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Coefficient Inequalities for Classes of Univalent Functions Defined by $q-$ Derivatives*

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ABSTRACT: Using the principal of subordination and the $q-$derivative, we obtain sharp bounds for some classes of univalent functions.

Key Words: Univalent functions, $q-$derivative , Subordination.

Contents

1. Introduction

Denote by $\mathcal{A}$ the class of analytic functions:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ (z \in \mathbb{U} = \{z : z \in \mathbb{C}\ ,\ |z| < 1\}). \tag{1.1}$$

For $0 < q < 1$, the q-derivative of $f \in \mathcal{A}$, is given by (see [4], [5])

$$\begin{aligned} D_q f(z) &= \frac{f(qz) - f(z)}{(q-1)z}, z \neq 0 \\ &= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \end{aligned} \tag{1.2}$$

where, $[n]_q = \dfrac{q^n - 1}{q - 1}$, as $q \to 1^-, [n]_q \to n,\ D_q f(0) = f'(0)$ and $D_q(D_q f(z)) = D_q^2 f(z)$. If $\eta(z) = z^n$, then

$$D_q \eta(z) = D_q(z^n) = \frac{q^n - 1}{q - 1} z^{n-1} = [n]_q z^{n-1},$$

$$\lim_{q \to 1^-} D_q \eta(z) = \lim_{q \to 1^-} [n]_q z^{n-1} = n z^{n-1} = \eta'(z).$$

Denote by $\mathcal{P}$ the class of analytic functions ϕ of positive real part on $\mathbb{U}$ with $\phi(0) = 1,\ \Re\{\phi(z)\} > 0$. Using the q-derivative $D_q f(z), f \in \mathcal{A}, \varkappa \in P, 0 \leq \lambda \leq 1, b \in \mathbb{C}^* = \mathbb{C}/\{0\}$, let

$$\mathcal{H}_{q,b}^{\lambda}(\varkappa) = \left\{ f : 1 + \frac{1}{b}\left[(1-\lambda)\left(\frac{z D_q f(z)}{f(z)} \right) + \lambda \frac{D_q(z D_q f(z))}{D_q f(z)} - 1 \right] \prec \varkappa(z) \right\}, \tag{1.3}$$

where $\prec$ denotes the usual subordination (see [7], [3], [2]).

For different choices of q, b, λ, in (1.3), the class $\mathcal{H}_{q,b}^{\lambda}(\varkappa)$, generalizes many classes studied earlier, for example (see Seoudy and Aouf [10], [11], Ravichandran et al. [9], Ali et al. [1] with $p = 1$

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and Ramachandran et al. [8], with $\alpha = 0$ and $\beta = 1$. Also, we obtain the new class $\mathcal{H}_{q,\theta}^{\lambda,\alpha}(\varkappa)$ for $b = e^{-i\theta}(1-\alpha)\cos\theta, 0 \leq \alpha < 1, |\theta| < \frac{\pi}{2}$, where

$$\mathcal{H}_{q,\theta}^{\lambda,\alpha}(\varkappa) = \left\{ f : \frac{e^{i\theta}\left[(1-\lambda)\left(\frac{zD_qf(z)}{f(z)}\right) + \lambda\frac{D_q(zD_qf(z))}{D_qf(z)}\right] - \alpha\cos\theta - i\sin\theta}{(1-\alpha)\cos\theta} \prec \varkappa(z) \right\}.$$

The following known lemma is needed to establish our results.

Lemma 1.1 [6]. *If $p(z) = 1 + r_1z + r_2z^2 + ... \in \mathcal{P}$ and δ is a complex number, then*

$$\left|r_2 - \delta r_1^2\right| \leq 2\max\{1; |2\delta - 1|\}. \tag{1.4}$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \textit{and} \quad p(z) = \frac{1+z}{1-z}.$$

Also, we note that

$$\left|r_2 - \xi r_1^2\right| \leq \begin{cases} -4\xi + 2 & if \quad \xi \leq 0, \\ 2 & if \quad 0 \leq \xi \leq 1, \\ 4\xi - 2 & if \quad \xi \geq 1, \end{cases} \tag{1.5}$$

when $\xi < 0$ or $\xi > 1$, the equality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \xi < 1$, then the equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\xi = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z} \qquad (0 \leq \gamma \leq 1)$$

or one of its rotations. If $\gamma = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that equality holds in the case of $\xi = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < \xi < 1$:

$$\left|r_2 - \xi r_1^2\right| + \xi\left|r_1\right|^2 \leq 2 \qquad \left(0 \leq \xi \leq \frac{1}{2}\right)$$

and

$$\left|r_2 - \xi r_1^2\right| + (1-\xi)\left|r_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} \leq \xi \leq 1\right).$$

2. Main results

We assume in the reminder of this paper that $f \in \mathcal{A}, \varkappa \in P, 0 < q < 1, 0 \leq \lambda \leq 1$ and $b \in \mathbb{C}^*$.

Theorem 2.1. *Let*

$$\varkappa(z) = 1 + d_1z + d_2z^2 + ... \tag{2.1}$$

with $d_1 > 0$. If $f(z) \in \mathcal{H}_{q,b}^{\lambda}(\varkappa)$, then

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| \leq \; & \frac{|b|\,|d_1|}{2([3]_q - 1)[1 + \lambda([3]_q - 1)]}\max\{1, \\ & \left|\frac{d_2}{d_1} + \frac{bd_1}{([2]_q-1)[1+\lambda([2]_q-1)]^2}\left[[1 + \lambda([2]_q^2 - 1) - \mu\frac{([3]_q-1)[1+\lambda([3]_q-1)}{([2]_q-1)}\right]\right|\right\}. \end{aligned} \tag{2.2}$$

The result is sharp.

Proof: If $f \in \mathcal{H}_{q,b}^{\lambda}(\varkappa)$, then there is a function ω, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$1+\frac{1}{b}\left[(1-\lambda)\tfrac{zD_qf(z)}{f(z)}+\lambda\tfrac{D_q(zD_qf(z))}{D_qf(z)}-1\right]=\varkappa(\omega(z)). \tag{2.3}$$

Define the function $p(z)$ by

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+r_1z+r_2z^2+\dots . \tag{2.4}$$

We see that $\Re\{p(z)\}>0$ and $p(0)=1$, since $\omega(z)$ is a Schwarz function.. Therefore,

$$\begin{aligned}\varkappa(\omega(z)) &= \varkappa\left(\frac{p(z)-1}{p(z)+1}\right)\\ &= \varkappa\left(\frac{1}{2}\left[r_1z+\left(r_2-\frac{r_1^2}{2}\right)z^2+\left(r_3-r_1r_2+\frac{r_1^3}{4}\right)z^3+\dots\right]\right)\\ &= 1+\frac{1}{2}d_1r_1z+\left[\frac{1}{2}d_1\left(r_2-\frac{r_1^2}{2}\right)+\frac{1}{4}d_2r_1^2\right]z^2+\dots..\end{aligned} \tag{2.5}$$

Equating the coefficients of (2.5) and (2.3), we have

$$[[2]_q-1+\lambda([2]_q-1)^2]a_2=\frac{1}{2}bd_1r_1,$$

$$\begin{aligned}&([3]_q-1)[1+\lambda([3]_q-1)]a_3-([2]_q-1)[1+\lambda([2]_q-1)]a_2^2\\ =\ &(\frac{1}{2}d_1r_2-\frac{1}{4}d_1r_1^2+\frac{1}{4}d_2r_1^2)b,\end{aligned}$$

or

$$a_2=\frac{bd_1r_1}{2([2]_q-1)[1+\lambda([2]_q-1)]},$$

$$a_3=\tfrac{bd_1}{2([3]_q-1)[1+\lambda([3]_q-1)]}\left\{d_2-\frac{d_1^2}{2}\left[1-\frac{d_2}{d_1}-\tfrac{[1+\lambda([2]_q^2-1)]bd_1}{([2]_q-1)[1+([2]_q-1)]^2}\right]\right\}.$$

Therefore,

$$a_3-\mu a_2^2=\tfrac{bd_1}{2([3]_q-1)[1+\lambda([3]_q-1)]}\left(d_2-\delta d_1^2\right), \tag{2.6}$$

where

$$\delta=\frac{1}{2}\left\{1-\frac{d_2}{d_1}-\tfrac{bd_1}{([2]_q-1)[1+\lambda([2]_q-1)]^2}\left[1+\lambda([2]_q^2-1)-\mu\tfrac{([3]_q-1)[1+\lambda([3]_q-1)]}{([2]_q-1)}\right]\right\}. \tag{2.7}$$

Our result now follows by an application of (1.4). The result is sharp for the functions

$$1+\frac{1}{b}\left[(1-\lambda)\frac{zD_qf(z)}{f(z)}+\lambda\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right]=\varkappa\left(z^2\right),$$

and

$$1+\frac{1}{b}\left[(1-\lambda)\frac{zD_qf(z)}{f(z)}+\lambda\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right]=\varkappa(z).$$

The proof of Theorem 1 is completed. □

Remark 2.1. *(i) Putting $\lambda=0$ in Theorem 1, we obtain the result of Seoudy and Aouf [10, Theorem 1];*

(ii) Putting $\lambda=1$ in Theorem 1, we obtain the result of Seoudy and Aouf [10, Theorem 2];

(iii) Theorem 1 for $b=1$, corrects the result of Ramachandram et al. [8, Theorem 2, $\alpha=0,\beta=1$].

Theorem 2.2. *Let* $\varkappa(z)$ in the form (2.1), *with* $d_1 > 0$ *and* $d_2 \geq 0$. *Let*

$$\alpha_1 = \frac{(d_2-d_1)([2]_q-1)^2[1+\lambda([2]_q-1)]^2+([2]_q-1)[1+\lambda([2]_q^2-1)]bd_1^2}{([3]_q-1)[1+\lambda([3]_q-1)]bd_1^2}, \quad (2.8)$$

$$\alpha_2 = \frac{(d_2+d_1)([2]_q-1)^2[1+\lambda([2]_q-1)]^2+([2]_q-1)[1+\lambda([2]_q^2-1)]bd_1^2}{([3]_q-1)[1+\lambda([3]_q-1)]bd_1^2}, \quad (2.9)$$

$$\alpha_3 = \frac{d_2([2]_q-1)^2[1+\lambda([2]_q-1)]^2+([2]_q-1)[1+\lambda([2]_q^2-1)]bd_1^2}{([3]_q-1)[1+\lambda([3]_q-1)]bd_1^2}. \quad (2.10)$$

If $f(z)\in\mathcal{H}_{q,b}^{\lambda}(\varkappa)$ *with* $b > 0$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{bd_2}{([3]_q-1)[1+\lambda([3]_q-1)]}+ \\ +\frac{b^2d_1^2}{([2]_q-1)[1+\lambda([2]_q-1)]^2}\left(\frac{[1+\lambda([2]_q^2-1)]}{([3]_q-1)[1+\lambda([3]_q-1)]}-\mu\frac{1}{([2]_q-1)}\right), & \mu\leq\alpha_1, \\ \frac{bd_1}{([3]_q-1)[1+\lambda([3]_q-1)]}, & \alpha_1\leq\mu\leq\alpha_2, \\ -\frac{bd_2}{([3]_q-1)[1+\lambda([3]_q-1)]}- \\ -\frac{b^2d_1^2}{([2]_q-1)[1+\lambda([2]_q-1)]^2}\left(\frac{[1+\lambda([2]_q^2-1)]}{([3]_q-1)[1+\lambda([3]_q-1)]}-\mu\frac{1}{([2]_q-1)}\right), & \mu\geq\alpha_2. \end{cases} \quad (2.11)$$

Further, if $\alpha_1 \leq \mu \leq \alpha_3$, *then*

$$\left|a_3-\mu a_2^2\right| + \frac{([2]_q-1)^2[1+\lambda([2]_q-1)]^2}{([3]_q-1)[1+\lambda([3]_q-1)]d_1^2b}\left[d_1-d_2-\frac{bd_1^2}{([2]_q-1)[1+\lambda([2]_q-1)]^2}\right.$$

$$\left.\times\left([1+\lambda([2]_q^2-1)]-\mu\frac{([3]_q-1)[1+\lambda([3]_q-1)]}{([2]_q-1)})\right)\right]|a_2|^2 \leq \frac{bd_1}{([3]_q-1)[1+\lambda([3]_q-1)]}, \quad (2.12)$$

and if $\alpha_3 \leq \mu \leq \alpha_2$, *then*

$$\left|a_3-\mu a_2^2\right| + \frac{([2]_q-1)^2[1+\lambda([2]_q-1)]^2}{([3]_q-1)[1+\lambda([3]_q-1)]d_1^2b}\left[d_1+d_2+\frac{bd_1^2}{([2]_q-1)[1+\lambda([2]_q-1)]^2}\right.$$

$$\left.\times\left([1+\lambda([2]_q^2-1)]-\mu\frac{([3]_q-1)[1+\lambda([3]_q-1)]}{([2]_q-1)})\right)\right]|a_2|^2 \leq \frac{bd_1}{([3]_q-1)[1+\lambda([3]_q-1)]}. \quad (2.13)$$

The result is sharp.

Proof: The proof follows by applying (1.5) to (2.6) and (2.7). To show that the bounds are sharp, we define the functions $\mathcal{K}_{\varkappa k}$ $(k = 2, 3, 4, ...)$ by

$$1+\frac{1}{b}\left[(1-\lambda)\frac{zD_q\mathcal{K}_{\varkappa_k}(z)}{\mathcal{K}_{\varkappa k}(z)}+\lambda\frac{D_q(zD_q\mathcal{K}_{\varkappa k}(z))}{D_q\mathcal{K}_{\varkappa k}(z)}-1\right]=\varkappa\left(z^{n-1}\right),$$

$$\mathcal{K}_{\varkappa k}(0)=0=\mathcal{K}_{\varkappa_k}(0)-1$$

and the functions $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ $(0 \leq \tau \leq 1)$ by

$$1+\frac{1}{b}\left[(1-\lambda)\frac{zD_q\mathcal{F}_\tau(z)}{\mathcal{F}_\tau(z)}+\lambda\frac{D_q(zD_q\mathcal{F}_\tau(z))}{D_q\tau(z)}-1\right]=\varkappa\left(\frac{z(z+\tau)}{1+\tau z}\right),$$

$$\mathcal{F}_\tau(0)=0=\mathcal{F}_\tau(0)-1$$

and

$$1+\frac{1}{b}\left[(1-\lambda)\frac{zD_q\mathcal{G}_\tau(z)}{\mathcal{G}_\tau(z)}+\lambda\frac{D_q(zD_q\mathcal{G}_\tau(z))}{D_q\mathcal{G}_\tau(z)}-1\right]=\varkappa\left(\frac{1+\tau z}{z(z+\tau)}\right),$$

$$\mathcal{G}_\tau(0)=0=\mathcal{G}_\tau^{'}(0)-1.$$

The functions $\mathcal{K}_{\varkappa k}, \mathcal{F}_\lambda$ and $\mathcal{G}_\lambda \in \mathcal{H}_{q,b}^{\lambda}(\varkappa)$. If $\mu < \alpha_1$ or $\mu > \alpha_2$, then the equality holds if and only if f is $\mathcal{K}_{\varkappa 2}$, or one of its rotations. When $\alpha_1 < \mu < \alpha_2$, the equality holds if and only if f is $\mathcal{K}_{\varkappa 3}$, or one of its rotations. If $\mu = \alpha_1$, then the equality holds if and only if f is $\mathcal{F}_\tau$, or one of its rotations. If $\mu = \alpha_2$, then the equality holds if and only if f is $\mathcal{G}_\tau$, or one of its rotations. □

Remark 2.2 *(i) Taking $q \to 1^-$ and $\lambda = \alpha$, in the above results, we obtain the results of [12, with $\lambda = 0$];*
(ii) Theorem 2 for $b = 1$, corrects the result of Ramachandram et al. [8, Theorem 1, $\alpha = 0, \beta = 1$];
(iii) Putting $\lambda = 0$ in Theorem 2, we obtain the result of Seoudy and Aouf [10, Theorem 3];
(iv) Putting $\lambda = 1$ in Theorem 2, we obtain the result of Seoudy and Aouf [10, Theorem 3];
(v) Taking $b = e^{-i\theta}(1-\alpha)\cos\theta$ in the above results, we obtain results for the class $\mathcal{H}_{q,\theta}^{\lambda,\alpha}(\varkappa)$.

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