Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–8.
ISSN-0037-8712 in press
doi:10.5269/bspm.45061

Fixed Point Theorems for Modified Generalized F-Contraction in G-Metric Spaces

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ABSTRACT: In this paper, we introduce new notions of generalized F-contractions of type (S) and type (M) in G-metric spaces. Some fixed point theorems are established using these new notions. A suitable example is also provided to support our results.

Key Words: Fixed point, G-metric space, Modified generalized F-contraction.

Contents

1. Introduction

A metric space is a nonempty set X equipped with a map d of two variables which helps us to measure the distance between two points. In higher mathematics, we need to find the distance not only between numbers and vectors, but also between sequences and functions. In order to find an appropriate concept of a metric space, numerous approaches exist in this sphere. A number of generalizations of a metric space have been discussed by many eminent mathematicians. In 2006, Sims and Mustafa introduced the perception of G-metric space and gave an important generalization of a metric space as follows:

Definition 1.1. *[3] Let X be a non empty set and $G : X^3 \to [0, \infty)$ be a map which satisfies the following properties:*
(i) $G(x, y, z) = 0$ if $x = y = z$;
(ii) $0 < G(x, x, y)$ whenever $x \neq y$;
(iii) $G(x, x, y) \leq G(x, y, z)$, $y \neq z$;
(iv) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$;
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, $\forall x, y, z, a \in X$.
Then, the function G is said to be G-metric on X and the pair (X, G) is known as G-metric space.

In 1922, Banach established a useful result in fixed point theory regarding a contraction mapping, known as the Banach contraction principle.

Definition 1.2. *[1] Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping. Let $d(fx, fy) < d(x, y)$ holds for all $x, y \in X$ with $x \neq y$. Then, f is called a contraction known as Banach contraction.*

In 2012, Wardowski [6] gave a new contraction known as F-contraction and proved fixed point theorem concerning F-contractions. In this manner, Wardowski conclude the Banach contraction principle in a different way from the eminent results from the literature. Piri and Kumam [5] also established Wardowski type fixed point theorems in complete metric spaces. Motivated by the perception of Dung and Hang [2], recently Piri and Kumam [5] generalized the concept of generalized F-contraction and established

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2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted October 23, 2018. Published November 28, 2019

some fixed point theorems for such kind of functions in complete metric spaces, by addition of four terms $d(f^2x, x), d(f^2x, fx), d(f^2x, y), d(f^2x, fy)$.

Wardowski [6] defined the F-contraction as follows:

Definition 1.3. *[6] Let (X, d) be a metric space and let $f : X \to X$ be a self-mapping. Then, f is called an F-contraction on (X, d), if there exist $F \in \Im$ such that*
$d(fx, fy) > 0 \Rightarrow \gamma + F(d(fx, fy)) \leq F(d(fx, fy))$ for all $x, y \in X$,
where $\Im$ is class of all mappings $F : (0, \infty) \to R$ such that
(F1) F is strictly increasing function, that is, for all $a, b \in (0, \infty)$,
if $a < b$, then $F(a) < F(b)$.
(F2) For every sequence $\{a_n\}$ of natural numbers,
$\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
(F3) There exists $q \in (0, 1)$ such that $\lim_{a\to 0^+}(a^q F(a)) = 0$.

Wardowski [6] gave some examples of $\Im$ as follows:
1.$F(\zeta) = \ln \zeta$.
2.$F(\zeta) = -\frac{1}{\zeta^{\frac{1}{2}}}$.
3.$F(\zeta) = \ln(\zeta) + \zeta$.
4.$F(\zeta) = \ln(\zeta^2 + \zeta)$.

Remark 1.4. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined as $F = \ln(\beta)$, then $F \in \Im$. Now, F-contraction changes to a Banach contraction. Consequently, the Banach contractions are special case of F-contractions. There are F-contractions which are not Banach contractions(see [6]).*

F-weak contraction was established by Wardowski and Dung in 2014, which is defined as follows:

Definition 1.5. *[7] Let (X, d) be a metric space and $\mathfrak{T} : X \to X$ be a function. $\mathfrak{T}$ is known as F-weak contraction on (X, d), if there exist $F \in \Im$ and $\gamma > 0$ such that for all $x, y \in X$,*
$d(\mathfrak{T}x, \mathfrak{T}y) > 0$ implies that
$\gamma + F(d(\mathfrak{T}x, \mathfrak{T}y)) \leq F(max\{d(x, y), d(\mathfrak{T}x, x), d(y, \mathfrak{T}y), \frac{d(x,\mathfrak{T}y)+d(y,\mathfrak{T}x)}{2}\})$.

Theorem 1.6. *[7] Let (X, d) be a complete metric space and let $\mathfrak{T} : X \to X$ be an F-weak contraction. If F or $\mathfrak{T}$ is continuous, then $\mathfrak{T}$ has a unique fixed point $x^* \in X$ and the sequence $\{\mathfrak{T}^n x\}$ converges to x^* for every $x \in X$, where n varies from 1 to ∞.*

Dung and Hang [2] investigated the concept of generalized F-contraction and proved useful fixed point results for such kind of functions.

Definition 1.7. *[2] Let (X, d) be a metric space and $f : X \to X$ be a self-mapping. Then, f is called a generalized F-contraction on (X, d), if there exist $F \in \Im$ and $\delta > 0$ such that for all $x, y \in X$,*
$d(fx, fy) > 0$ implies that

$$\delta + F(d(fx, fy)) \leq F(max\{d(x, y), d(x, fx), d(y, fy),$$
$$\frac{d(x, fy) + d(y, fx)}{2}, \frac{d(f^2x, x) + d(f^2x, fy)}{2}, d(f^2x, fx), d(f^2x, y), d(f^2x, fy)\}).$$

Subsequently, Piri and Kumam [4] replace the condition $(F3)$ with $(F3^{'})$ in the definition of F-contraction given by Wardowski [6].
$(F3^{'})$: F is continuous on $(0, \infty)$.

They gave the notation $\mathfrak{F}$ to denote the class of all maps $F : \mathbb{R}_+ \to \mathbb{R}$ which fulfil the conditions $(F1), (F2)$ and $(F3^{'})$. Piri and Kumam also proved some useful fixed point results for metric spaces. Now, the conditions $(F3)$ and $(F3^{'})$ are not associated with each other. For example, for $q \geq 1, F(\beta) = \frac{-1}{\beta^q}$, then F meet the conditions (F1) and (F2) but it does not fulfil (F3), while it fulfils the condition $(F3^{'})$. In view of this, it is significant to observe the sequel of Wardowski [6] with the functions $F \in \Im$ rather than F $\in \mathfrak{F}$.

The goal of our paper is to propose new notions of modified generalized F-contraction of type (S) and type (M) in G-metric spaces and prove fixed point theorems for such functions.

2. Main Results

Throughout the paper, we use the following notations.
S_G is the class of all functions $F : (0, \infty) \to \mathbb{R}$ such that
(S1) F is strictly increasing, that means $x < y$ implies that $Fx < Fy$, where x, y are positive reals.
(S2) $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$, for every sequence $\{a_n\}$ of positive numbers.
(S3) F is continuous on(0,∞).
M_G is the class of all maps $F : (0, \infty) \to \mathbb{R}$ such that
(M1) F is strictly increasing, that means $x < y$ implies that $Fx < Fy$, where x, y are positive reals.
(M2) $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$, for every sequence $\{a_n\}$ of positive numbers.
(M3) $\exists\, m \in (0, 1)$ such that $\lim_{a\to 0^+} a^m F(a) = 0$.

Definition 2.1. *Let (X, G) be a G-metric space and $f : X \to X$ be a mapping. Then, f is known as modified generalized F-contraction of type (S), if $\exists\, F \in S_G$ and $\lambda > 0$ such that $G(fx, fy, fz) > 0$, then*

$$\lambda + F(G(fx, fy, fz)) \leq F(S_f(x, y, z)), \tag{2.1}$$

where

$$S_f(x, y, z) = max\{G(x, fy, fy), G(y, fx, fx), G(y, fz, fz), G(z, fy, fy), G(z, fx, fx), G(x, fz, fz)\}.$$

Definition 2.2. *Let (X, G) be a G-metric space and $f : X \to X$ be a mapping. Then, f is known as modified generalized F-contraction of type (M), if $\exists\, F \in M_G$ and $\lambda > 0$ such that $G(fx, fy, fz) > 0$, then*

$$\lambda + F(G(fx, fy, fz)) \leq F(S_f(x, y, z)),$$

where

$$S_f(x, y, z) = max\{G(x, fy, fy), G(y, fx, fx), G(y, fz, fz), G(z, fy, fy), G(z, fx, fx), G(x, fz, fz)\}.$$

Example 2.3. *Let $X = [0, 2]$.*
We define G on X by $G(x, y, z) =| x - y | + | y - z | + | z - x |$.
Let $f : X \to X$ be defined as
$fx = 3$, when $x \in [0, 2)$ and $fx = \frac{1}{2}$, if $x = 5$.
Now, (X, G) is complete metric space. By choosing $Fx = \ln x$ and $\lambda = \ln \frac{1}{2}$, we get that f is modified generalized F-contraction of type (S) and type (M).

Theorem 2.4. *Let (X, G) be a complete G-metric space and $f : X \to X$ be a modified generalized F-contraction of type (S). Then, f has a unique fixed point $u \in X$ and the sequence $\{f^n(x_0)\}$, where $n \in N$, converges to u for each $u \in X$.*

Proof. Let $x_0 \in X$ and $\{x_n\}$ be the Picard sequence, that is, $x_n = fx_{n-1}$, where n ∈ N. If ∃ n ∈ N such that $x_{n+1} = x_n$, then, $fx_n = x_n$. So, x_n is fixed point of f. Let us suppose that $x_n \neq x_{n+1}\, \forall\, n \in N$. Then, $G(x_{n+1}, x_n, x_n) > 0\ \forall\, n \in N$. From equation (2.1), we have

$$G(fx_{n-1}, fx_n, fx_n) > 0,$$

which implies that,

$$\begin{aligned}\lambda + F(G(fx_{n-1}, fx_n, fx_n) &\leq F(max\{G(x_{n-1}, fx_n, fx_n), G(x_n, fx_{n-1}, fx_{n-1}),\\ &G(x_n, fx_n, fx_n), G(x_n, fx_n, fx_n), G(x_n, fx_{n-1}, fx_{n-1}), G(x_{n-1}, fx_n, fx_n\})\\ &= F(max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n),\\ &G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}\})\\ &= F(max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\}).\end{aligned}$$

Therefore,

$$\lambda + F(G(fx_{n-1}, fx_n, fx_n) \leq F(max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\}). \tag{2.2}$$

If there exists $n \in N$ such that

$$\max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\} = G(x_n, x_{n+1}, x_{n+1}).$$

From (2.2), we get

$$\lambda + F(G(x_n, x_{n+1}, x_{n+1}) \leq F(G(x_n, x_{n+1}, x_{n+1}),$$

we get a contradiction, because $\lambda > 0$. Therefore,

$$\max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n) \forall n \in N.$$

From (2.2), we obtain

$$\lambda + F(G(fx_{n-1}, fx_n, fx_n) \leq F(G(x_{n-1}, x_n, x_n)),$$

which implies that

$$F(G(fx_{n-1}, fx_n, fx_n) \leq F(G(x_{n-1}, x_n, x_n)) - \lambda.$$

Therefore,

$$F(G(x_n, x_{n+1}, x_{n+1}) \leq F(G(x_{n-1}, x_n, x_n)) - \lambda. \tag{2.3}$$

Since, $\lambda > 0$. Thus,

$$F(G(x_n, x_{n+1}, x_{n+1}) < F(G(x_{n-1}, x_n, x_n)).$$

Using the condition of (S1), F is strictly increasing. Therefore,

$$G(x_n, x_{n+1}, x_{n+1}) < G(x_{n-1}, x_n, x_n) \forall\, n \in N.$$

So, $\{G(x_{n+1}, x_n, x_n)\}$ is non negative decreasing sequence of real numbers, where $n \in N$. Thus, we conclude that $\lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = \mu \geq 0$. Now, we claim that $\mu = 0$. Let us suppose that $\mu > 0$. Also, $\{G(x_{n+1}, x_n, x_n)\}$ is non negative decreasing sequence of real numbers, where $n \in N$. Therefore,

$$\mu \leq G(x_{n+1}, x_n, x_n).$$

Again, by using the assumption (S1), F is is strictly increasing. Therefore, $F\mu \leq H(G(x_{n+1}, x_n, x_n))$. Using equation (2.3), we obtain

$$\begin{aligned} F(\mu) &\leq F(G(x_{n-1}, x_n, x_n)) - \lambda \\ &\leq G(x_{n-2}, x_{n-1}, x_{n-1})) - 2\lambda \\ &\leq G(x_{n-3}, x_{n-2}, x_{n-2})) - 3\lambda \\ &\vdots \\ &\leq G(x_0, x_1, x_1)) - n\lambda. \end{aligned}$$

Therefore,

$$F(\mu) \leq F(G(x_0, x_1, x_1)) - n\lambda \;\forall\; n \in N. \tag{2.4}$$

Also, $F(\mu)$ is a real number and $\lim_{n\to\infty}[F(G(x_0, x_1, x_1)) - n\lambda)] = -\infty$. Therefore, $\exists\, m \in N$such that

$$[F(G(x_0, x_1, x_1)) - n\lambda)] < F\mu \;\; \forall\, n > m. \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$F(\mu) \leq [F(G(x_0, x_1, x_1)) - n\lambda)] < F\mu \;\forall\, n > m.$$

This contradiction establishes that $\mu = 0$. Now, we have

$$\lim_{n\to\infty} G(x_n, fx_n, fx_n) = \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Further, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, let us suppose that there exist $\delta > 0$ and sequences $\{a(n)\}_{n=1}^{\infty}$, $\{b(n)\}_{n=1}^{\infty}$ such that for each $n \in N$,

$$a(n) > b(n) > n, \qquad G(x_{a(n)},\ x_{b(n)}, x_{b(n)}) \geq \delta, \qquad G(x_{a(n)-1},\ x_{b(n)}, x_{b(n)}) < \delta. \tag{2.6}$$

Therefore,

$$\begin{aligned}\delta &\leq G(x_{a(n)},\ x_{b(n)}, x_{b(n)})\\ &\leq G(x_{a(n)},\ x_{a(n)-1}, x_{a(n)-1}) + G(x_{a(n)-1},\ x_{b(n)}, x_{b(n)})\\ &\leq G(x_{a(n)},\ x_{a(n)-1}, x_{a(n)-1}) + \in\\ &= G(x_{a(n)-1},\ fx_{a(n)-1}, fx_{a(n)-1}).\end{aligned}$$

Since, $\mu = 0$, we obtain

$$\lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0. \tag{2.7}$$

So, the above inequality becomes

$$\lim_{n\to\infty} G(x_{a(n)}, x_{b(n)}, x_{b(n)}) = \delta. \tag{2.8}$$

From (2.7), $\exists\ p \in N$ such that

$$G(x_{a(n)}, fx_{a(n)}, fx_{a(n)}) < \frac{\delta}{8}\ and\ G(x_{b(n)}, fx_{b(n)}, fx_{b(n)}) < \frac{\delta}{8}\,\forall\ n\ \geq p. \tag{2.9}$$

Now, we claim that

$$G(fx_{a(n)}, fx_{b(n)}, fx_{b(n)}) = G(x_{a(n)+1}, x_{b(n)+1}, x_{b(n)+1}) > 0\,\forall\, n\ \geq p. \tag{2.10}$$

Again by contradiction, let us suppose that $\exists\ q \geq N$, such that

$$G(x_{a(q)+1}, fx_{b(q)+1}, fx_{b(q)+1}) = 0. \tag{2.11}$$

Combining (2.6),(2.9),(2.11), we get

$$\begin{aligned}\delta &\leq G(x_{a(q)}, x_{b(q)}, x_{b(q)})\\ &\leq G(x_{a(q)}, x_{a(q)+1}, x_{a(q)+1}) + G(x_{a(q)+1}, x_{b(q)}, x_{b(q)})\\ &\leq G(x_{a(q)}, x_{a(q)+1}, x_{a(q)+1}) + G(x_{a(q)+1}, x_{b(q)+1}, x_{b(q)+1}) + G(x_{b(q)+1}, x_{b(q)}, x_{b(q)})\\ &= G(x_{a(q)}, fx_{a(q)}, fx_{a(q)}) + G(x_{a(q)+1}, x_{b(q)+1}, x_{b(q)+1}) + G(x_{b(q)}, fx_{b(q)}, fx_{b(q)})\\ &< \frac{\delta}{8} + 0 + \frac{\delta}{8} = \frac{\delta}{4},\end{aligned}$$

which is contradiction and hence our supposition is wrong.

Thus, we get

$$G(fx_{a(n)}, fx_{b(n)}, fx_{b(n)}) = G(x_{a(n)+1}, x_{b(n)+1}, x_{b(n)+1}) > 0\,\forall\, n\ \geq p.$$

From (2.10) and assumption of the theorem, we obtain

$$\lambda + F(G(fx_{a(n)}, fx_{b(n)}, fx_{b(n)})) \leq F(G(x_{a(n)}, x_{b(n)}, fx_{b(n)}))\ \forall\ n \geq N. \tag{2.12}$$

From (S3), (2.8) and (2.12), we get

$$\lambda + H(\delta) \leq H(\delta),$$

which is contradiction. Hence,$\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence. By completeness property of (X, G), $\{x_n\}_{n=1}^{\infty}$ converges to a point u in X. Therefore,

$$\lim_{n\to\infty} G(x_n, u, u) = 0. \tag{2.13}$$

Finally, we show that $fu = u$. Two cases arise,

(i) $\forall n \in N \ \exists k_n \in N,\ k_n > k_{n-1}, k_0 = 1$ and $x_{k_n+1} = fu$.

(ii) $\exists m_3 \in N, \forall n \geq m_3, G(fx_n, fu, fu) > 0$.

In the first case

$$u = \lim_{n\to\infty} x_{k_{n+1}} = \lim_{n\to\infty} fu = fu.$$

In the second case, using the assumption of the Theorem 2.4, we get

$$\begin{aligned}
\lambda + F(G(x_{n+1}, fu, fu)) &= \lambda + F(G(fx_n, fu, fu)) \\
&\leq F(max\{G(x_n, fu, fu), G(u, fx_n, fx_n), G(u, fu, fu), \\
&\quad G(u, fu, fu), G(u, fx_n, fx_n), G(x_n, fu, fu)\},
\end{aligned}$$

for each $n \geq m_3$.

From (S3), (2.13) and taking limit when $n \to \infty$, the above inequality becomes $\lambda + H(G(u, fu, fu)) \leq H(G(u, fu, fu))$, which is contradiction. So our supposition is wrong. Therefore, $fu = u$. Next, we show that f has atmost one fixed point. On the contrary, we suppose that u and v are two fixed points of f, such that $fu \doteq u \neq v \doteq fv$. Now, $G(fu, fv, fv) = G(u, v, v) > 0$. From (2.1), we get

$$\begin{aligned}
F(G(u, v, v)) &< \lambda + F(G(u, v, v)) \\
&= \lambda + F(G(fu, fv, fv)) \\
&\leq F(max\{G(u, fv, fv), G(v, fu, fu), G(v, fv, fv), G(v, fv, fv), \\
&\quad G(v, fu, fu), G(u, fv, fv)\}) \\
&= F(max\{G(u, v, v), G(v, u, u), G(v, v, v), G(v, v, v), G(v, u, u), G(u, v, v)\}) \\
&= F(G(u, v, v).
\end{aligned} \tag{2.14}$$

It is a contradiction. Therefore, $G(u, v, v) = 0$, that means $u = v$. This establishes that the fixed point of f is unique. □

Theorem 2.5. *Let (X, G) be a complete G-metric space and $f : X \to X$ be a continuous modified generalized F-contraction of type (M). Then, f has a unique fixed point $u \in X$ and the sequence $f^n(x_0)$, where $n \in N$ converges to u, for each $u \in X$.*

Proof. By using identical procedure which is used in Theorem 2.4, we get

$$\begin{aligned}
H(G(x_n, x_{n+1}, x_{n+1})) &= H(G(fx_{n-1}, fx_n, fx_n)) \\
&\leq H(G(x_{n-1}, x_n, x_n)) - \lambda \\
&< H(G(x_{n-1}, x_n, x_n)).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} G(x_n, fx_n, fx_n) = \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

As in proof of Theorem 2.4, we can prove that $\{x_n\}$ is a Cauchy sequence. Also, (X, G) is complete metric space. Therefore, $\{x_n\}$ converges to some point u $\in X$. Since, f is continuous. Therefore,

$$G(u, fu, fu) = \lim_{n\to\infty} G(x_n, fx_n, fx_n) = \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.15}$$

By using identical steps used in proof of Theorem 2.4, we can prove that u is unique fixed point of f. □

Example 2.6. *Let $X = [0,5]$. We define G on X by $G(x,y,z) =\mid x-y \mid + \mid y-z \mid + \mid z-x \mid$. Let $f : X \to X$ be defined as $fx = 3$, when $x \in [0,5)$ and $fx = \frac{1}{3}$, if $x = 5$. Now, (X,G) is complete metric space. Since, f is not continuous. Therefore, f is not F-contraction. For $x \in [0,5]$ and $y = 5$,*

$$G(fx, f2, f2) = G(3, \frac{1}{3}, \frac{1}{3}) =\mid 3 - \frac{1}{3} \mid + \mid \frac{1}{3} - 3 \mid + \mid \frac{1}{3} - 3 \mid = \frac{16}{3} > 0.$$

Further,

$$\begin{aligned} &max\{G(x,fy,fy), G(y,fx,fx), G(y,fz,fz), \\ &\quad G(z,fy,fy), G(z,fx,fx), G(x,fz,fz)\} \geq G(y,fx,fx) + G(y,fz,fz) \\ &\qquad = G(5,3,3) + G(5,\frac{1}{3},\frac{1}{3}) \\ &\qquad = 4 + \frac{28}{3} = \frac{40}{3} > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} G(fx,f5,f5) \quad \leq \quad & \frac{1}{3} max\left\{G(x,fy,fy), G(y,fx,fx), G(y,fz,fz),\right. \\ & \left. G(z,fy,fy), G(z,fx,fx), G(x,fz,fz)\right\}. \end{aligned}$$

Now, by choosing $Fx = \ln x$ and $\lambda = \ln \frac{1}{5}$, we obtain that f is modified generalized F-contraction of type (S) and type (M). Let (X,d) and f be defined as in the above Example. Since, f is not an F-weak contraction, because f is not continuous. So, Theorem 1.6 is not implemented to f on (X,d). Also, f is a generalized F-contraction of type (S)and type (M) and (X,d) is complete, hence Theorem 2.4 and 2.5 are implemented to f on (X,d).

Theorem 2.7. *Let (X,G) be a complete G-metric space and $f : X \to X$ be a map such that $G(fx,fy,fy) \leq \ell\ G(x,y,y) + mG(x,fx,fx) + nG(y,fy,fy), \forall\, x,y \in X$, where ℓ,m,$n \geq 0$ with $\ell + m + n < 1$. Then,*

(i) f has a unique fixed point $u \in X$.

(ii) For each $x \in X$, if $f^{n+1}x = f^n x\ \forall\, n \in \mathbb{N} \cup \{0\}$, then $\lim_{n\to\infty} f^n x = u$.

Proof. It is given that $\forall\ x,y \in X$,

$$\begin{aligned} G(fx,fy,fy) \quad &\leq \quad \ell G(x,y,y) + mG(x,fx,fx) + nG(y,fy,fy) \\ &\leq \quad (\ell + m + n) max\{G(x,fy,fy), G(y,fx,fx), G(y,fz,fz), \\ &\qquad + G(z,fy,fy), G(z,fx,fx), G(x,fz,fz)\} \\ &\leq \quad k\, max\{G(x,fy,fy), G(y,fx,fx), G(y,fz,fz), \\ &\qquad + G(z,fy,fy), G(z,fx,fx), G(x,fz,fz)\}, \end{aligned}$$

where k $= \ell + m + n \in [0,1)$.

If $G(fx,fy,fy) > 0$, we get

$$\begin{aligned} \ln \frac{1}{k} + \ln(G(fx,fy,fy)) \quad \leq \quad & \ln(max\{G(x,fy,fy), G(y,fx,fx), G(y,fz,fz), \\ & + G(z,fy,fy), G(z,fx,fx), G(x,fz,fz)\}). \end{aligned}$$

By taking $F = \ln(\ell)$ and $\lambda = \ln \frac{1}{k}$ in Theorem 2.4 or 2.5, we get the result. Hence the proof. □

3. Conclusion

In this manuscipt, we improved the results of [7] by conferring examples of F-contraction of kind (S) and kind (M) by discarding the continuity condition of the self mapping in the framework of G-metric spaces. The notions of generalized F-contraction of kind (S) and kind (M) extend other well known metrical fixed point theorems within the literature.

4. Acknowledgment

The authors are extremely grateful to the anonymous editor and reviewers for their insightful reading the manuscript and their valuable suggestions, which substantially improved the standard of the paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this manuscript.

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