Bol. Soc. Paran. Mat. (3s.) **v. 2022 (40)** : 1–9.
©SPM –ISSN-2175-1188 ON LINE ISSN-0037-8712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.44202

Some Topological Properties and Asymptotic Behavior of the Higher Eigencurves for the p-Laplacian Operator with Weight

Mimoun Moussaoui, Ahmed Dakkak and Omar Chakrone

ABSTRACT: In this paper, we show that for each real α there exists a unique real $t_n(\alpha)$ such that $\lambda_n(\alpha m_1 + t_n(\alpha)m_2) = 1$, where m_1 and m_2 are bounded weight functions and $\lambda_n(m)$ is the n^{th} Ljusternik-Schinerlmann eigenvalue of the p-Laplacian operator with weight m. We also study the asymptotic behavior, the variational formulation and some topological properties of the eigencurve $t_n(\cdot)$.

Key Words: Higher eigencurves, Topological properties, Variational formulation, Asymptotic behavior.

Contents

1. Introduction

The study of differential equations and variational problems has become an important topic of modern nonlinear analysis because of their important applications, we refer the reader to [8] for more. Consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = & \lambda m(x)|u|^{p-2}u \quad \text{in} \quad \Omega, \\ \quad u = & 0 \quad\quad\quad\quad\quad\ \text{on} \quad \partial\Omega \end{cases} \tag{$\mathcal{P}$}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < +\infty, m(\cdot) \in M^+(\Omega)$, with

$$M^+(\Omega) = \big\{\varphi \in L^\infty(\Omega) : \text{meas}\,\{x \in \Omega : \varphi(x) > 0\} \neq 0\big\}.$$

We say that λ is an eigenvalue of the p-Laplacian with weight $m(.)$ when the problem $(\mathcal{P})$ has at least a nontrivial solution $u \in W_0^{1,p}(\Omega)$. The set of positive eigenvalues constitutes the spectrum $\sigma_p^+(-\Delta_p, m, \Omega)$. For $p = 2$ ($\Delta_p = \Delta$ is the Laplacian operator), it is well known (see [6,7]), that $\sigma_2^+(-\Delta, m, \Omega) = \{\mu_k(m), k = 1, 2, \ldots\}$, with

$$0 < \mu_1(m) < \mu_2(m) \leq \mu_3(m) \ldots \to +\infty,$$

each eigenvalue $\mu_k(m)$ is repeated as many times as its multiplicity. For $p \neq 2$, the critical point theory of Ljusternik-Schnirelmann (see [9]) provides a sequence in $\sigma_p^+(-\Delta_p, m, \Omega)$ given by $\lambda_1(m) < \lambda_2(m) \leq \lambda_3(m) \leq \ldots \leq \lambda_n(m), \ldots \to +\infty$ and formulated as follows

$$\frac{1}{\lambda_n(m)} = \sup_{K\in\Gamma_n} \min_{u\in K} \int_\Omega m|u|^p \tag{1.1}$$

where Γ_n is defined by:

$$\Gamma_n = \{K \subset S : K \text{ is symmetrical, compact and} \quad \gamma(K) \geq n\},$$

2010 *Mathematics Subject Classification:* 47A16, 47D06, 47D03.
Submitted August 20, 2018. Published January 02, 2019

where S is the unit sphere of $W_0^{1,p}(\Omega)$ and γ is the genus function (see [9]). We may also define the negative spectrum by $\lambda_{-n}(m) = -\lambda_n(-m)$ (See [3]). Whether or not this sequence of both the positive and negative eigenvalues, denoted $\lambda_k(m)$, constitutes the whol set of all eigenvalues remains an open question when $N > 1$ and $p \neq 2$.
Consider two weight functions m_1, $m_2 \in M^+(\Omega)$, it is rather desirable to gather more information about the question "Whether or not $C_n = \{(\alpha, \beta) \in \mathbb{R}^2 : \lambda_n(\alpha m_1 + \beta m_2) = 1\}$ constitutes a curve?" . Several applications related to these problems can be found in the bifurcation domain, we refer the reader to [2].
On the other hand, this is a kind of inverse problem in the following sense:
For $\delta > 0$ given, we look for a weight $m(\cdot) \in \text{span}\{m_1, m_2\}$ such that $\lambda_n(m) = \delta$. By the homogeneity of λ_n we take $\delta = 1$.
Existence results for the curves C_n with $n \in \{1, 2\}$ were studied in [1,4,5] among other. In [1] the authors considered the case where $n = 1$ and m_2 is a constant, they established some properties relating to the first eigencurve C_1 such as concavity, differentiability and the asymptotic behavior. The authors in [4] showed that $C_n \neq \emptyset$ under the assumption $ess \inf_{\Omega} m_2 > 0$, the technique used is based on the strict monotonicity property, which is not applicable in the general case where $ess \inf_{\Omega} m_2 = 0$. In [5], the authors considered the case where $n = 2$ they showed that for each $\alpha \in \mathbb{R}$ there exists a real number $\beta_2(\alpha)$ such that $(\alpha, \beta_2(\alpha)) \in C_2$. They proved the asymptotic behavior of $\beta_2(\cdot)$. The techniques used are not adaptable when $n \geq 3$.
In this paper, we assume that

$$m_1, m_2 \in M^+(\Omega),\ m_2 \geq 0\ a.e.\text{in}\,\Omega \quad \text{and} \quad ess \inf_{\Omega^\star_{m_1}} m_2 > 0 \tag{H_0}$$

where $\Omega^\star_{m_1} = \{x \in \Omega : m_1(x) \neq 0\}$. For each $\alpha \in \mathbb{R}$, we prove the existence of a unique real number $t_n(\alpha)$ such that $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$, we give the variational formulation of $t_n(\alpha)$, we also study its monotonicity, continuity properties and its asymptotic behavior .
This paper is organized as follows. In section 2, we present our main results. In section 3, we introduce some basic preliminary results. In section 4, we give the proofs of our main results.

2. Main results

We will use below the notation $\Omega^+_{m_1} = \{x \in \Omega : m_1(x) > 0\}$, $\Omega^-_{m_1} = \{x \in \Omega : m_1(x) < 0\}$ and $\Omega^\star_{m_2} = \{x \in \Omega : m_2(x) \neq 0\}$.
Our main results are the following.

Theorem 2.1. *Assume (H_0) holds, then we have:*

1. *For $\alpha \in [0, \lambda_n(m_1)]$, there exists a unique real $t_n(\alpha) \in \mathbb{R}^+$ such that $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$.*
2. *For $\alpha \in]\lambda_n(m_1), +\infty[$, there exists a unique real $t_n(\alpha) \in \mathbb{R}^-$ such that $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$.*
3. *If $m_1 \geq 0$ a.e.in Ω, then for $\alpha \in] - \infty, \lambda_n(m_1)]$, there exists a unique real $t_n(\alpha) \in \mathbb{R}^+$ such that $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$.*
4. *If $meas(\Omega^-_{m_1}) > 0$, then*
   - *For $\alpha \in [\lambda_{-n}(m_1), 0[$, there exists a unique real $t_n(\alpha) \in \mathbb{R}^+$ such that $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$.*
   - *For $\alpha \in]-\infty, \lambda_{-n}(m_1)[$, there exists a unique real $t_n(\alpha) \in \mathbb{R}^-$ such that $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$.*

Denoting by $\Gamma^1_n = \{K \in \Gamma_n : K \subset S'\}$, $S' = \{u \in S : \int_\Omega m_2 |u|^p \neq 0\}$, we have the following results.

Theorem 2.2. *Assume (H_0) holds, then we have:*

1. *For $\alpha \in \mathbb{R}$, the unique real $t_n(\alpha)$ such that $\lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$, is given by*

$$t_n(\alpha) = \inf_{K \in \Gamma^1_n} \max_{u \in K} \frac{1 - \alpha \int_\Omega m_1 |u|^p}{\int_\Omega m_2 |u|^p}$$

2. $t_n(\cdot)$ *is continuous in* $\mathbb{R}$.
3. *If meas*$(\Omega^-_{m_1}) > 0$, *then* $t_n(\cdot)$ *is decreasing in* $[\lambda_n(m_1), +\infty[$ *and increasing in* $]-\infty, \lambda_{-n}(m_1)]$.
4. *If* $m_1(\cdot) \geq 0$ *a.e.in* Ω, *then* $t_n(\cdot)$ *is decreasing in* $\mathbb{R}$.

Theorem 2.3. *Assume* (H_0) *holds, then we have:*

1. $\lim\limits_{\alpha \to +\infty} \frac{t_n(\alpha)}{\alpha} = -ess \sup\limits_{\Omega^+_{m_1}} \frac{m_1}{m_2}$.
2. *If meas* $(\Omega^-_{m_1}) > 0$, *then* $\lim\limits_{\alpha \to -\infty} \frac{t_n(\alpha)}{\alpha} = -ess \inf\limits_{\Omega^-_{m_1}} \frac{m_1}{m_2}$.
3. *If* $m_1 \geq 0$ *in* Ω, *then* $\lim\limits_{\alpha \to -\infty} \frac{t_n(\alpha)}{\alpha} = -ess \inf\limits_{\Omega^\star_{m_2}} \frac{m_1}{m_2}$.

3. Preliminary results

First we recall the following results which will be used later.

Proposition 3.1. *If* $m, m' \in M^+(\Omega)$ *such that* $m'(x) \geq m(x)$ *a.e.*$x \in \Omega$ *and* $m'(x) > m(x)$ *for a.e.* $x \in \Omega^+_m$, *then for each* n *in* $\mathbb{N}^\star$ *we have* $\lambda_n(m) > \lambda_n(m')$.

Proof. we have

$$\frac{1}{\lambda_n(m)} = \sup_{K \in \Gamma_n} \min_{u \in K} \int_\Omega m|u|^p.$$

Let (K_j) a sequence in Γ_n such that

$$\lim_{j \to +\infty} \min_{K_j} \int_\Omega m|u|^p = \sup_{K \in \Gamma_n} \min_{u \in K} \int_\Omega m|u|^p = \frac{1}{\lambda_n(m)},$$

since K_j is compact we have

$$\min_{K_j} \int_\Omega m'|u|^p = \int_\Omega m'|u_{K_j}|^p \quad u_{K_j} \in K_j.$$

The sequence (u_{K_j}) is bounded, so $u_{K_j} \rightharpoonup \tilde{u}$ in $W^{1,p}_0(\Omega)$ and $u_{K_j} \to \tilde{u}$ in $L^p(\Omega)$, in other hand we have

$$\min_{K_j} \int_\Omega m|u|^p \leq \int_\Omega m|u_{K_j}|^p = \int_\Omega m'|u_{K_j}|^p - \int_\Omega (m'-m)|u_{K_j}|^p, \tag{3.1}$$

passing to the limit in (3.1) we get

$$\frac{1}{\lambda_n(m)} \leq \int_\Omega m|\tilde{u}|^p \leq \frac{1}{\lambda_n(m')} - \int_\Omega (m'-m)|\tilde{u}|^p. \tag{3.2}$$

We claim that $\delta = \int_\Omega (m'-m)|\tilde{u}|^p > 0$, indeed if $\delta = 0$ then $\tilde{u} = 0$ in Ω^+_m hence from (3.2) we get $\frac{1}{\lambda_n(m)} \leq 0$ contradiction, so we conclude that

$$\frac{1}{\lambda_n(m)} < \frac{1}{\lambda_n(m')}$$

that is $\lambda_n(m') < \lambda_n(m)$.

Proposition 3.2. *([5]) We have*

1. *If* $m, m^{'} \in M^+(\Omega)$ *and* $m(x) \leq m^{'}(x)$ *for a.e.* $x \in \Omega$, *then* $\lambda_n(m) \geq \lambda_n(m^{'})$.
2. *The mapping* $\lambda_n \colon m \to \lambda_n(m)$ *is continuous in* $M^+(\Omega)$ *for the distance* $d(m, m') = ||m - m'||_\infty$.

Proposition 3.3. *([5]) Let* (m_k) *be a sequence in* $M^+(\Omega)$ *such that* $m_k \to m$ *in* $L^\infty(\Omega)$ *then,*

$$\lim_{k \to +\infty} \lambda_n(m_k) = +\infty \quad \textit{if and only if} \quad m(x) \leq 0 \, \textit{for a.e.} x \in \Omega.$$

4. Proofs of the main results

Proof of theorem 2.1.

1. To show the first result, we distinguish several cases.
if $\alpha = 0$, the unique real $t_n(0)$ such that: $\lambda_n(0m_1 + t_n(0)m_2) = 1$ is $t_n(0) = \lambda_n(m_2)$.
If $0 < \alpha < \lambda_n(m_1)$, we consider the function $h_\alpha(\cdot)$ defined by $h_\alpha(t) = \lambda_n(\alpha m_1 + tm_2)$. It is clear that $h_\alpha(\cdot)$ is well defined on $[0, +\infty[$, decreasing and continuous (see proposition 3.2). In other hands, we have
$$h_\alpha(0) = \frac{\lambda_n(m_1)}{\alpha} > 1 \tag{4.1}$$
and
$$\lim_{t\to+\infty} h_\alpha(t) = \lim_{t\to+\infty} \frac{\lambda_n(\frac{\alpha}{t}m_1 + m_2)}{t} = 0. \tag{4.2}$$
Using (4.1), (4.2) and the fact that h_α is continuous, we deduce that there exists a real $t_n(\alpha) \in]0, +\infty[$ such that $h_\alpha(t_n(\alpha)) = 1$, i.e. $\lambda_n(\alpha m_1 + t_n(\alpha)m_2) = 1$.
If $\alpha = \lambda_n(m_1)$, we take $t_n(\alpha) = 0$.
To show the uniqueness, we proceed as follows, let $\beta < \beta^{'}$, assume $\lambda_n(\alpha m_1 + \beta m_2) = \lambda_n(\alpha m_1 + \beta^{'} m_2) = 1$, denote $m = \alpha m_1 + \beta m_2$ and $m^{'} = \alpha m_1 + \beta^{'} m_2$. If $x \in \Omega_m^+$ By (H_0) we deduce that $m_2(x) > 0$, hence $\beta m_2 < \beta^{'} m_2$, so we conclude that $m^{'}(x) \geq m(x)$ for a.e.x$\in \Omega$ and $m^{'}(x) > m(x)$ for a.e.$x \in \Omega_m^+$, then by proposition 3.1 we get $\lambda_n(m) > \lambda_n(m^{'})$ which gives a contraduction.

2. Since $\alpha > \lambda_n(m_1)$ we deduce that
$$0 < h_\alpha(0) < 1. \tag{4.3}$$
Let $A_\alpha = \{t \leq 0 : \alpha m_1 + tm_2 \leq 0 \,\text{in}\, \Omega\}$, we have $d = \dfrac{-\alpha||m_1||_\infty}{ess \inf\limits_{\Omega_{m_1}^+} m_2} \in A_\alpha$, hence $A_\alpha \neq \emptyset$. Set $\tau_\alpha = \sup A_\alpha$, we will show that $A_\alpha =]-\infty, \tau_\alpha]$. Indeed, for $k \in \mathbb{N}^\star$, there exists $t_k \in A_\alpha$ such that $\tau_\alpha - \dfrac{1}{k} \leq t_k$, it follows that $\alpha m_1 + \tau_\alpha m_2 \leq \alpha m_1 + t_k m_2 + \dfrac{1}{k} m_2$, then $\alpha m_1 + \tau_\alpha m_2 \leq \dfrac{1}{k}||m_2||_\infty$. Using the fact that $k \in \mathbb{N}^\star$ is arbitrary, we deduce that $\alpha m_1 + \tau_\alpha m_2 \leq 0$, so $\tau_\alpha \in A_\alpha$, hence $A_\alpha =]-\infty, \tau_\alpha]$ (since $0 \notin A_\alpha$ then $\tau_\alpha < 0$).
Let $(t_i)_i$ be a sequence in $]\tau_\alpha, 0[$ such that $\lim\limits_{i\to+\infty}(t_i) = \tau_\alpha$. Then we have
$$\alpha m_1 + t_i m_2 \to \alpha m_1 + \tau_\alpha m_2 \quad \text{in} \quad L^\infty(\Omega). \tag{4.4}$$
The function $h_\alpha(\cdot)$ is well defined on $]\tau_\alpha, 0[$, hence by (4.4) and proposition 3.3 we deduce that
$$\lim_{i\to+\infty} h_\alpha(t_i) = +\infty. \tag{4.5}$$
So relations (4.3) and (4.5) imply that there exists $t_n(\alpha) \in]\tau_\alpha, 0[$ such that $h_\alpha(t_n(\alpha)) = 1$, i.e., $\lambda_n(\alpha m_1 + t_n(\alpha)m_2) = 1$. As in the first result, we show the uniqueness.

3. For the third result, we prove only the case $\alpha < 0$, the case $\alpha \in [0, \lambda_n(m_1)]$ has been already treated. For this, we consider the set $B_\alpha = \{t > 0 : \alpha m_1 + tm_2 \in M^+(\Omega)\}$. It is easy to see that
$$t > \frac{|\alpha|||m_1||_\infty}{ess \inf\limits_{\Omega_{m_1}^+} m_2} \quad \text{implies that} \quad t \in B_\alpha.$$
Let $\eta_\alpha = \inf B_\alpha$. We show that $\eta_\alpha \notin B_\alpha$. Indeed, for $k \in \mathbb{N}^\star$, $\eta_\alpha - \frac{1}{k} \notin B_\alpha$. Hence $\alpha m_1 + \eta_\alpha m_2 - \dfrac{1}{k} m_2 \leq 0 \,\text{in}\, \Omega$. It follows that $\alpha m_1 + \eta_\alpha m_2 \leq \dfrac{1}{k}||m_2||_\infty$. Since $k \in \mathbb{N}^\star$ is arbitrary we get $\alpha m_1 + \eta_\alpha m_2 \leq 0$. Hence $\eta_\alpha \notin B_\alpha$ and $B_\alpha =]\eta_\alpha, +\infty[$. Let $(t_j)_j$ be a sequence in B_α such that

$\lim_{j\to+\infty} t_j = \eta_\alpha$, then $\alpha m_1 + t_j m_2 \to \alpha m_1 + \eta_\alpha m_2$ in $L^\infty(\Omega)$. According to proposition 3.3, we obtain

$$\lim_{t\to\eta_\alpha^+} h_\alpha(t) = +\infty. \tag{4.6}$$

On the other hand we have

$$\lim_{t\to+\infty} h_\alpha(t) = \lim_{t\to+\infty} \frac{\lambda_n(\frac{\alpha}{t} m_1 + m_2)}{t} = 0,$$

then from (4.6) and the previous results, we deduce that there exists a unique real $t_n(\alpha) \in]\eta_\alpha, +\infty[$ such that $h_\alpha(t_n(\alpha)) = 1, i.e., \lambda_n(\alpha m_1 + t_n(\alpha) m_2) = 1$.

4. This case is treated in the same way.

Proof of Theorem 2.2.

1. First we claim that $\Gamma_n^1 \neq \emptyset$. Indeed, assume by contradiction that $\Gamma_n^1 = \emptyset$, then for all $K \in \Gamma_n$ there exists $u \in K$ such that $u \notin S'$. Hence, taking into account that $\Omega^\star_{m_1} \subset \Omega^\star_{m_2}$, we deduce that $\int_\Omega (\alpha m_1 |u|^p + \beta m_2 |u|^p) dx = 0$ for each $(\alpha, \beta) \in C_n$, which gives

$$\min_K \int_\Omega (\alpha m_1 |u|^p + \beta m_2 |u|^p) dx \leq 0 \quad \forall K \in \Gamma_n.$$

It follows that

$$1 = \frac{1}{\lambda_n(\alpha m_1 + \beta m_2)} = \sup_{\Gamma_n} \min_K \int_\Omega \alpha m_1 |u|^p + \beta m_2 |u|^p dx \leq 0.$$

Which is a contradiction, so $\Gamma_n^1 \neq \emptyset$.
Let $\theta_n(\alpha) = \inf_{K\in\Gamma_n^1} \max_K \dfrac{1 - \alpha \int_\Omega m_1 |u|^p}{\int_\Omega m_2 |u|^p}$, so for each $K \in \Gamma_n^1$ we have

$$\theta_n(\alpha) \leq \max_K \frac{1 - \alpha \int_\Omega m_1 |u|^p}{\int_\Omega m_2 |u|^p}.$$

Since K is compact, there exists $u_k \in K$ such that

$$\max_K \frac{1 - \alpha \int_\Omega m_1 |u|^p}{\int_\Omega m_2 |u|^p} = \frac{1 - \alpha \int_\Omega m_1 |u_k|^p}{\int_\Omega m_2 |u_k|^p}.$$

Then

$$\theta_n(\alpha) \int_\Omega m_2 |u_k|^p + \alpha \int_\Omega m_1 |u_k|^p \leq 1,$$

hence

$$\min_K \left(\theta_n(\alpha) \int_\Omega m_2 |u|^p + \alpha \int_\Omega m_1 |u|^p \right) \leq 1. \tag{4.7}$$

On the other hand, if $K \notin \Gamma_n^1$ we have

$$\min_K \left(\theta_n(\alpha) \int_\Omega m_2 |u|^p + \alpha \int_\Omega m_1 |u|^p \right) \leq 0. \tag{4.8}$$

From (4.7) and (4.8), we get

$$\min_K \left(\theta_n(\alpha) \int_\Omega m_2 |u|^p + \alpha \int_\Omega m_1 |u|^p \right) \leq 1, \quad \forall K \in \Gamma_n$$

thus

$$\sup_{K\in\Gamma_n}\min_{u\in K}\left(\theta_n(\alpha)\int_\Omega m_2|u_k|^p+\alpha\int_\Omega m_1|u_k|^p\right)\leq 1.$$

Hence

$$\frac{1}{\lambda_n(\theta_n(\alpha)m_2+\alpha m_1)}\leq 1,$$

which gives

$$\lambda_n(\theta_n(\alpha)m_2+\alpha m_1)\geq 1. \tag{4.9}$$

On other hand, for $K\notin\Gamma_n^1$, we have

$$\min_K\left(t_n(\alpha)\int_\Omega m_2|u|^p+\alpha\int_\Omega m_1|u|^p\right)\leq 0,$$

hence

$$\sup_{K\notin\Gamma_n^1}\min_{u\in K}\left(t_n(\alpha)\int_\Omega m_2|u|^p+\alpha\int_\Omega m_1|u|^p\right)\leq 0. \tag{4.10}$$

Since $\lambda_n(\alpha m_1+t_n(\alpha)m_2)=1$, we have

$$\sup_{K\in\Gamma_n}\min_{u\in K}\left(t_n(\alpha)\int_\Omega m_2|u|^p+\alpha\int_\Omega m_1|u|^p\right)=1. \tag{4.11}$$

From (4.10) and (4.11) we deduce that

$$\sup_{K\in\Gamma_n^1}\min_{u\in K}\left(t_n(\alpha)\int_\Omega m_2|u|^p+\alpha\int_\Omega m_1|u|^p\right)=1. \tag{4.12}$$

Assume by contradiction that, there exists $K_1\in\Gamma_n^1$ such that

$$t_n(\alpha)>\max_{u\in K_1}\frac{1-\int_\Omega\alpha m_1|u|^p}{\int_\Omega m_2|u|^p},$$

then for all $u\in K_1$ we have

$$t_n(\alpha)\int_\Omega m_2|u|^p+\int_\Omega\alpha m_1|u|^p>1.$$

Since K_1 is compact, we get

$$\min_{u\in K_1}(\int_\Omega\alpha m_1|u|^p+t_n(\alpha)\int_\Omega m_2|u|^p)>1,$$

so we conclude that

$$\sup_{K\in\Gamma_n^1}\min_{u\in K}\left(\int_\Omega\alpha m_1|u|^p+t_n(\alpha)\int_\Omega m_2|u|^p\right)>1. \tag{4.13}$$

This contradicts the equality (4.12). So for all $K\in\Gamma_n^1$ we have

$$t_n(\alpha)\leq\max_{u\in K}\frac{1-\int_\Omega\alpha m_1|u|^p}{\int_\Omega m_2|u|^p}.$$

Hence

$$t_n(\alpha)\leq\inf_{\Gamma_n^1}\max_{u\in K}\frac{1-\int_\Omega\alpha m_1|u|^p}{\int_\Omega m_2|u|^p}=\theta_n(\alpha). \tag{4.14}$$

Using the monotonicity of λ_n with respect to the weight (see Proposition 3.2), (4.9) and (4.14) we get $1=\lambda_n(\alpha m_1+t_n(\alpha)m_2)\geq\lambda_n(\alpha m_1+\theta_n(\alpha)m_2)\geq 1$. Hence we deduce that $t_n(\alpha)=\theta_n(\alpha)$.

2. Let $K \in \Gamma_n^1$, we define a functional $h(\cdot)$ in $K \times \mathbb{R}$ by
$$h(u,\alpha) = \frac{1-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p}.$$
For $(\alpha,\alpha') \in \mathbb{R}^2$, we have
$$h(u,\alpha) - h(u,\alpha') = \frac{(\alpha'-\alpha)\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p},$$
hence
$$|h(u,\alpha) - h(u,\alpha')| \leq \delta|\alpha-\alpha'| \quad \text{where} \quad \delta = \frac{||m_1||_\infty}{ess \inf\limits_{\Omega^\star_{m_2}} m_2}.$$
It follows that
$$h(u,\alpha') - \delta|\alpha-\alpha'| \leq h(u,\alpha) \leq h(u,\alpha') + \delta|\alpha-\alpha'|.$$
So we conclude that we have
$$\sup_K h(u,\alpha') - \delta|\alpha-\alpha'| \leq \sup_K h(u,\alpha) \leq \sup_K h(u,\alpha') + \delta|\alpha-\alpha'|.$$
Since K is arbitrary,
$$\inf_{K\in\Gamma_n^1}\sup_K h(u,\alpha') - \delta|\alpha-\alpha'| \leq \inf_{K\in\Gamma_n^1}\sup_K h(u,\alpha) \leq \inf_{K\in\Gamma_n^1}\sup_K h(u,\alpha') + \delta|\alpha-\alpha'|.$$
Hence we get
$$|t_n(\alpha) - t_n(\alpha')| \leq \delta|\alpha-\alpha'|.$$

3. For $\alpha \in]\lambda_n(m_1), +\infty[$, we have $t_n(\alpha) < 0$. Denote $\Gamma_n^{1+} = \{K \in \Gamma_n^1 : \inf\limits_K \int_\Omega m_1|u|^p \geq 0\}$. Since $t_n(\alpha) = \inf\limits_{\Gamma_n^1}\max\limits_K \frac{1-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p} < 0$, we conclude that there exists $K \in \Gamma_n^1$ such that $\max\limits_K \frac{1-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p} < 0$. Hence $\max\limits_K(1-\alpha\int_\Omega m_1|u|^p) < 0$, it follows that $\Gamma_n^{1+} \neq \emptyset$ and $t_n(\alpha) = \inf\limits_{\Gamma_n^{1+}}\max\limits_K \frac{1-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p}$.
Let $K \in \Gamma_n^{1+}$ and α, $\alpha' \in]\lambda_n(m_1), +\infty[$, assume $\alpha \geq \alpha'$, we get
$$\frac{1-\alpha'\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p} \geq \frac{1-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p} \quad \forall u \in K.$$
It follows that
$$\max_{u\in K}\frac{1-\alpha'\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p} \geq \max_{u\in K}\frac{1-\alpha\int_\Omega m_1|u|^p}{\int_\Omega m_2|u|^p}.$$
Since $K \in \Gamma_n^{1+}$ is arbitrary, we conclude that $t_n(\alpha') \geq t_n(\alpha)$. Hence $t_n(\cdot)$ is decreasing.
Similarly we show that $t_n(\cdot)$ is increasing in $]-\infty, \lambda_{-n}(m_1)[$.

4. The case $m_1 \geq 0$ is treated in the same way. ■

Proof of Theorem 2.3.

1. For $\alpha > \lambda_n(m_1)$, set $g(\alpha) = \frac{-t_n(\alpha)}{\alpha}$. We will show that $g(\cdot)$ is an increasing function on $]\lambda_n(m_1), +\infty[$. Indeed let α, $\alpha^{'} \in]\lambda_n(m_1), +\infty[$ such that $\alpha > \alpha^{'}$. Assume by contradiction that $\frac{t_n(\alpha)}{\alpha} \geq \frac{t_n(\alpha^{'})}{\alpha^{'}}$. Hence we have $m_1 + \frac{t_n(\alpha)}{\alpha}m_2 \geq m_1 + \frac{t_n(\alpha^{'})}{\alpha^{'}}m_2$. By proposition 3.2, we get
$$\alpha = \lambda_n(m_1 + \frac{t_n(\alpha)}{\alpha}m_2) \leq \lambda_n(m_1 + \frac{t_n(\alpha^{'})}{\alpha^{'}}m_2) = \alpha^{'},$$

which gives a contradiction. Then $\dfrac{t_n(\alpha)}{\alpha} < \dfrac{t_n(\alpha')}{\alpha'}$, this implies $g(\alpha) > g(\alpha')$, i.e., $g(.)$ is increasing. On the other hand, we have

$$\alpha m_1(x) + t_n(\alpha) m_2(x) > 0 \text{ in } \Omega_\alpha \text{ with } \mathrm{meas}(\Omega_\alpha) > 0.$$

Since $\alpha > 0$ and $t_n(\alpha) \le 0$ (see theorem 2.2). The inequality above implies that $\Omega_\alpha \subset \Omega^+_{m_1}$. Hence

$$\frac{-t_n(\alpha)}{\alpha} < \frac{m_1}{m_2} \quad \forall x \in \Omega_\alpha \subset \Omega^+_{m_1}.$$

So we conclude that

$$g(\alpha) \le ess\sup_{\Omega^+_{m_1}} \frac{m_1}{m_2} \le \frac{||m_1||_\infty}{ess\inf\limits_{\Omega^\star_{m_1}} m_2}.$$

It follows that g is bounded from above and is an increasing function.
Let $l = \lim\limits_{\alpha\to+\infty} g(\alpha)$, we have

$$l \le ess\sup_{\Omega^+_{m_1}} \frac{m_1}{m_2} \tag{4.15}$$

and

$$m_1 + \frac{t_n(\alpha)}{\alpha} m_2 \to m_1 - l m_2 \quad \text{in} \quad L^\infty(\Omega). \tag{4.16}$$

Since $\lambda_n(m_1 + \dfrac{t_n(\alpha)}{\alpha} m_2) = \alpha \to +\infty$, from proposition 3.3 and (4.16), we deduce that $m_1 - l m_2 \le 0 \quad \forall x \in \Omega$, thus

$$ess\sup_{\Omega^+_{m_1}} \frac{m_1}{m_2} \le l. \tag{4.17}$$

The inequalities (4.15) and (4.17) yield the result.

2. The proof can be carried out as we did in the first result. We consider the mapping $f(\alpha) = \dfrac{-t_n(\alpha)}{\alpha}$, we affirm that f is decreasing on $]-\infty, \lambda_{-n}(m_1)[$. Taking into account that $t_n(\alpha) \le 0$, we conclude that

$$f(\alpha) \ge ess\inf_{\Omega^-_{m_1}} \frac{m_1}{m_2}.$$

Hence, f is bounded from below. Let $k = \lim\limits_{\alpha\to-\infty} f(\alpha)$, we have

$$k \ge ess\inf_{\Omega^-_{m_1}} \frac{m_1}{m_2}$$

and

$$-m_1 - \frac{t_n(\alpha)}{\alpha} m_2 \to -m_1 + k m_2 \quad \text{in} \quad L^\infty(\Omega).$$

Since $\lambda_n\left(-m_1 - \frac{t_n(\alpha)}{\alpha} m_2\right) = |\alpha| \to +\infty$, we get $-m_1 + k m_2 \le 0$ in Ω. This yields $k \le ess\inf\limits_{\Omega^-_{m_1}} \dfrac{m_1}{m_2}$. Hence, we get

$$\lim_{\alpha\to-\infty} \frac{t_n(\alpha)}{\alpha} = -ess\inf_{\Omega^-_{m_1}} \frac{m_1}{m_2}.$$

3. We show the third result in a similar way. ■

References

1. P. A. Binding, Y. X. Huang. The principal eigencurve of the p-Laplacian. Differntial and Integral Equations 1995; 405–414.

2. P. A. Binding, Y. X. Huang. Bifurcation from eigencurves of the p-laplacian. Differntial and Integral Equations 1995; 8: 415–428.

3. M. Cuesta. Eigenvalue problems for the p-Laplacian with indefinite weights. EJDE 2001; 33: 1–9.

4. A. Dakkak. M. Hadda, Eigencurves of the p-Laplacian weights and their asymptotic behavior, EJDE 2017; 35: 1–7.

5. A. Dakkak, M. Moussaoui. On the second eigencurve for the p-Laplacian operator with weight. BSPM 2017; 35: 281–289.

6. D. G. De Figueiredo. Positive solution of semi-linear elliptic equation. Lecture Notes in Mathematics 1982; 957: 34–87.

7. D. G. De Figueiredo, J. P. Gossez. Strict monotonicity of eigenvalues and unique continuation. Comm Par Diff Equat 1992; 17: 339–346.

8. M R○užička. Electrorheological Fluids: Modeling and Mathematical Theory, Springer-verlag, Berlin, 2000.

9. A. Szulkin. Ljusternik-Schnirelmann theory on C^1-manifolds. Ann I H Poincaré Anal non linéaire 1988; 5: 119–139.

M. Moussaoui,
LANOL Laboratory,
Faculty of sciences,
Department of Mathematics,
Oujda, Morocco.
E-mail address: mmoussaoui_2@yahoo.fr

and

A. Dakkak,
LSI Laboratory,
Sidi Mohamed ben Abdellah University,
Polydisciplinary Faculty,
Taza, Morocco
E-mail address: dakkakahmed@hotmail.com

and

O. Chakrone,
LANOL Laboratory,
Faculty of sciences,
Department of Mathematics,
Oujda, Morocco.
E-mail address: chakrone@yahoo.fr