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# On the Iterations of Generalized Bi-derivation on Prime Ring

Faiza Shujat and Ahlam Fallatah

ABSTRACT: In the present paper we obtain some results in connection of the symmetric generalized biderivations on prime ring which are the generalization of the results of existing literature in [1], [2], [7].

Key Words: Prime ring, Generalized biderivation, Multiplier.

## Contents



## 1. Introduction

Throughout the paper $R$ will denote a ring with centre $Z(R)$. A ring $R$ is said to be prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. We shall write $[x, y]$ the commutator $xy - yx$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Maksa [4] introduced the concept of a symmetric biderivation. It was shown in [4] that symmetric biderivations are related to general solution of some functional equations. There has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. Some results on symmetric biderivation in prime and semiprime rings can be found in [3], [4], [5] and [6].

A mapping $D : R \times R \longrightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x, x)$ for all $x \in R$, where $D : R \times R \longrightarrow R$ is a symmetric and biadditive (i.e. additive in both arguments) mapping, is called the trace of $D$. The trace $f$ of $D$ satisfies the relation $f(x + y) = f(x) + f(y) + 2D(x, y)$, for all $x, y \in R$. A biadditive mapping $D : R \times R \longrightarrow R$ is called a biderivation if for every $x \in R$, the map $y \mapsto D(x, y)$ as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is a derivation of $R$, i.e., $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$ and $D(x, yz) = D(x, y)z + yD(x, z)$ satisfied for all $x, y, z \in R$.

The notion of generalized symmetric biderivations introduced in [8], which is defined as follows: Let $R$ be a ring and $D : R \times R \longrightarrow R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \longrightarrow R$ is said to be generalized biderivation if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $y \mapsto D(x, y)$ as well as if for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $x \mapsto D(x, y)$ for all $x, y \in R$. It also satisfies $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + xD(y, z)$ for all $x, y, z \in R$. The trace $g$ of $\Delta$ is defined as $\Delta(x, x) = g(x)$, which satisfies $g(x + y) = g(x) + g(y) + 2\Delta(x, y)$ for all $x, y \in R$.

An additive mapping $h : R \longrightarrow R$ is called left (resp. right) multiplier of $R$ if $h(xy) = h(x)y$ (resp. $h(xy) = xh(y)$) for all $x, y \in R$. A biadditive mapping $\zeta : R \times R \longrightarrow R$ is said to be a left (resp. right) bi-multiplier of $R$ if $\zeta(x, yz) = \zeta(x, y)z$ (resp. $\zeta(xz, y) = x\zeta(z, y)$) for all $x, y, z \in R$.
In this paper, we prove some theorems on symmetric generalized biderivations of prime ring in order to generalizes the results proved in [1,2,7].

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## 2. Main Theorems

Author in [2] established that: let $R$ be a prime ring of characteristic not two and $I$ be a nonzero ideal of $R$. If $\Delta$ is a symmetric generalized biderivation on $R$ with associated biderivation $D$ such that $[\Delta(x,x),\Delta(y,y)]=0$ for all $x,y\in I$, then one of the following conditions hold

1. $R$ is commutative.
2. $\Delta$ acts as a left bimultiplier on $R$.

Motivated by the above idea, we extend the result for $n$ iterations of symmetric generalized biderivations. Infact we prove the following:

**Theorem 2.1.** *Let $R$ be a prime ring of characteristic not two, $I$ be an ideal of $R$ and $n\geq 1$ a fixed integer. Consider $\Delta_1,\Delta_2,...,\Delta_n : R\times R\to R$ is a generalized biderivation with associated biderivation $D_1,D_2,...,D_n : R\times R\to R$ respectively such that $\Delta_1(x,x)\cdot\Delta_2(y,y)\cdots\Delta_n(u,u)=0$ for all $x,y,\cdots,u\in I$. Then one of the following holds:*

1. *$\Delta_1(x,x)=0$ for $x\in I$,*
2. *All $\Delta_{n+1}(y,y)$ acts as a left bi-multiplier on $R$, for all $n\geq 1$.*

**Proof** We shall prove it by induction. If $n=1$, then it is obvious we get $\Delta_1(x,x)=0$ for $x\in I$. Consider now $n=1,2$, we have by hypothesis

$$\Delta_1(x,x)\Delta_2(y,y)=0 \text{ for all } x,y\in I. \tag{2.1}$$

Linearize in $y$ to get

$$\Delta_1(x,x)\{\Delta_2(y,y)+2\Delta_2(y,z)+\Delta_2(z,z)\}=0 \text{ for all } x,y,z\in I. \tag{2.2}$$

Comparing (2.1) and (2.2) and using characteristic condition on $R$, we find

$$\Delta_1(x,x)\Delta_2(y,z)=0 \text{ for all } x,y,z\in I. \tag{2.3}$$

Replacing $z$ by $zr$ in (2.3), we obtain

$$\Delta_1(x,x)\Delta_2(y,z)r+\Delta_1(x,x)zD_2(y,r)=0 \text{ for all } x,y\in I, r\in R. \tag{2.4}$$

In view of (2.3), (2.4) takes the form

$$\Delta_1(x,x)zD_2(y,r)=0 \text{ for all } x,y,z\in I, r\in R. \tag{2.5}$$

Since $R$ is prime, we can find either $\Delta_1(x,x)=0$ or $D_2(y,r)=0$ for all $x,y\in I, r\in R$. Now consider the later case $D_2(y,r)=0$ for all $x,y\in I, r\in R$. A simple manipulation shows that $\Delta_2(y,xr)=\Delta_2(y,x)r$ for all $x,y\in I, r\in R$. Hence $\Delta_2$ acts as a left bimultiplier as desired.

If $n=1,2,3$, then by hypothesis we can write

$$\Delta_1(x,x)\Delta_2(y,y)\Delta_3(z,z)=0 \text{ for all } x,y,z\in I. \tag{2.6}$$

Linearizing (2.6) in $z$ and applying characteristic condition on $R$, to get

$$\Delta_1(x,x)\Delta_2(y,y)\Delta_3(z,u)=0 \text{ for all } x,y,z,u\in I. \tag{2.7}$$

Substituting $us$ in place of $u$ in (2.7) and using (2.7), we have

$$\Delta_1(x,x)\Delta_2(y,y)uD_3(z,s)=0. \text{ for all } x,y,z,u\in I, s\in R. \tag{2.8}$$

We conclude from equation (2.8) and primeness argument of $R$ that either $\Delta_1(x,x)\Delta_2(y,y)=0$ or $D_3(z,s)=0$ for all $x,y,z\in I, s\in R$. If we take $\Delta_1(x,x)\Delta_2(y,y)=0$, then we are done by previous case for $n = 2$. Now consider $D_3(z, s) = 0$ for all $z \in I, s \in R$, we can find $\Delta_3(z, ts) = \Delta_3(z, t)s$, that is, $\Delta_3$ acts as a left bimultiplier.

Next suppose it's true for $n$ and we shall prove it for $n + 1$. Let us assume the hypothesis

$$\Delta_1(x, x)\Delta_2(y, y)\cdots\Delta_n(z, z)\Delta_{n+1}(w, w) = 0 \text{ for all } x, y, z, w \in I. \tag{2.9}$$

Linearizing (2.9) in $w$ and applying characteristic condition on $R$, to get

$$\Delta_1(x, x)\Delta_2(y, y)\cdots\Delta_n(z, z)\Delta_{n+1}(w, v) = 0 \text{ for all } x, y, z, w, v \in I. \tag{2.10}$$

Substituting $vt$ for $v$ in (2.10) and using (2.10), we arrive at $\Delta_1(x, x)\Delta_2(y, y)\cdots\Delta_n(z, z)vD_{n+1}(w, t) = 0$ for all $x, y, z, u, w \in I, t \in R$. Primeness of $R$ yields that we have either $\Delta_1(x, x)\Delta_2(y, y)\cdots\Delta_n(z, z) = 0$ or $D_{n+1}(w, t) = 0$. If $\Delta_1(x, x)\Delta_2(y, y)\cdots\Delta_n(z, z) = 0$, then we are done by the later case. If $D_{n+1}(w, t) = 0$ for all $w \in I, t \in R$, then we can easily conclude that $\Delta_{n+1}(w, \rho t) = \Delta_{n+1}(w, \rho)t$. Hence $\Delta_{n+1}$ acts as a left bimultiplier on $R$ as desired. This complete the assertion of the theorem.

**Example 2.1**Let $R = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in S\right\}$ where $S$ is any commutative ring. Consider $\Delta_1, \Delta_2 : R \times R \longrightarrow R$ be two generalized biderivation with associated map $D : R \times R \longrightarrow R$ defined as $\Delta_1\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_1a_2 & 0 \\ 0 & 0 \end{pmatrix}$, $\Delta_2\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a_1a_2 & 0 \end{pmatrix}$ and $D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & b_1b_2 \end{pmatrix}$. We can easily show that if $\Delta_1 \cdot \Delta_2 = 0$, then $\Delta_2$ acts as a left bi-multiplier and $\Delta_1 \neq 0$.

In [7], author prove that: Let $R$ be a prime ring of characteristic not two and three. If $D_1, D_2$ are the symmetric biderivations of $R$ with trace $f_1, f_2$, respectively, such that $f_1(x)f_2(x) = 0$ for all $x \in R$, then either $D_1 = 0$ or $D_2 = 0$. Authors in [1] extends the previously cited results for an ideal of a prime ring. Now we generalizes the idea for $n$ iteration of symmetric biderivations.

**Theorem 2.2.** *Let $R$ be a prime ring of characteristic not two, $k \geq 1$ a fixed positive integer and $D_n : R \times R \longrightarrow R$, $n = 1, 2, ..., k$ be biderivations on $R$ such that*

$$D_1(x, x)D_2(y, y)....D_n(z, z) = 0$$

*for all $x, y, z \in R$. Then any one of $D_n = 0$.*

**Proof** We will prove it by induction. For $n = 1$, it is obvious, that is, we get $D_1(x, x) = 0$ for all $x \in R$. Consider $n = 2$, we have by hypothesis

$$D_1(x, x)D_2(y, y) = 0 \text{ for all } x, y \in R. \tag{2.11}$$

Linearization of (2.11) in $y$ yields that

$$D_1(x, x)D_2(y, y) + 2D_1(x, x)D_2(y, z) + D_1(x, x)D_2(z, z) = 0 \text{ for all } x, y, z \in R. \tag{2.12}$$

In view of char $R \neq 2$ and (2.11), (2.12) takes the form

$$D_1(x, x)D_2(y, z) = 0 \text{ for all } x, y, z \in R. \tag{2.13}$$

Substituting $zu$ for $z$ in (2.13) and applying (2.13), we find

$$D_1(x, x)zD_2(y, u) = 0 \text{ for all } x, y, z, u \in R. \tag{2.14}$$

In particular, we can get $D_1(x, x)zD_2(x, x) = 0$ for all $x, z \in R$. Since $R$ is prime, we get either $D_1 = 0$ or $D_2 = 0$ on $R$. Let us consider $n = 3$, we have

$$D_1(x, x)D_2(y, y)D_3(z, z) = 0 \text{ for all } x, y, z \in R. \tag{2.15}$$

Linearization of (2.15) in $z$ and use of (2.15) yields that

$$2D_1(x,x)D_2(y,y)D_3(z,w) = 0 \text{ for all } x,y,z,w \in R \tag{2.16}$$

In view of char $R \neq 2$, (2.16) takes the form

$$D_1(x,x)D_2(y,y)D_3(z,w) = 0 \text{ for all } x,y,z,w \in R. \tag{2.17}$$

Replacing $zu$ in place of $z$ in (2.17) and using (2.17), we obtain

$$D_1(x,x)D_2(y,y)zD_3(u,w) = 0 \text{ for all } x,y,z,u,w \in R. \tag{2.18}$$

Since $R$ is prime, we get from above equation either $D_1(x,x)D_2(y,y) = 0$ or $D_3(u,w) = 0$ for all $x,y,z,u,w \in R$. If we take $D_1(x,x).D_2(y,y) = 0$ for all $x,y \in R$, then we get the desired result from above discussion. Hence we obtain either $D_1 = 0$, or $D_2 = 0$ or $D_3 = 0$ on $R$.

Suppose that it is true for $n = k$. That is, if we consider

$$D_1(x,x)D_2(y,y)D_3(z,z)....D_k(w,w) = 0 \text{ for all } x,y,z,w \in R, \tag{2.19}$$

then we have either $D_1 = 0$ or $D_2 = 0$ or $D_3 = 0$ or $...D_k = 0$ on $R$.
Next we consider $n = k+1$, we have by hypothesis

$$D_1(x,x)D_2(y,y)D_3(z,z)...D_k(w,w)D_{k+1}(u,u) = 0 \text{ for all } x,y,z,w,u \in R. \tag{2.20}$$

Linearization of (2.20) in $u$ and applying (2.20) yields that

$$2D_1(x,x)D_2(y,y)....D_k(w,w)D_{k+1}(u,v) = 0 \text{ for all } x,y,w,u,v \in R. \tag{2.21}$$

In view of char $R \neq 2$, (2.21) reduces to the form

$$D_1(x,x)D_2(y,y)....D_k(w,w)D_{k+1}(u,v) = 0 \text{ for all } x,y,w,v,u \in R. \tag{2.22}$$

Substituting $zu$ for $u$ in (2.22), we find

$$D_1(x,x)D_2(y,y)....D_k(w,w)zD_{k+1}(u,v) = 0 \text{ for all } x,y,z,u,w,v \in R. \tag{2.23}$$

Primeness of $R$ implies that either $D_1(x,x)D_2(y,y)....D_k(w,w) = 0$ or $D_{k+1}(u,v) = 0$ for all $x,y,u,w,v \in R$. If $D_1(x,x)D_2(y,y)...D_k(w,u) = 0$ for all $x,y,u,w,v \in R$, then we get the desired result by the last assumption. Hence we get any one of $D_n = 0$ on $R$. This complete the proof.

**Example 2.2** Consider $R = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mid a,b,c,d \in S \right\}$ where $S$ is any ring. Define

$$D_1\left( \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ b_1b_2 & 0 \end{pmatrix}$$

and

$$D_2\left( \begin{pmatrix} 0 & a_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & b_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & b_1b_2 \\ 0 & 0 \end{pmatrix}.$$

We can find that if $D_1(x,y) \cdot D_2(x,y) \neq 0$ then $D_1 \neq 0$ and $D_2 \neq 0$. This indicates the negation of our Theorem 2.2, which shows that to achieve $D_1(x,y)D_2(x,y) = 0$, we must have either $D_1 = 0$ or $D_2 = 0$ for all $x,y \in R$. We think of the negation in this example because if we take any one of our biderivation is zero, then it will reduces to the trivial form that product of some biderivation with zero will be zero.

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*Faiza Shujat,*
*Department of Mathematics,*
*College of Science, Taibah University, Madinah*
*Saudi Arabia.*
*E-mail address:* faiza.shujat@gmail.com

*and*

*Ahlam Fallatah,*
*Department of Mathematics,*
*College of Science, Taibah University, Madinah*
*Saudi Arabia.*
*E-mail address:* afallatah@taibahu.edu.sa