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On a Class of Ikeda-Nakayama Rings

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ABSTRACT: In this work we introduce the notion of P-Ikeda-Nakayama rings (P-IN-rings) which is in some way a generalization of the notion of Ikeda-Nakayama rings (IN-rings). Then, we study the transfer of this property to trivial ring extension, localization, homomorphic image and to the direct product.

Key Words: P-IN-ring, trivial ring extensions, Localisation, Homomorphic image, Direct product.

Contents

1. Introduction and Preliminaries

In this part, R denotes a nonzero associative ring with identity. V. Camillo, W. K. Nicholson and M. F. Yousif (2000) introduced the Ikeda-Nakayama ring (right IN-ring). A ring is said to be IN-ring if $l(I) + l(J) = l(I \cap J)$ for all ideals I, J of R where $l(X)$ denotes the left annihilator of X(see [7]). Examples of IN-ring are the ring $\mathbb{Z}$ of integers, right self-injective rings and right uniserial rings. In [5], the authors have introduced and investigated the concept of a right SA-ring. A ring R is called a right SA-ring, if for any ideals I and J of R there is an ideal K of R such that $r(I)+r(J) = r(K)$, where $r(I)$ (resp. $l(I)$) denotes the right annihilator (resp. the left annihilator) of I. QF-rings, left IN-rings and quasi-Baer rings are examples of right SA-rings (see for instance [5], [6]).

All rings considered below are commutative with unit, and all modules are unital.

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by: $(a, e)(a', e') = (aa', ae' + a'e)$. R is called the trivial ring extension of A by E. Considerable work has been concerned with trivial ring extensions. These rings have proven to be useful in solving many open problems and conjectures for various contexts in commutative and non-commutative ring theory (see for instance ([9], [10] and [13]). This construction was first introduced in 1962 by Nagata [11] in order to facilitate interaction between rings and their modules and also to provide various families of examples of commutative rings containing zero-divisors. The literature abounds of papers on trivial extensions dealing with the transfer of ring-theoretic notions in various settings of these constructions (see for instance [1], [4] and [8]). For more details on commutative trivial extensions (or idealizations) we refer the reader to Glaz"s and Huckaba"s respective books [[9], [10]], and also to Anderson and Winders relatively recent and comprehensive survey paper [2].

In this paper, we introduce a particular class of IN-rings that we call P-IN-rings. We call a ring R a P-IN-ring if the annihilator of the intersection of any two principal ideals is the sum of the annihilators of these two ideals. If R is a IN-ring, then R is naturally a P-IN-ring. Then we investigate the possible transfer of a P-IN-ring to various trivial extension constructions. Also, we examine the transfer of a P-IN-ring property to localization, homomorphic image and the direct product of rings.

2. Transfert of the P-IN-ring to trivial ring extension

In this section, we study the possible transfer of the P-IN-ring to various trivial extension contexts. First, we explore a different context, namely, the trivial ring extension of a local ring (A, M) by an

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A-module E such that $ME = 0$.

Proposition 2.1. *Let (A, M) be a local with maximal ideal M, E be an A-module such that $ME = 0$, and $R := A \propto E$ be the trivial ring extension of A by E. If R is a P-IN-ring then so is A.*

The proof of this Proposition requires the following Lemma.

Lemma 2.2. *Let (A, M) be a local ring with maximal ideal M, E be an A-module such that $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of A by E. Then : $Ann_R(I \propto 0)$=$Ann_A(I) \propto E$ for all proper ideals I of A.*

Proof. If $(a, e) \in Ann_R(I \propto 0)$, then $\forall (i, 0) \in (I \propto 0)$: $(a, e)(i, 0) = (0, 0)$, so $ai = 0$ $\forall i \in I$ (since $ie \in IE \subseteq ME$ and $ME = 0$).
Conversely, let $(a, 0) \in Ann_A(I) \propto 0$. Our aim is to show that $(a, 0) \in Ann_R(I \propto 0)$. Indeed, we have $\forall i \in I$: $(a, 0)(i, 0) = (ai, 0) = (0, 0)$ (since $a \in Ann_A(I)$ and $i \in I$, so $Ann_A(I) \propto 0 \subseteq Ann_R(I \propto 0)$ and this completes the proof. □

Proof of Proposition 2.1. Let $I = Aa$, $J = Ab$ be two principal ideals of A, where $a \in I$ and $b \in J$. We claim that $Ann_A(I) + Ann_A(J) = Ann_A(I \cap J)$. Two cases are then possible:
• **case 1.** If $I = A$ or $J = A$ then $Ann_A(I) + Ann_A(J) = Ann_A(I \cap J)$.
• **case 2.** If I and J two principal ideals of A, hence $I \propto 0$=$Aa \propto 0$=$R(a, 0)$ and $J \propto 0$=$Ab \propto 0$=$R(b, 0)$ (since $ME = 0$) are two principal ideals of R, hence :
a) $Ann_R(I \propto 0) + Ann_R(J \propto 0)$=$Ann_R((I \propto 0) \cap (J \propto 0))$ (since R is P-IN-ring)= $Ann_R((I \cap J) \propto 0)$=$Ann_A(I \cap J) \propto 0$ (by lemma 2.2).
b) $Ann_R(I \propto 0) + Ann_R(J \propto 0)$=$Ann_A(I) \propto 0$+$Ann_A(J) \propto 0$ (by lemma 2.2) =$(Ann_A(I)$+$Ann_A(J)) \propto 0$. Therefore, by (a) and (b) we have $(Ann_A(I) + Ann_A(J)) \propto 0$= $Ann_A(I \cap J) \propto 0$. Thus, $Ann_A(I) + Ann_A(J) = Ann_A(I \cap J)$.
✓ ***Question 1:*** If A is a P-IN-ring then so is $R := A \propto E$?.
So that we can respond to this question, we are in need of the results of the following theorem.

Theorem 2.3. *Let (A, M) be a local domain with maximal ideal M, E be an A-module such that $ME = 0$, and $R := A \propto E$ be the trivial ring extension of A by E. Let $I = R(a, e)$, $J = R(b, f)$ be two principal ideals of R, where (a, e), $(b, f) \in R$. Two cases are then possible:*
• ***case 1.*** *If $I = A$ or $J = A$ then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*
• ***case 2.*** *Let $I = R(a, e)$, $J = R(b, f)$ be two principal proper ideals of R, where a, $b \in M$. Three cases are then possible:*
• ***case 1.*** *$a = b = 0$. Two cases are then possible:*
1) *If $\{e, f\}$ are linearly independent then $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$.*
2) *If $\{e, f\}$ are linearly dependent, so $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*
• ***case 2.*** *a and b are comparable. Assume for example that $a = cb$, where $c \in A$. Two cases are then possible:*
1) *If $c \in M$, two cases are then possible:*
i) *If $e = 0$, then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*
ii) *If $e \neq 0$ then two cases are possible:*
α) *If $a \neq 0$, so $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*
β) *If $a = 0$ then $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$.*
2) *If $c \notin M$, then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*
• ***case 3.*** *a and b are not comparable.*
Then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.

Proof. • **case 1.** clear.
• **case 2.** Let $I = R(a, e)$, $J = R(b, f)$ be two principal proper ideals of R, where a, $b \in M$. Three cases are then possible:
• **case 1.** $a = b = 0$. Two cases are then possible:

1) If $\{e,f\}$ are linearly independent, then :

i) If $e\neq 0$ and $f\neq 0$ assume that $(l,m)(0,e)=(u,v)(0,f)\in R(0,e)\cap R(0,f)$, where (l,m), $(u,v)\in R$. Since $(l,m)(0,e)=(0,\overline{l}e)$ and $(u,v)(0,f)=(0,\overline{u}f)$, then $\overline{l}e=\overline{u}f$, hence $\overline{l}=\overline{u}=0$ since $\{e,f\}$ are linearly independent. Therefore, $R(0,e)\cap R(0,f)=0$ hence $Ann_R(I\cap J)=R$. On the other hand, if $e\neq 0$ and $f\neq 0$. Our aim is to show that $Ann_R(I)=M\propto E$ and $Ann_R(J)=M\propto E$. Indeed,

• Clearly $Ann_R(I)\subseteq M\propto E$ (since R is a local ring with maximal ideal $M\propto E$ and $e\neq 0$). Conversely, let $(m,g)\in M\propto E$, we claim that $(m,g)\in Ann_R(0,e)$ i.e $(m,g)(0,e)=(0,0)$. Indeed, $(m,g)(0,e)=(0,me)=)(0,0)$ (since $me\in ME$ and $ME=0$).

• Clearly $Ann_R(J)=M\propto E$.

Consequently, $Ann_R(I)+Ann_R(J)\neq Ann_R(I\cap J)$.

2) If $\{e,f\}$ are linearly dependent, assume that $e=\overline{w}f$, where $w\in A$. Then $(0,e)=(w,0)(0,f)\in R(0,f)$ and so $R(0,e)\cap R(0,f)=R(0,e)$. We have two cases possible :

i) If $e=0$ or $f=0$ then $Ann_R(I)+Ann_R(J)=Ann_R(I\cap J)$.

ii) If $e\neq 0$ et $f\neq 0$ then $Ann_R(I)=M\propto E$ and $Ann_R(J)=M\propto E$. On the other hand $Ann_R(I\cap J)=Ann_R(R(0,e))=M\propto E$. Therefore, $Ann_R(I)+Ann_R(J)=Ann_R(I\cap J)$.

• **case 2.** a and b are comparable. Assume for example that $a=cb$, where $c\in A$. Two cases are then possible:

1) If $c\in M$, let $(l,m)(a,e)$=$(u,v)(b,f)\in R(a,e)\cap R(b,f)$, where (l,m), $(u,v)\in R$. Then, $cbl=al=ub$ and $\overline{l}e=\overline{u}f$ since $a,b\in M$. But, $cbl=ub$ implies $u=cl\in M$ (since A is a domain); so $\overline{l}e=\overline{u}f=0$. Two cases are then possible: $e=0$ or $e\neq 0$.

i) Assume that $e=0$. Hence $\overline{l}e=0$ for each $l\in A$ and so $R(a,0)\cap R(b,f)\subseteq R(a,0)$. Conversely, let $(u,v)(a,0)\in R(a,0)$. Clearly, $(u,v)(a,0)=(u,v)(cb,0)=(uc,0)(b,f)$ since $c\in M$, hence $(u,v)(a,0)\in R(a,0)\cap R(b,f)$. Therefore, $R(a,0)\cap R(b,f)=R(a,0)=I$, so two cases are possible :

$*$) If $a=0$ then $Ann_R(I)+Ann_R(J)=Ann_R(I\cap J)$.

$**$) If $a\neq 0$. we claim that $Ann_R(I)=Ann_R(a,0)=0\propto E$ and $Ann_R(J)=Ann_R(b,f)=0\propto E$. Indeed,

• Let $(d,g)\in Ann_R(a,0)$ implies $(d,g)(a,0)=(0,0)$ implies $(da,0)=(0,0)$, so $da=0$ then $d\in Ann_A(a)$ implies $d=0$ (since A is a domain and $a\neq 0$) then $(d,g)\in 0\propto E$ hence $Ann_R(I)\subseteq 0\propto E$. Conversely, clearly $0\propto E\subseteq Ann_R(I)$. Thus, $Ann_R(I)=0\propto E$.

• Let $(d,g)\in Ann_R(b,f)$ then $(d,g)(b,f)=(0,0)$ implies $db=0$ and $df=0$ hence $d\in Ann_A(b)\cap Ann_A(f)$ so $d=0$ (since A is a domain and $b\neq 0$), therefore $(d,g)\in 0\propto E$ thus $Ann_R(b,f)\subseteq 0\propto E$. Conversely, clearly $0\propto E\subseteq Ann_R(I)$.

Consequently, $Ann_R(I)+Ann_R(J)=Ann_R(I\cap J)$.

ii) Assume that $e\neq 0$. Hence, $l\in M$ since $\overline{l}e=0$ and so $R(a,e)\cap R(b,f)\subseteq aM\propto 0$. Conversely, let $(au,0)\in aM\propto 0$, where $u\in M$. Then $(au,0)=(u,0)(a,e)=(uc,0)(b,f)\in R(a,e)\cap R(b,f)$. Therefore, $R(a,e)\cap R(b,f)=aM\propto 0$.

α) If $a\neq 0$ We have $Ann_R(I)=Ann_R(a,e)=0\propto E$, $Ann_R(J)=0\propto E$ and $Ann_R(aM\propto 0)=(Ann_A(aM))\propto E=0\propto E$ (since A is a domain and by lemma 2.2) so $Ann_R(I)+Ann_R(J)=Ann_R(I\cap J)$.

β) If $a=0$ then $Ann_R(I\cap J)=R$. Moreover, $Ann_R(I)=M\propto E$ and $Ann_R(J)=0\propto E$. Therefore, $Ann_R(I)+Ann_R(J)\neq Ann_R(I\cap J)$

2) If $c\notin M$, then c is invertible. Clearly, $R(a,e)=R(bc,\overline{c}\overline{c^{-1}}e)=R(c,0)(b,\overline{c^{-1}}e)=R(b,\overline{c^{-1}}e)$ since $(c,0)$ is invertible in R (since c is invertible in A). So, we may assume that $a=b$. Then we have two cases possible:

• If $e=f$ then $Ann_R(I)+Ann_R(J)=Ann_R(I\cap J)$.

• If $e\neq f$. Our aim is to show that $R(a,e)\cap R(b,f)=aM\propto 0$. Indeed, let $e\neq f\in E$. Assume $(l,m)(a,e)$=$(u,v)(a,f)\in R(a,e)\cap R(b,f)$, where (l,m), $(u,v)\in R$. Hence, $la=ua$ and $\overline{l}e=\overline{u}f$ since $a\in M$ and $ME=0$. Therefore, $l=u$ since a is a regular element, so $\overline{l}(e-f)=0$. Hence $l\in M$ since $(e-f)\neq 0$ and E is an (A/M)-vector space. Therefore $R(a,e)\cap R(b,f)\subseteq aM\propto 0$. Conversely, let $(au,0)\in aM\propto 0$, where $u\in M$. Clearly, $(au,0)=(u,0)(a,e)=(u,0)(a,f)$ since $u\in M$ and so $(au,0)\in R(a,e)\cap R(a,f)$. Consequently, $Ann_R(I)=Ann_R(a,e)=0\propto E$, $Ann_R(J)=0\propto E$ and $Ann_R(aM\propto 0)=(Ann_A(aM))\propto E=0\propto E$ (since A is a domain and by lemma 2.2) so $Ann_R(I)+Ann_R(J)=Ann_R(I\cap J)$.

• **case 3.** a and b are not comparable.
Clearly, $Ann_R(I) = 0 \propto E$ and $Ann_R(J) = 0 \propto E$ (since a and b are not comparable so $a \neq 0$ and $b \neq 0$). On the other hand, we have $0 \propto E = Ann_R(I) \subseteq Ann_R(I \cap J)$. Conversely, we have $(a,e) \in I$ and $(b,f) \in J$ then $(a,e)(b,f) \in IJ \subseteq I \cap J$ so $(ab,0) \in I \cap J$ and then $Ann_R(I \cap J) \subseteq Ann_R(ab,0) = 0 \propto E(ab \neq 0$ since A is a domain). Consequently, $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$. □

The next example illustrates the failure of question (1), in general.

Example 2.4. *Let (A, M) be a local domain ring with maximal ideal M, E be an A-module such that $ME = 0$. Let $R := A \propto E$ be the trivial ring extension of A by E. Then*

1. *A is a P-IN-ring.*
2. *R is not a P-IN-ring.*

Proof. 1. A is a P-IN-ring (since A is a domain).

2. Let $I = R(0,e)$, $J = R(b,f)$ where $e \neq 0$ and $b \neq 0$, then $Ann_R(I) = M \propto E$ and $Ann_R(J) = 0 \propto E$. On the other hand, we have by theorem 2.3 [case 2.$ii).\beta$)] that $I \cap J = R(0,e) \cap R(b,f) = 0$ then $Ann_R(I \cap J) = R$, so $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$. Thus, R is not a P-IN-ring. □

Next, we examine the context of trivial ring extensions of a domain by its quotient field.

Theorem 2.5. *Let A be a domain, $Q = qf(A)$ be the quotient field of A, and $R := A \propto Q$ be the trivial ring extension of A by Q and let $I = R(a,e)$, $J = R(b,f)$ be two principal ideals of R, where (a,e), $(b,f) \in R$. Three cases are then possible:*
• ***case 1.*** *$a = b = 0$. Two cases are then possible:*
1) *If $e = 0$ or $f = 0$ then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*
2) *If $e \neq 0$ and $f \neq 0$. Two cases are then possible:*
i) if $Ae \cap Af = 0$ then $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$.
ii) if $Ae \cap Af \neq 0$ then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.
• ***case 2.*** *If $a \neq 0$ and $b = 0$, or $a = 0$ and $b \neq 0$ then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*
• ***case 3.*** *$a \neq 0$ and $b \neq 0$ then $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.*

To facilitate the proof of this theorem we shall need a sequence of lemmas.

Lemma 2.6. *With the notation of Theorem 2.5, let $I = R(a,e)$ be a principal ideal of R, where $a \in A\backslash\{0\}$ and $e \in Q$. Then, $I = Aa \propto Q = R(a,0)$.*

Proof. Clearly, $I = R(a,e) = \{(b,f)(a,e)/b \in A, f \in Q\} = \{(ba, fa+be)/b \in A, f \in Q\}$. But, $\{af/f \in Q\} = Q$, hence $I = Aa \propto Q = R(a,0)$. □

Lemma 2.7. *Let A be a domain and $R := A \propto E$ be the trivial ring extension of A by E. Then*

1. *$Ann_R(I \propto E)$=$0 \propto Ann_E(I)$ for any nonzero ideal I of A.*
2. *$Ann_R(0 \propto E')$=$Ann_A(E') \propto E$ for any submodule E' of E.*

Proof. 1. If $(a,e) \in Ann_R(I \propto E)$ then $(a,e)(i,e') = (0,0)$ for each $(i,e') \in (I \propto E)$ and so $ai = 0$ and $ae' + ei = 0$. Hence, $a = 0$ (since A is a domain) and $e \in Ann_E(I)$ which means that $(a,e) \in 0 \propto Ann_E(I)$.
Conversely, let $(0,e) \in 0 \propto Ann_E(I)$. Our aim is to show that $(0,e) \in Ann_R(I \propto E)$. Indeed, $(0,e)(i,e')$=$(0,ei)$=$(0,0)$ (since $e \in Ann_E(I)$ and $i \in I$) for each $(i,e') \in (I \propto E)$. Therefore, $(0,e) \in Ann_R(I \propto E)$.

2. Let $(a, e) \in Ann_R(0 \propto E')$ then $(a, e)(0, e') = (0, 0)$ for each $(0, e') \in (0 \propto E')$ and so, $ae' = 0$ for all $e' \in E'$ so, $a \in Ann_A(E')$. Thus, $Ann_R(0 \propto E') \subseteq Ann_A(E') \propto E$.
   Conversely, let $(a, e) \in Ann_A(E') \propto E$. It remains to show that $(a, e) \in Ann_R(0 \propto E')$. Indeed, $(a, e)(0, e')$=$(0, ae')$=$(0, 0)$ (since $e' \in Ann_A(E')$ and $a \in A$). Therefore, $Ann_A(E') \propto E \subseteq Ann_R(0 \propto E')$.

□

Lemma 2.8. *Let A be a domain, $Q = qf(A)$ be the quotient field of A. Then:*

1. *$Ann_Q(I \cap J) = 0$ for each nonzero ideals I, J of A.*
2. *$Ann_A(Ae) = Ann_A(e) = 0$ for each $e \in Q\backslash\{0\}$.*

Proof. clearly, since A is a domain and Q is a torsion-free. □

Proof of Theorem 2.5. Let $I = R(a, e)$, $J = R(b, f)$ be two principal ideals of R, where (a, e), $(b, f) \in R$. Three cases are then possible:

• **case 1.** $a = b = 0$.Hence, $I = R(0, e) = 0 \propto Ae$ and $J = R(0, f) = 0 \propto Af$. Two cases are then possible:

1) Clear.

2) If $e \neq 0$ and $f \neq 0$. Hence, $Ann_R(I) = Ann_A(Ae) \propto Q = 0 \propto Q$ by lemma 2.7 and lemma 2.8 and $Ann_R(J) = 0 \propto Q$, so $Ann_R(I) + Ann_R(J) = 0 \propto Q$.

On the other hand, $I \cap J = (0 \propto Ae) \cap (0 \propto Af) = 0 \propto (Ae \cap Af)$. Two cases are then possible:

i) if $Ae \cap Af = 0$ then $Ann_R(I \cap J) = R$. Consequently, $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$.

ii) if $Ae \cap Af \neq 0$ then $Ann_R(I \cap J) = 0 \propto Q$ by lemma 2.8. Therefore, $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.

• **case 2.** If $a \neq 0$ and $b = 0$, or $a = 0$ and $b \neq 0$.

By symmetry, we may assume that $a \neq 0$ and $b = 0$. Then, $I = R(a, e) = Aa \propto Q$ by Lemma 2.6 and $J = R(0, f) = 0 \propto Af$, so $J \subseteq I$ and $I \cap J = J$. Consequently, $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.

• **case 3.** $a \neq 0$ and $b \neq 0$. Hence, $I = R(a, e) = Aa \propto Q$ and $J = R(b, f) = Ab \propto Q$ then, $Ann_R(I) = 0 \propto Ann_Q(Aa) = 0$ and $Ann_R(J) = 0 \propto Ann_Q(Ab) = 0$. On the other hand, $I \cap J = (Aa \cap Ab) \propto Q$, hence $Ann_R(I \cap J) = 0 \propto Ann_Q(Aa \cap Ab) = 0$ by lemma 2.8 (since $Aa \cap Ab \neq 0$. Deny, $ab \in Aa \cap Ab$, so $ab = 0$ hence $a = 0$ or $b = 0$ since A is a domain, contradiction.). Thus, $Ann_R(I) + Ann_R(J) = Ann_R(I \cap J)$.

In the following example, we prove that under the same conditions as in Theorem 2.5, we can't transfer the P-IN-ring property from A to R.

Example 2.9. *Let A be a domain, $Q = qf(A)$ be the quotient field of A, and $R := A \propto Q$ be the trivial ring extension of A by Q. Then:*

1. *A is a P-IN-ring.*
2. *R is not a P-IN-ring.*

Proof.

1. A is a P-IN-ring (since A is a domain).
2. Let $I = R(0, e)$, $J = R(0, f)$ where e, $f \in Q\backslash\{0\}$ and $Ae \cap Af = 0$ Then, by the proof of Theorem 2.5 (case1.2)i)), $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$. Therefore, R is not a P-IN-ring and this completes the proof of Example 2.9.

□

3. Localization and quotient of a P-IN-ring

In this section, we present the following result which states a condition under which the P-IN-ring is stable under localization.

Proposition 3.1. *Let R be a P-IN-ring and S a multiplicative subset of R which is contained in $R\backslash Z(R)$. Then $S^{-1}R$ is a P-IN-ring.*

The proof will use the following Lemma.

Lemma 3.2. *Let R be a commutative ring and S a multiplicative subset of R which is contained in $R\backslash Z(R)$. Then:*

$$Ann_{S^{-1}R}(S^{-1}I) = S^{-1}(Ann_R(I)).$$

Proof. $\supseteq$) Let $\frac{a}{s} \in S^{-1}(Ann_R(I))$, we can assume that $a \in Ann_R(I)$, then $\frac{a}{s} \in Ann_{S^{-1}R}(S^{-1}I)$. Indeed, $\forall \frac{b}{t} \in S^{-1}I$: $\frac{b}{t}.\frac{a}{s} = \frac{ba}{ts} = \frac{0}{ts} = \frac{0}{1}$ [since : $b \in I$ and $a \in Ann_R(I)$]. Thus, $S^{-1}(Ann_R(I)) \subseteq Ann_{S^{-1}R}(S^{-1}I)$.
$\supseteq$) Let $\frac{a}{s} \in Ann_{S^{-1}R}(S^{-1}I)$ and we claim that $a \in Ann_R(I)$ i.e. we prove that : $\forall i \in I : ia = 0$. Indeed, $\forall i \in I$, $\frac{i}{1}.\frac{a}{s} = \frac{ia}{s} = \frac{0}{1}$, then there exists $t \in S$ such that $tia = 0$ and hence $\forall i \in I : ia = 0$ [since $t \in S \subseteq R\backslash Z(R)$]. So, $\frac{a}{s} \in S^{-1}(Ann_R(I))$. □

Proof of Proposition 3.1. Let $I = S^{-1}R\frac{a}{s}$,$J = S^{-1}R\frac{b}{t}$ be two principal ideals of $S^{-1}R$, where $a, b \in R$ and $s, t \in S$ we have R is a P-IN-ring so, $Ann_R(aR) + Ann_R(bR) = Ann_R(aR \cap bR)$. This means that : $S^{-1}(Ann_R(aR) + Ann_R(bR)) = S^{-1}(Ann_R(aR \cap bR))$ then by lemma 2.2, we have $Ann_{S^{-1}R}(S^{-1}aR) + Ann_{S^{-1}R}(S^{-1}bR) = Ann_{S^{-1}R}(S^{-1}(aR \cap bR)) = Ann_{S^{-1}R}((S^{-1}aR) \cap (S^{-1}bR)) = Ann_{S^{-1}R}((S^{-1}R\frac{a}{s}) \cap (S^{-1}R\frac{b}{t})) = Ann_{S^{-1}R}(I \cap J)$. Thus, $Ann_{S^{-1}R}(I) + Ann_{S^{-1}R}(J) = Ann_{S^{-1}R}(I \cap J)$ and so $S^{-1}R$ is a P-IN-ring.
Recall that a ring R is called a weakly finite conductor ring if $Ra \cap Rb$ is a finitely generated ideal of R (see [12]).

Proposition 3.3. *Let R be a weakly finite conductor ring, P-IN-ring and S a multiplicative subset of R then $S^{-1}R$ is a P-IN-ring*

Proof. Trivial [since, if I is a finitely generated ideal of R then $Ann_{S^{-1}R}(S^{-1}I) = S^{-1}(Ann_R(I))$ by [3]]. □

✓ ***Question 2:*** If $S^{-1}R$ is a P-IN-ring then so is R ?.
The example below answers this question.

Example 3.4. *Let $A = K[[X_1, X_2, X_3]] = K + M$ be a power series ring over a field K and $M := (X_1, X_2, X_3)$. Let E be an A-module such that $ME = 0$ and $dim_{A/M}(E) \geq 2$. Let $R := A \propto E$ be the trivial ring extension of A by E and let S be the multiplicative subset of R given by $S := \{(X_1, 0)^n/n \in N\}$ and S_0 the multiplicative subset of A given by $S_0 := \{X_1^n/n \in N\}$. Then:*

1. *R is not a P-IN-ring.*
2. *$S_0^{-1}A$ is a P-IN-ring.*
3. *$S^{-1}R$ and $S_0^{-1}A$ are isomorphic rings. In particular, $S^{-1}R$ is a P-IN-ring.*

Proof.
1. Let $I = R(0, e)$, $J = R(0, f)$ where $\{e, f\}$ are linearly independent, then $Ann_R(I) = M \propto E$ and $Ann_R(J) = M \propto E$. On the other hand, we have by theorem 2.3 $I \cap J = R(0, e) \cap R(0, f) = 0$ then $Ann_R(I \cap J) = R$ so, $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$. Thus, R is not a P-IN-ring.
2. Since K is a domain then A is a domain hence $S_0^{-1}A$ is a domain. Thus, $S_0^{-1}A$ is a P-IN-ring.
3. Since $X_1E \subseteq ME = 0$ and $X_1 \in S_0$, then $S_0^{-1}E = 0$. Thus, $S^{-1}(0 \propto E) = 0$ and so $S^{-1}R = \{\frac{(a,0)}{(s,0)}/a \in A \text{ and } s \in S_0\}$. Now, we easily check that:

$$f\colon S_0^{-1}A \to S^{-1}R$$
$$\tfrac{a}{s} \mapsto \tfrac{(a,0)}{(s,0)}$$

is a ring isomorphism. In particular, $S^{-1}R$ is a P-IN-ring by 2).

□

✓ ***Question 3:*** If R/I is a P-IN-ring then so is R ?.
The ring in our next example illustrates the failure of this question, in general.

Example 3.5. *Let (A, M) be a local domain with maximal ideal M, E be an A-module such that $ME = 0$ and $dim_{A/M}(E) \geq 2$. Let$R := A \propto E$ be the trivial extension of A by E. Then:*

1. *$R/(0 \propto E)$ is a P-IN-ring .*
2. *R is not a P-IN-ring.*

Proof. 1. since $0 \propto E$ is a prime ideal of R hence $R/0 \propto E$ is a domain so it's a P-IN-ring.

2. Let $I = R(0, e)$, $J = R(0, f)$ where $\{e, f\}$ are linearly independent, then $Ann_R(I) = M \propto E$ and $Ann_R(J) = M \propto E$. On the other hand, we have by theorem 2.3 $I \cap J = R(0, e) \cap R(0, f) = 0$ then $Ann_R(I \cap J) = R$ so, $Ann_R(I) + Ann_R(J) \neq Ann_R(I \cap J)$. Thus, R is not a P-IN-ring.

□

Finally, we study a particular case of homomorphic images, that is, the direct product of *P-IN*-rings.

Theorem 3.6. *Let $(R_i)_{1\leq i\leq n}$ be a family of rings and $R := \prod_{i=1}^{n} R_i$ a direct product of rings. Then R is a* P-IN*-ring if and only if so is R_i for each $i = 1, \cdots, n$.*

Before proving Theorem 3.6, we establish the following lemma.

Lemma 3.7. *Let R_1 and R_2 be two rings and $I := I_1 \times I_2$ be an ideal of $R_1 \times R_2$ where I_i is an ideal of R_i for each $i = 1, 2$. Then :*

$$Ann_{R_1\times R_2}(I_1 \times I_2) = Ann_{R_1}(I_1) \times Ann_{R_2}(I_2).$$

Proof. Trivial. □

Proof of Theorem 3.6. The proof is done by induction on n and it suffices to check it for $n = 2$. Assume that $R := R_1 \times R_2$ is a P-IN-ring. Let I_1 and I_2 be two principal ideals of R_1. Then :

$$\begin{aligned}
(Ann_{R_1}(I_1 \cap I_2)) \times R_2 &= (Ann_{R_1}(I_1 \cap I_2)) \times (Ann_{R_2}(0)) \\
&= Ann_{R_1\times R_2}((I_1 \cap I_2) \times \{0\}) \quad \text{(By Lemma 3.7)} \\
&= Ann_{R_1\times R_2}((I_1 \times \{0\}) \cap (I_2 \times \{0\})) \\
&= Ann_{R_1\times R_2}(I_1 \times \{0\}) + Ann_{R_1\times R_2}(I_2 \times \{0\}) \\
&= (Ann_{R_1}(I_1) \times R_2) + (Ann_{R_1}(I_2) \times R_2) \\
&= (Ann_{R_1}(I_1) + Ann_{R_1}(I_2)) \times R_2
\end{aligned}$$

So, $Ann_{R_1}(I_1 \cap I_2) = Ann_{R_1}(I_1 + Ann_{R_1}(I_2)$. Thus, R_1 is a P-IN-ring (the case that R_2 is a P-IN-ring is similar).

Conversely, Assume that R_1 and R_2 are a P-IN-rings. Let $I := I_1 \times I_2$ and $J := J_1 \times J_2$ be two principal ideals of $R_1 \times R_2$ where I_1, J_1 and I_2, J_2 are principal ideals of R_1 and R_2 respectively.

$$Ann_{R_1\times R_2}((I_1 \times I_2) \cap (J_1 \times J_2)) = Ann_{R_1\times R_2}((I_1 \cap J_1) \times (I_2 \cap J_2))$$

$$\begin{aligned}
&= Ann_{R_1}(I_1 \cap J_1) \times Ann_{R_2}(I_2 \cap J_2) \\
&= (Ann_{R_1}(I_1) + Ann_{R_1}(J_1)) \times (Ann_{R_2}(I_2) + Ann_{R_2}(J_2)) \\
&= (Ann_{R_1}(I_1) \times Ann_{R_2}(I_2)) + (Ann_{R_1}(J_1) \times Ann_{R_2}(J_2)) \\
&= (Ann_{R_1\times R_2}(I_1 \times I_2)) + (Ann_{R_1\times R_2}(J_1 \times J_2)).
\end{aligned}$$

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