$\begin{aligned}
\frac{|P_2x(t)|}{h(t)} &= \frac{1}{h(t)}\left|\int_0^t e^{-\rho(t-s)}f(s,x(s))ds+\int_0^t K(t-u)g(u,x(u))du\right|.\\
&\leq \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\frac{|f(s,x(s))|}{h(s)}ds+\int_0^t \frac{K(t-u)}{h(t-u)}\frac{|g(u,x(u))|}{h(u)}du.\\
&\leq \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\overline{f}\Big(s,\frac{|x(s)|}{h(s)}\Big)ds+\int_0^t \frac{K(t-u)}{h(t-u)}\overline{g}\Big(u,\frac{|x(u)|}{h(u)}\Big)du.\\
&\leq c_3\|x\|+c_2\|x\|\\
&\leq (c_3+c_2)\epsilon,\\
&< +\infty. \qquad (2.7)
\end{aligned}$$

which implies that $P_2\mathfrak{B}(\epsilon)\subseteq E$. For any $x,y\in\mathfrak{B}(\epsilon)$, we have

$$\begin{aligned}
\sup_{t\geq 0}\left|\frac{P_1x(t)}{h(t)}-\frac{P_1y(t)}{h(t)}\right| &\leq \sup_{t\geq 0}|\rho|\int_0^t \frac{e^{-\rho(t-u)}|x(u)-y(u)|}{h(t)}du.\\
&\leq \sup_{t\geq 0}|\rho|\int_0^t \frac{e^{-\rho(t-u)}}{h(t-u)}.\frac{|x(u)-y(u)|}{h(u)}du.\\
&\leq |\rho|\int_0^t \frac{e^{\rho s}}{h(s)}ds\|x-y\|\leq c_1\|x-y\|.
\end{aligned}$$

Since $c_1<1$, hence P_1 is a contraction.
Secondly, for every $x,y\in\mathfrak{B}(\epsilon)$, we have

$$\begin{aligned}
&\sup_{t\geq 0}\frac{|P_2x(t)+P_1y(t)|}{h(t)}\\
=\;&\sup_{t\geq 0}\left\{\frac{1}{h(t)}\left|e^{-\rho t}x_0+\frac{(1-e^{-\rho t})}{\rho}\Big(x_1-f(0,x_0)\Big)+\rho\int_0^t e^{-\rho(t-u)}y(u)du\right.\right.\\
&\left.\left.+\int_0^t e^{-\rho(t-s)}f(s,x(s))ds+\int_0^t K(t-u)g(u,x(u))du\right|\right\}.\\
\leq\;&\sup_{t\geq 0}\left\{\frac{e^{-\rho t}}{h(t)}|x_0|+\frac{1}{|\rho|}\left(\frac{1}{h(t)}+\frac{e^{-\rho t}}{h(t)}\right)\Big(|x_1|+|f(0,x_0)|\Big)+\rho\int_0^t \frac{e^{-\rho(t-u)}}{h(t-u)}\frac{|y(u)|}{h(u)}du\right.\\
&\left.+\int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\frac{|f(s,x(s))|}{h(s)}ds+\int_0^t \frac{K(t-u)}{h(t-u)}\frac{|g(u,x(u))|}{h(u)}du\right\}.\\
\leq\;& c_4|x_0|+\frac{1+c_4}{|\rho|}\Big(|x_1|+|f(0,x_0)|\Big)+c_1\|y\|+c_3\|x\|+c_2\|x\|.\\
\leq\;& \frac{c_4|\rho|+1+c_4}{|\rho|}\delta+(c_1+c_3+c_2)\epsilon.\\
\leq\;& \epsilon.
\end{aligned}$$

Thus, $P_1+P_2\in\mathfrak{B}(\epsilon)$.
From the assumption $\dfrac{|P_2x(t)|}{h(t)}<+\infty$, we find that the set $\left\{\dfrac{x(t)}{h(t)}:\;\; x(t)\in\mathfrak{B}(\epsilon)\right\}$ is uniformly bounded in E. Furthermore, the convolution product of two functions where the first one is of L^1 and the other tends to zero also tends to zero. Therefore, for $t-u\geq 0$, we have:

$$\begin{aligned}
0\leq \lim_{t\longrightarrow+\infty}\frac{k(t-u)}{h(t-u)} &\leq \lim_{t\longrightarrow+\infty}\frac{1}{\Gamma(p-1)}\int_u^t \frac{e^{-\rho(t-s)}(s-u)^{p-2}}{h(t-s)h(s-u)}ds\\
&= \lim_{t\longrightarrow+\infty}\frac{1}{\Gamma(p-1)}\int_0^t \frac{e^{-\rho(t-u-s)}}{h(t-u-s)}.\frac{(s)^{p-2}}{h(s)}ds=0,
\end{aligned}$$

because, $\dfrac{t^{p-2}}{h(t)} \longrightarrow 0$ as $t \longrightarrow +\infty$ for $1 < p < 2$. In addition, by the continuity of the functions $k(t)$ and $h(t)$, it follows that there exists a positive constant c_5 such that $\left|\dfrac{k(t-u)}{h(t-u)}\right| \leq c_5$ and for any $x \in \mathfrak{B}(\epsilon)$ and for all $t_1, t_2 \in [0, T^\star]$, $T^\star \in \mathbb{R}_+$, $t_1 < t_2$, we have:

$$\begin{aligned}
&\left|\frac{P_2x(t_2)}{h(t_2)} - \frac{P_2x(t_1)}{h(t_1)}\right| \\
\leq\ & \int_0^{t_1} \left|\frac{k(t_2-u)}{h(t_2)} - \frac{k(t_1-u)}{h(t_1)}\right| |g(u,x(u))|du + \int_{t_1}^{t_2} \frac{k(t_2-u)}{h(t_2)} |g(u,x(u))|du. \\
&+ \int_0^{t_1} \left|\frac{e^{-\rho(t_2-s)}}{h(t_2)} - \frac{e^{-\rho(t_1-s)}}{h(t_1)}\right| |f(s,x(s))|ds + \int_{t_1}^{t_2} \frac{e^{-\rho(t_2-s)}}{h(t_2)} |f(s,x(s))|ds \\
\leq\ & \int_0^{t_1} \left|\frac{k(t_2-u)h(u)}{h(t_2)} - \frac{k(t_1-u)h(u)}{h(t_1)}\right| \overline{g}\Big(u, \frac{|x(u)|}{h(u)}\Big)du + \int_{t_1}^{t_2} \frac{k(t_2-u)}{h(t_2-u)} . \frac{|g(u,x(u))|}{h(u)}du. \\
&+ \int_0^{t_1} \left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)} - \frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right| \overline{f}\Big(s, \frac{|x(s)|}{h(s)}\Big)ds + \int_{t_1}^{t_2} \frac{e^{-\rho(t_2-s)}}{h(t_2-s)} . \frac{|f(s,x(s))|}{h(s)}ds \\
\leq\ & \int_0^{t_1} \left|\frac{k(t_2-u)h(u)}{h(t_2)} - \frac{k(t_1-u)h(u)}{h(t_1)}\right| \overline{g}(u,\epsilon)du + c_5 \int_{t_1}^{t_2} \overline{g}(u,\epsilon)du. \\
&+ \int_0^{t_1} \left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)} - \frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right| \overline{f}(s,\epsilon)ds + c_4 \int_{t_1}^{t_2} \overline{f}(s,\epsilon)ds.
\end{aligned}$$

Thus, $\left|\dfrac{P_2x(t_2)}{h(t_2)} - \dfrac{P_2x(t_1)}{h(t_1)}\right| \longrightarrow 0$, as $t_2 \longrightarrow t_1$, which means that $\left\{\frac{x(t)}{h(t)} : \ x(t) \in \mathfrak{B}(\epsilon)\right\}$ is equicontinuous on any compact of $\mathbb{R}_+$.

Now, based on Lemma 1.4 to show that $P_2\mathfrak{B}(\epsilon)$ is relatively compact it suffices to prove that $\{\frac{x(t)}{h(t)} : x(t) \in \mathfrak{B}(\epsilon)\}$ is equiconvergent at infinity. Indeed, for any $\epsilon^\star > 0$, there exists $M > 0$ such that

$$c_5 \int_M^{+\infty} \overline{g}(u,\epsilon)du \leq \frac{\epsilon^\star}{6}, \qquad c_4 \int_M^{+\infty} \overline{f}(s,\epsilon)du \leq \frac{\epsilon^\star}{6}.$$

Then there exists $T > M$ such that for all $t_1, t_2 \geq T$, we get

$$\begin{aligned}
\sup_{u\in[0,M]} \left|\frac{k(t_2-u)h(u)}{h(t_2)} - \frac{k(t_1-u)h(u)}{h(t_1)}\right| &\leq \sup_{u\in[0,M]} \left|\frac{k(t_2-u)}{h(t_2-u)}\right| + \sup_{u\in[0,M]} \left|\frac{k(t_1-u)}{h(t_1-u)}\right| \\
&\leq \frac{\epsilon^\star}{6A}, \\
\sup_{s\in[0,M]} \left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)} - \frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right| &\leq \sup_{s\in[0,M]} \left|\frac{e^{-\rho(t_2-s)}}{h(t_2-s)}\right| + \sup_{s\in[0,M]} \left|\frac{e^{-\rho(t_1-s)}}{h(t_1-s)}\right| \\
&\leq \frac{\epsilon^\star}{6B},
\end{aligned}$$

where

$$A = \int_0^{+\infty} \overline{g}(u,\epsilon)du, \quad B = \int_0^{+\infty} \overline{f}(s,\epsilon)ds.$$

Then, we have

$$\begin{aligned}
&\left|\frac{P_2x(t_2)}{h(t_2)}-\frac{P_2x(t_1)}{h(t_1)}\right| \\
\leq\;& \int_0^M\left|\frac{e^{-\rho(t_2-s)}}{h(t_2)}-\frac{e^{-\rho(t_1-s)}}{h(t_1)}\right|f(s,x(s))ds+\int_M^{t_2}\frac{e^{-\rho(t_2-s)}}{h(t_2)}f(s,x(s))ds \\
&+\int_0^M\left|\frac{k(t_2-u)}{h(t_2)}-\frac{k(t_1-u)}{h(t_1)}\right|g(u,x(u))du+\int_M^{t_2}\frac{k(t_2-u)}{h(t_2)}g(u,x(u))du \\
&+\int_M^{t_1}\frac{e^{-\rho(t_1-s)}}{h(t_1)}f(s,x(s))ds+\int_M^{t_1}\frac{k(t_1-u)}{h(t_1)}g(u,x(u))du \\
\leq\;& \int_0^M\left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)}-\frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right|\overline{f}(s,x(s))ds+\int_M^{t_2}\frac{e^{-\rho(t_2-s)}}{h(t_2-u)}\overline{f}(s,x(s))ds \\
&+\int_0^M\left|\frac{k(t_2-u)h(u)}{h(t_2)}-\frac{k(t_1-u)h(u)}{h(t_1)}\right|\overline{g}(u,x(u))du+\int_M^{t_2}\frac{k(t_2-u)}{h(t_2-u)}\overline{g}(u,x(u))du \\
&+\int_M^{t_1}\frac{e^{-\rho(t_1-s)}}{h(t_1-s)}\overline{f}(s,x(s))ds+\int_M^{t_1}\frac{k(t_1-u)}{h(t_1-u)}\overline{g}(u,x(u))du \\
\leq\;& \frac{\epsilon^\star}{6}+\frac{\epsilon^\star}{6}+2c_5\int_M^{+\infty}\overline{f}(s,x(s))ds+2c_4\int_M^{+\infty}\overline{g}(u,x(u))du \\
\leq\;& \frac{\epsilon^\star}{6}+\frac{\epsilon^\star}{6}+\frac{\epsilon^\star}{3}+\frac{\epsilon^\star}{3}=\epsilon^\star.
\end{aligned}$$

Finally, from krasnoselskii fixed point Theorem, we conclude that the problem (1.1) has at least one solution. □

3. Stability Result

Before stating and proving our main stability results, we need the following definitions:

Definition 3.1. *The trivial solution $x=0$ of fractional order system (1.1) is said to be*

1) *Stable in Banach space E, if for every $\epsilon>0$, there exists $\delta=\delta(\epsilon)$ such that $\sum_{i=0}^{i=1}|x_i|\leq\delta$ implies that the solution $x(t)=x(t,x_0,x_1)$ exists for all $t\geq 0$ and satisfies $\|x\|\leq\epsilon$.*

2) *Asymptotically stable, if it is stable in E and there exists a number $\mu>0$ such that $\sum_{i=0}^{i=1}|x_i|\leq\mu$ implies that $\lim_{t\to+\infty}\|x(t)\|=0$.*

Theorem 3.2. *Assume that all assumptions of Theorem 2.1 hold such that $|x_0|\geq|f(0,x_0)|$. Then the trivial solution $x=0$ of the system (1.1) is stable in the Banach space E.*

Proof. Let for any $\epsilon>0$

$$0<\delta_1\leq\frac{\{1-(c_1+c_2+c_3)\}|\rho|}{c_4|\rho|+1+c_4}\epsilon. \tag{3.1}$$

From the assumption $|x_0|+|x_1|+|f(0,x_0|\leq\delta$ it follows that

$$|x_0|+|x_1|\leq\delta-|f(0,x_0|=\delta_1>0.$$

Then, we get

$$\begin{aligned}
\|x\| &= \sup_{t\geq 0}\Big|\frac{e^{-\rho t}}{h(t)}x_0 + \frac{1-e^{-\rho t}}{\rho h(t)}(x_1 - f(0,x_0)) + \rho\int_0^t \frac{e^{-\rho(t-u)}}{h(t)}x(u)du \\
&\quad + \int_0^t \frac{e^{-\rho(t-s)}}{h(t)}f(s,x(s))ds + \int_0^t \frac{k(t-u)}{h(t)}g(u,x(u))du\Big| \\
&\leq \sup_{t\geq 0}\Big\{\frac{e^{-\rho t}}{h(t)}|x_0| + \frac{1+e^{-\rho t}}{\rho h(t)}(|x_1| + |f(0,x_0)|) + |\rho|\int_0^t \frac{e^{-\rho(t-u)}}{h(t-u)}\frac{|x(u)|}{h(u)}du \\
&\quad + \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\frac{|f(s,x(s))|}{h(s)}ds + \int_0^t \frac{k(t-u)}{h(t-u)}\frac{|g(u,x(u))|}{h(u)}du\Big\} \\
&\leq c_4\delta_1 + \frac{1+c_4}{|\rho|}\delta_1 + c_1\|x\| + c_3\|x\| + c_2\|x\|.
\end{aligned}$$

Hence,

$$\|x\| \leq \frac{c_4|\rho| + 1 + c_4}{|\rho|}\delta_1 \leq \epsilon,$$

therefore, the trivial solution $x = 0$ of the problem (1.1) is stable in the Banach space E. □

Theorem 3.3. *Suppose that all assumptions of Theorem 2.1 are satisfied with*

$$\lim_{t\to+\infty}\frac{e^{-\rho t}}{h(t)} = 0, \tag{3.2}$$

and for any $r > 0$ there exist two strictly positive functions $\varphi_r(t), \psi_r(t) \in L^1[0,+\infty)$ such that $|u| \leq r$ implies

$$\frac{|g(t,u)|}{h(t)} \leq \varphi_r(t), \quad \frac{|f(t,u)|}{h(t)} \leq \psi_r(t), \quad a.e.\colon\ t \in [0,+\infty). \tag{3.3}$$

Then the trivial solution $x = 0$ of the system (1.1) is asymptotically stable in E.

Proof. From Theorem 3.2 it follows that the trivial solution $x = 0$ of problem (1.1) is stable in the Banach space E. So, it suffices to show that $x = 0$ is attractive. For this fact, we define for any $r > 0$

$$\widetilde{\mathfrak{B}}(r) = \{x \in \mathfrak{B}(r) : \lim_{t\to+\infty}\frac{x(t)}{h(t)} = 0\}.$$

We only show that $P_2x + P_1y \in \widetilde{\mathfrak{B}}(r)$ for any $x, y \in \widetilde{\mathfrak{B}}(r)$, in other words,

$$\lim_{t\to+\infty}\frac{P_2x(t) + P_1y(t)}{h(t)} = 0.$$

For all $x, y \in \widetilde{\mathfrak{B}}(r)$, we have:

$$\begin{aligned}
&\frac{|P_2x(t)+P_1y(t)|}{h(t)} \\
=\ &\frac{1}{h(t)}\Big|e^{-\rho t}x_0+\frac{(1-e^{-\rho t})}{\rho}\Big(x_1-f(0,x_0)\Big)+\rho\int_0^t e^{-\rho(t-u)}y(u)du \\
&+\int_0^t e^{-\rho(t-s)}f(s,x(s))ds+\int_0^t K(t-u)g(u,x(u))du\Big|. \\
\leq\ &\frac{e^{-\rho t}}{h(t)}|x_0|+\frac{1}{|\rho|}\Big(\frac{1}{h(t)}+\frac{e^{-\rho t}}{h(t)}\Big)\Big(|x_1|+|f(0,x_0)|\Big)+|\rho|\int_0^t\frac{e^{-\rho(t-u)}}{h(t-u)}\frac{|y(u)|}{h(u)}du \\
&+\int_0^t\frac{e^{-\rho(t-s)}}{h(t-s)}\frac{|f(s,x(s))|}{h(s)}ds+\int_0^t\frac{K(t-u)}{h(t-u)}\frac{|g(u,x(u))|}{h(u)}du. \\
\leq\ &\frac{e^{-\rho t}}{h(t)}|x_0|+\frac{1}{|\rho|}\Big(\frac{1}{h(t)}+\frac{e^{-\rho t}}{h(t)}\Big)\Big(|x_1|+|f(0,x_0)|\Big)+\rho\int_0^t\frac{e^{-\rho(t-u)}}{h(t-u)}\frac{|y(u)|}{h(u)}du \\
&+\int_0^t\frac{e^{-\rho(t-s)}}{h(t-s)}\psi_r(s)ds+\int_0^t\frac{K(t-u)}{h(t-u)}\varphi_r(u)du.
\end{aligned}$$

From (2.5) and (3.2), we have:

$$\int_0^t\frac{e^{-\rho(t-u)}}{h(t-u)}\frac{|y(u)|}{h(u)}du\longrightarrow 0,\quad \text{as } t\longrightarrow+\infty,$$

and

$$\frac{k(t-u)}{h(t-u)}=\frac{1}{\Gamma(p-1)}\int_0^t\frac{e^{-\rho(t-u)}}{h(t-u)}(s-u)^{p-2}ds\longrightarrow 0,\quad \text{as } t\longrightarrow+\infty.$$

Together with the hypotheses $\varphi_r(t), \psi_r(t)\in L^1[0,+\infty)$, we find

$$\int_0^t\frac{K(t-u)}{h(t-u)}\varphi_r(u)du\longrightarrow 0,\quad \text{as } t\longrightarrow+\infty,$$

and

$$\int_0^t\frac{e^{-\rho(t-s)}}{h(t-s)}\psi_r(s)ds\longrightarrow 0,\quad \text{as } t\longrightarrow+\infty.$$

Moreover, since $h(t)\longrightarrow+\infty$ as $t\longrightarrow+\infty$, we conclude that

$$\frac{P_2x(t)+P_1y(t)}{h(t)}\longrightarrow 0,\quad \text{as } t\longrightarrow+\infty.$$

Therefore, $P_2x+P_1y\in\widetilde{\mathfrak{B}}(r)$ which implies that the trivial solution $x=0$ of problem (1.1) is asymptotically stable. □

Example 3.4.

$$\begin{cases}{}^CD^{\frac{4}{3}}_{0^+}x(t)=\dfrac{t^3x^3}{e^{(\sigma+2)t}}+{}^CD^{\frac{1}{3}}_{0^+}\Big(\dfrac{x^{\frac{4}{3}}}{(1+t^4)e^{(\sigma+1)t}}\Big),\quad t\in[0,+\infty)\\[2ex] x(0)=x_0,\quad x'(0)=x_1,\end{cases}\tag{3.4}$$

where $\sigma>0$. Suppose $0<|\rho|\leq\frac{\sigma}{2}$. Let $h(t)=e^{(\sigma+1)t}$ and $c_1=\frac{|\rho|}{\sigma+1+\rho}$.
Then, (2.5) holds i.e.,

$$e^{-\rho t}/h(t)=\frac{e^{-\rho t}}{e^{(\sigma+1)t}}=e^{-(\rho+\sigma+1)t}\in BC(\mathbb{R}_+)\cap L^1(\mathbb{R}_+),$$

and

$$|\rho|\int_0^{+\infty}\frac{e^{-\rho s}}{e^{(\sigma+1)s}}ds=|\rho|\int_0^{+\infty}e^{-(\rho+\sigma+1)s}\leq\frac{|\rho|}{\sigma+1+\rho}=c_1.$$

The Banach space is

$$E_1=\{x(t)\in C(\mathbb{R}_+):\ \sup_{t\geq 0}\frac{|x(t)|}{e^{(\sigma+1)t}}<\infty\},$$

equipped with the norm

$$\|x\|=\sup_{t\geq 0}\frac{|x(t)|}{e^{(\sigma+1)t}}.$$

Let

$$\overline{g}(t,r)=\frac{t^3r^3}{e^t},\qquad \overline{f}(t,r)=\frac{r^{\frac{4}{3}}}{1+t^4}.$$

We get $\overline{f}(t,r),\overline{g}(t,r)\in L^1(\mathbb{R}_+)$ *in* t *for fixed* r.
After some computations, we find

$$\frac{k(t-u)}{h(t-u)}\leq\frac{1}{\Gamma(\frac{1}{3})}\int_u^t\frac{(s-u)^{-\frac{2}{3}}}{e^{(\sigma+1)(s-u)}}ds=\frac{1}{\Gamma(\frac{1}{3})}\int_0^{t-u}\frac{\tau^{-\frac{2}{3}}}{e^{(\sigma+1)\tau}}d\tau\leq(\sigma+1)^{\frac{1}{3}},$$

$$\int_0^t\frac{e^{-\rho(t-s)}}{h(t-s)}\frac{\overline{f}(s,r)}{r}ds=\int_0^t\frac{e^{-\rho(t-s)}}{h(t-s)}\frac{r^{\frac{1}{3}}}{1+t^4}ds\leq c_3,$$

and

$$\int_0^t\frac{k(t-u)}{h(t-u)}\frac{\overline{g}(u,r)}{r}du=\int_0^t\frac{k(t-u)}{h(t-u)}\frac{t^3r^2}{e^t}du\leq c_2.$$

Therefore, all assumptions of Theorem 3.2 are satisfied, then the trivial solution of (3.4) is stable in the Banach space E_1.
Let $\varphi_r,\ \psi_r\in L^1(\mathbb{R}_+)$ *where*

$$\varphi_r(t)=\frac{t^3r^3}{e^{(\sigma+2)t}},\quad \psi_r(t)=\frac{r^{\frac{4}{3}}}{(1+t^4)e^{(\sigma+1)t}},$$

satisfies the following inequalities

$$|\overline{g}(t,r)|\leq\varphi_r(t),\quad |\overline{f}(t,r)|\leq\psi_r(t),$$

and

$$\lim_{t\longrightarrow+\infty}\frac{e^{-\rho t}}{h(t)}=0.$$

Then, from Theorem 3.3 we conclude that trivial solution of (3.4) is asymptotically stable.

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References

1. N. Abdellouahab, B. Tellab and Kh. Zennir, Existence and Stability results of a nonlinear fractional integro-differential equation with integral boundary conditions, Kragujevac J. Math., 46(2), (2022), 685-699.
2. N. Abdellouahab, B. Tellab and Kh. Zennir, Existence and Stability results for the solution of Neutral fractional integro-differential equation with nonlocal conditions, (2019), submitted.
3. B. Ahmed, A. Alsaedi, S. Salem and S. K. Ntouyas, Fractional Differential Equation Involving Mixed Nonlinearities with Nonlocal multi-point and Reimann-steiljes integral-multi-strip conditions, Fractal Fract., 34(3), (2019).

4. T. M. Atanackovic, B. Stankovic, On a differential equation with left and right fractional derivatives, Fractional calc. Appl. Anal., 10(2), (2007), 139-150.

5. T.A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, Dover publications INC, Mineola, New York, 2006.

6. F. Ge, C. Kou, Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations, Appl. Math. Comput., 257, (2015), 308-316.

7. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations. Elsevier, Amsterdam, 539,(2006).

8. C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, Nonl. Anal., 74, (2011), 5975-5986.

9. M. A. Kranoselskii, Two remarks on the method of successive approximations. Uspekhi Mat. Nauk., 10, (1955), 123-127.

10. B. Stankovic, An equation with left and right fractional derivatives, Publications de l'institute mathematique, Nouvelle série, 80(94), (2006), 259-272.

11. Y. Zhou, Basic theory of fractional differential equations, 6, Singapore: World Scientific, (2014).

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