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On a Class Of h-Fourier Integral Operators With The Complex Phase

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ABSTRACT: In this work, we study the L^2-boundedness and L^2-compactness of a class of h-Fourier integral operators with the complex phase. These operators are bounded (respectively compact) if the weight of the amplitude is bounded (respectively tends to 0).

Key Words: h-Fourier integral operators, h-pseudodifferential operators, complex function, Symbol and phase.

Contents

1. Introduction

Since 1970, numerous mathematicians are interested in these types of operators:

$$F\varphi(x) = (2\pi h)^{-n} \iint e^{\frac{i}{h}(S(x,\theta)-y\theta)} a(x,\theta)\varphi(y)\,dyd\theta, \varphi \in S(\mathbb{R}^n)\ . \tag{1.1}$$

like [6,12,1,9,7,18]. The integral operators (1.1) appear naturally in the expression of the solutions of the semiclassical hyperbolic partial differential equations and when expressing the C^∞ solution of the associated Cauchy's problem. Two C^∞ functions appear in (1.1): the phase function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ and the amplitude a.

In 1974 Melin and Sjostrand [15] studied an extension of the computation of the Fourier integral in the case where the phase functions assume complex values.

Our work consist a spectral study the L^2-boundedness and L^2-compactness of a class of h-Fourier integral operators with the complex phase; we're more particularly interested in continuity studies and on compactness on $L^2(\mathbb{R}^n)$.

It was proven in [1] by a very elaborate demonstration and under certains conditions (relatively strong) on the phase function ϕ and the amplitude a that all operators of the form:

$$(I(a,\phi;h)\psi)(x) = (2\pi h)^{-n} \iint\limits_{\mathbb{R}^n_y\ \mathbb{R}^N_\theta} e^{\frac{i}{h}\phi(x,\theta,y)} a(x,\theta,y)\psi(y)\,dyd\theta$$

are bounded on L^2 , where $\psi \in \mathcal{S}(\mathbb{R}^n)$ (*the Schwartz space*), $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$.

The used technique is to show that $I(a,\phi)I^*(a,\phi)$, $I^*(a,\phi)\ I(a,\phi)$ are $h-$pseudodifferential and apply the Calderòn-Vaillancourt's theorem (here $I^*(a,\phi)$ is the adjoint of $I(a,\phi)$).

In this paper, we will apply the same technic of [1] to establish L^2-boundedness and L^2-compactness of form (1.1) operators. That's why we will give brief demonstrations .

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We mainly prove the continuity of the operator F_h on $L^2(\mathbb{R}^n)$ when the weight of the amplitude a is bounded. Moreover, F_h is compact on $L^2(\mathbb{R}^n)$ if this weight tends to zero. Using the estimate given in [17,19] for $h-$pseudodifferential ($h-$admissible) operators, we also establish an L^2-estimate of $\|F_h\|$.

We note that if the amplitude a is just bounded, the Fourier integral operator F is not necessarily bounded on $L^2(\mathbb{R}^n)$.

2. A general class of h-Fourier integral operators with the complex phase

We consider the following integral transformations

$$\left(I\left(a,\phi;h\right)\psi\right)(x) = (2\pi h)^{-n} \iint_{\mathbb{R}^n_y\ \mathbb{R}^N_\theta} e^{\frac{i}{h}\phi(x,\theta,y)} a\left(x,\theta,y\right)\psi\left(y\right)dyd\theta \qquad (2.1)$$

for $\psi \in \mathcal{S}\left(\mathbb{R}^n\right)$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$ ((if $N = 0$, θ doesn't appear in (2.1)).

In general, the integral (2.1) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander [13]. The phase function and the amplitude a are assumed to satisfy the following hypothesis:

$(H1)$

$$\phi \in C^\infty\left(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y, \mathbb{C}\right)$$

when ϕ *is* a complex function , $Im(\phi)$ is non negative.

$(H2)$ For all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$, there exists $C_{\alpha,\beta,\gamma} > 0$, such that :

$$\left|\partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha \phi\left(x,\theta,y\right)\right| \le C_{\alpha,\beta,\gamma}\lambda^{(2-|\alpha|-|\beta|-|\gamma|)_+}\left(x,\theta,y\right)$$

where

$$\lambda\left(x,\theta,y\right) = \left(1+|x|^2+|\theta|^2+|y|^2\right)^{1/2},$$
$$\left(2-|\alpha|-|\beta|-|\gamma|\right)_+ = \max\left(2-|\alpha|-|\beta|-|\gamma|,0\right)$$

$(H3)$ There exists $K_1, K_2 > 0$, such that:

$$K_1\lambda(x,\theta,y) \le \lambda(\partial_y\phi,\partial_\theta\phi,y) \le K_2\lambda(x,\theta,y), \text{ for all } (x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$$

$(H3)^*$ There exists $K_1^*, K_2^* > 0$, such that:

$$K_1^*\lambda(x,\theta,y) \le \lambda\left(x,\partial_\theta\phi,\partial_x\phi\right) \le K_2^*\lambda(x,\theta,y), \text{ for all } (x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$$

For any open subset Ω of $\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0,1]$; we set:

$$\Gamma^\mu_\rho\left(\Omega\right) = \left\{ \begin{array}{c} a \in C^\infty\left(\Omega\right);\ \forall\ (\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n, \exists\ C_{\alpha,\beta,\gamma} > 0; \\ \left|\partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha a(x,\theta,y)\right| \le C_{\alpha,\beta,\gamma}\lambda^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)}(x,\theta,y) \end{array} \right\}$$

When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, we denote $\Gamma^\mu_\rho\left(\Omega\right) = \Gamma^\mu_\rho$. To give a meaning to the right hand side of (2.1), we consider $g \in \mathcal{S}\left(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y\right)$, $g\left(0\right) = 1$.

If $a \in \Gamma^\mu_0$, we define

$$a_\sigma\left(x,\theta,y\right) = g\left(x/\sigma,\theta/\sigma,y/\sigma\right)a\left(x,\theta,y\right),\ \sigma > 0$$

Theorem 2.1. *If ϕ satisfies $(H1), (H2), (H3)$, $(H3)^*$ and if $a \in \Gamma^\mu_0$ then:*

1. *For all $\psi \in \mathcal{S}\left(\mathbb{R}^n\right)$, $\lim\limits_{\sigma\to+\infty}\left[I\left(a_\sigma,\phi;h\right)\psi\right](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g. We define:*

$$\left(I\left(a,\phi;h\right)\psi\right)(x) := \lim_{\sigma\to+\infty}\left(I\left(a_\sigma,\phi;h\right)\psi\right)(x),$$

2. *$I(a,\phi;h) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ and $I(a,\phi;h) \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$ (here $\mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ (resp. $\mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$ is the space of bounded linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$) and $\mathcal{S}'(\mathbb{R}^n)$ the space of all distributions with temperate growth on $\mathbb{R}^n$).*

Proof. Let $\eta \in C^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}\eta \subseteq [-1,2]$ and $\eta \equiv 1$ on $[0,1]$. For all $\epsilon > 0$, we set

$$\omega_\epsilon(x,\theta,y) = \eta\left(\frac{|\partial_y\phi|^2 + |\partial_\theta\phi|^2}{\varepsilon\lambda(x,\theta,y)^2}\right)$$

The hypothesis $(H3)$ implies that there exsits $C>0$ such that we have on the support of ω_ϵ

$$\lambda(x,\theta,y) \le C\left[(1+|y|^2)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\lambda(x,\theta,y)\right]$$

Therefore, there exists ε_0 and a constant C_0, such that $\forall\varepsilon \le \varepsilon_0$ we have on the support of ω_ϵ

$$\lambda(x,\theta,y) \le C_0(1+|y|^2)^{\frac{1}{2}}.$$

In the sequel, we fix $\epsilon = \epsilon_0$. Then it is immediate that $I(\omega_\epsilon a_\sigma,\phi;h)\psi$ is an absolutely convergent integral and we have

$$I(\omega_\epsilon a,\phi;h)\psi = \lim_{\sigma\to+\infty} I(\omega_\epsilon a_\sigma,\phi;h)\psi. \tag{2.2}$$

Using $(H2)$ we prove also that $I(\omega_\epsilon a,\phi;h)\psi$ is a continuous operator from $\mathcal{S}(\mathbb{R}^n)$ into itself. To study $\lim\limits_{\sigma\to+\infty} I((1-\omega_\epsilon)a_\sigma,\phi;h)\psi$ we introduce the operator

$$L = -ih\left(|\partial_y\phi|^2 + |\partial_\theta\phi|^2\right)^{-1}\sum_{l=1}^{n}\left[(\partial_{y_l}\phi)\,\partial_{y_l} - (\partial_{\theta_l}\phi)\,(\partial_{\theta_l})\right].$$

Clearly we have

$$L(e^{\frac{i}{h}\phi}) = e^{\frac{i}{h}\phi}. \tag{2.3}$$

Let Ω_0 be the open subset of $\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n$ defined by

$$\Omega_0 = \left\{(x,\theta,y)\in\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n; |\partial_y\phi|^2 + |\partial_\theta\phi|^2 > \frac{\epsilon_0}{2}\lambda(x,\theta,y)^2\right\}.$$

We need the following lemma.

Lemma 2.2. *For all integer $q\ge 0$, and $b\in C^\infty(\mathbb{R}^n_y\times\mathbb{R}^N_\theta)$, we have*

$$\left({}^tL\right)^q((1-\omega_{\epsilon_0})b) = \sum_{|\alpha|+|\beta|\le q} g^q_{\alpha,\beta}\partial_y^\alpha\partial_\theta^\beta\left((1-\omega_{\epsilon_0})b\right),$$

${}^tL$ designantes the transpose of L, $g^q_{\alpha,\beta}\in\Gamma_0^{-q}(\Omega_0)$ and depend only on ϕ.

We prove the lemma by recurrence. It is obvious for $q=0$. Now we see easily that

$${}^tL = \sum_l (F_l\partial_{y_l} + G_l\partial_{\theta_l}) + H, \tag{2.4}$$

where F_l, G_l in $\Gamma_0^{-1}(\Omega_0)$, and $H\in\Gamma_0^{-2}(\Omega_0)$ (wich results from $(H2)$). Therefore, the recurrence is immediately proved.

We have from (2.3), $\forall q\ge 0$

$$\begin{aligned}&I((1-\omega_{\epsilon_0})a_\sigma,\phi;h)\psi(x)\\ &= \frac{1}{(2\pi h)^n}\iint\limits_{\mathbb{R}^n_y\ \mathbb{R}^N_\theta} e^{\frac{i}{h}\phi(x,\theta,y)}\left({}^tL\right)^q\left((1-\omega_{\epsilon_0})a_\sigma\psi;h\right)(x,\theta,y)\,dyd\theta\ .\end{aligned} \tag{2.5}$$

Now $\left({}^tL\right)^q((1-\omega_{\epsilon_0})a_\sigma\psi)$ described (when q varies) a bound of $\Gamma_0^{\mu-q}$, and for all $(x,\theta,y)\in\mathbb{R}_x^n\times\mathbb{R}_\theta^N\times\mathbb{R}_y^n$

$$\lim_{\sigma\to\infty}\left({}^tL\right)^q((1-\omega_{\epsilon_0})a_\sigma\psi)(x,\theta,y)=\left({}^tL\right)^q((1-\omega_{\epsilon_0})a\psi)(x,\theta,y)\,. \tag{2.6}$$

Finally, $\forall s>n+N$ we have

$$\iint\limits_{\mathbb{R}_y^n\ \mathbb{R}_\theta^N}\lambda^{-s}(x,\theta,y)\,dyd\theta\le C_s\lambda^{n+N-s}(x). \tag{2.7}$$

So it results from (2.5),(2.7) and using Lebesgue's theorem we have

$$\begin{aligned}&\lim_{\sigma\to\infty}I((1-\omega_{\epsilon_0})a_\sigma,\phi;h)\,\psi(x)\\&=(2\pi h)^{-n}\iint\limits_{\mathbb{R}_y^n\ \mathbb{R}_\theta^N}e^{\frac{i}{h}\phi(x,\theta,y)}\left({}^tL\right)^q((1-\omega_{\epsilon_0})a\psi;h)(x,\theta,y)\,dyd\theta\ .\end{aligned} \tag{2.8}$$

where $q>n+N+\mu$. From (2.2)and(2.8) we can prove the first part of the theorem.

Now let us show that $I((1-\omega_{\epsilon_0})a_\sigma,\phi;h)$ is continuons. Taking account of (2.4)and(2.8) , we get

$$I((1-\omega_{\epsilon_0})a,\phi;h)\,\psi(x)=(2\pi h)^{-n}\sum_{|\gamma|\le q}\ \iint\limits_{\mathbb{R}_y^n\ \mathbb{R}_\theta^N}e^{\frac{i}{h}\phi(x,\theta,y)}b_\gamma^{(q)}(x,\theta,y)\,\partial_y^\gamma\psi(y)dyd\theta\ , \tag{2.9}$$

with $b_\gamma^{(q)}\in\Gamma_0^{\mu-q}$. On the other hand, we have

$$x^\alpha\partial_x^\beta(e^{\frac{i}{h}\phi(x,\theta,y)}b_\gamma^{(q)}(x,\theta,y))\in\Gamma_0^{\mu-q+|\alpha|+|\beta|}. \tag{2.10}$$

We deduce from (2.9)and(2.10) that, for all $q>n+N+\mu+|\alpha|+|\beta|$, there exists a constant $C_{\alpha,\beta,q}$ such that

$$\left|x^\alpha\partial_x^\beta I((1-\omega_{\epsilon_0})a,\phi;h)\,\psi(x)\right|\le C_{\alpha,\beta,q}\sup_{\substack{x\in\mathbb{R}^n\\|\gamma|\le q}}|\partial_x^\gamma\psi(x)|\,,$$

which proves the continuity of $I((1-\omega_{\epsilon_0})a,\phi;h)$. □

Example 2.3. *Let us give two examples of operators of the form* (1.1) *which satisfy* $(H1)$ *to* $(H3)^*$*:*

1. *The Fourier transform*

$$\mathcal{S}(\mathbb{R}^n)\ni\psi\longmapsto\mathcal{F}\psi(x)=\int\limits_{\mathbb{R}^n}e^{-\frac{i}{h}xy}\psi(y)\,dy,$$

2. *Pseudodifferential operators*

$$\mathcal{S}(\mathbb{R}^n)\ni\psi\longmapsto Op\psi(x)=(2\pi h)^{-n}\int\limits_{\mathbb{R}^{2n}}e^{\frac{i}{h}(x-y)\theta}a(x,y,\theta)\,\psi(y)\,dyd\theta,$$

$a\in\Gamma_0^\mu\left(\mathbb{R}^{3n}\right)$.

3. Special form of the phase function

We consider the phase function $\phi(x,y,\theta)=S(x,\theta)-y\theta$. Where

$$S(x,\theta)=f(x,\theta)+iT(x,\theta), \tag{3.1}$$

and S satisfies: $(G1)$ $S\in C^{\infty}\left(\mathbb{R}_x^n\times\mathbb{R}_\theta^n,\mathbb{C}\right)$, ($S$ is a complex function)

$(G2)$ For all $(\alpha,\beta)\in\mathbb{N}^{2n}$, there exists $C'_{\alpha,\beta}>0$,

$$\left|\partial_x^\alpha\partial_\theta^\beta f(x,\theta)\right|\leq C'_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)}$$

$(G3)$ For all $(x,\theta)\in\mathbb{R}^{2n}$, $T(x,\theta)$ is nonnegative, and for all $(\alpha,\beta)\in\mathbb{N}^{2n}$, there exists $C''_{\alpha,\beta}>0$,

$$\left|\partial_x^\alpha\partial_\theta^\beta T(x,\theta)\right|\leq C''_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)}$$

$(G4)$ There exists $\delta_0>0$,

$$\inf_{x,\theta\in\mathbb{R}^n}\left|\det\frac{\partial^2 f}{\partial x\partial\theta}(x,\theta)\right|\geq\delta_0.$$

Lemma 3.1. *[16] If S satisfies $(G1)$, $(G2)$, $(G3)$ and $(G4)$. Then the function $\phi(x,y,\theta)=S(x,\theta)-y\theta$ satisfies $(H1)$, $(H2)$, $(H3)$ and $(H3)^*$.*

Lemma 3.2. *[16] If S satisfies $(G1)$, $(G2)$, $(G3)$ and $(G4)$, then there exists $C_2>0$, such that for all $(x,\theta),(x',\theta')\in\mathbb{R}^{2n}$*

$$|x-x'|+|\theta-\theta'| \quad\leq\quad C_2\left[\left|(\partial_\theta f)(x,\theta)-(\partial_\theta f)(x',\theta')\right|+\left|\theta-\theta'\right|\right] \tag{3.2}$$

Lemma 3.3. *[8] If S satisfies $(G1),(G2)$ et $(G3)$. Then there exists a constant $\varepsilon_0>0$, such that the phase function ϕ belongs to $\Gamma_1^2(\Omega_{\phi,\varepsilon_0})$, where*

$$\Omega_{\phi,\varepsilon_0}=\left\{(x,\theta,y)\in\mathbb{R}^{3n};\ |\partial_\theta\phi(x,\theta,y)|^2<\varepsilon_0\left(|x|^2+|y|^2+|\theta|^2\right)\right\}$$

Proposition 3.4. *[8] If $(x,\theta)\longmapsto a(x,\theta)$ belongs to $\Gamma_k^m\left(\mathbb{R}_x^n\times\mathbb{R}_\theta^n\right)$, then the function $(x,\theta,y)\to a(x,\theta)$ belongs to $\Gamma_k^m\left(\mathbb{R}_x^n\times\mathbb{R}_\theta^n\times\mathbb{R}_y^n\right)\cap\Gamma_k^m\left(\Omega_{\phi,\varepsilon_0}\right)$, $k\in\{0,1\}$.*

4. L^2-boundedness and L^2-compactness of F_h with the complex phase

Theorem 4.1. *Let F_h be the integral operator of distribution kernel*

$$K(x,y)=\int_{\mathbb{R}^n} e^{\frac{i}{h}f(x,\theta)-\frac{T(x,\theta)}{h}}a(x,\theta)\widehat{d\theta} \tag{4.1}$$

where $\widehat{d\theta}=(2\pi)^{-n}d\theta$, $a\in\Gamma_k^m(\mathbb{R}_{x,\theta}^{2n})$, $k=0,1$ and S satisfies $(G1)$, $(G2)$, $(G3)$ and $(G4)$. Then FF^* and F^*F are h-pseudodifferential operators with symbol in $\Gamma_k^{2m}(\mathbb{R}^{2n})$, $k=0,1$, given by

$$\begin{aligned}\sigma(FF^*)(x,\partial_x f(x,\theta)) &\equiv e^{-\frac{2T(x,\theta)}{h}}|a(x,\theta)|^2\left|(\det\frac{\partial^2 f}{\partial\theta\partial x})^{-1}(x,\theta)\right|\\ \sigma(F^*F)(\partial_\theta f(x,\theta),\theta) &\equiv e^{-\frac{2T(x,\theta)}{h}}|a(x,\theta)|^2\left|(\det\frac{\partial^2 f}{\partial\theta\partial x})^{-1}(x,\theta)\right|\end{aligned}$$

We denote here $a\equiv b$ for $a,b\in\Gamma_k^{2p}(\mathbb{R}^{2n})$ if $(a-b)\in\Gamma_k^{2p-2}(\mathbb{R}^{2n})$ and σ stands for the symbol.

Proof. If $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} F_h u(x) &= \int\limits_{\mathbb{R}^n} K(x,y)\, u(y)\, dy = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(f(x,\theta)+iT(x,\theta)-y\theta\,)} a(x,\theta)u(y)dy\widehat{d\theta} \\ &= \int\limits_{\mathbb{R}^n} e^{\frac{i}{h}f(x,\theta)-\frac{T(x,\theta)}{h}}\, a(x,\theta)\, (\int\limits_{\mathbb{R}^n} e^{-\frac{i}{h}y\theta}\, u(y)dy)\widehat{d\theta} \\ &= \int\limits_{\mathbb{R}^n} e^{\frac{i}{h}f(x,\theta)-\frac{T(x,\theta)}{h}}\, a(x,\theta)\, \mathcal{F}u(\theta)\widehat{d\theta}, \end{aligned} \tag{4.2}$$

where $\mathcal{F}$ the Fourier transform and for all $v \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} < F_h u, v >_{L^2(\mathbb{R}^n)} &= \int\limits_{\mathbb{R}^n}(\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}f(x,\theta)-\frac{T(x,\theta)}{h}}\, a(x,\theta)\mathcal{F}u(\theta)\widehat{d\theta})\, \overline{v(x)}\, dx \\ < F_h u, v >_{L^2(\mathbb{R}^n)} &= \int\limits_{\mathbb{R}^n} \widehat{u}(\theta)(\int\limits_{\mathbb{R}^n} \overline{e^{-\frac{i}{h}f(x,\theta)-\frac{T(x,\theta)}{h}}\, \overline{a(x,\theta)}\, v(x)\, dx})\widehat{d\theta}, \end{aligned}$$

then

$$< Fu\,(x)\,, v\,(x) >_{L^2(\mathbb{R}^n)} = (2\pi h)^{-n} < \mathcal{F}u\,(\theta)\,, \mathcal{F}\left((F^* v)\right)(\theta) >_{L^2(\mathbb{R}^n)},$$

and,

$$\mathcal{F}\left((F_h^* v)\right)(\theta) = \int\limits_{\mathbb{R}^n} e^{-\frac{i}{h}f(\widetilde{x},\theta)-\frac{T(\widetilde{x},\theta)}{h}}\, \overline{a}\,(\widetilde{x},\theta)\, v(\widetilde{x})d\widetilde{x}. \tag{4.3}$$

We have,

$$(FF^* v)\,(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(f(x,\theta)-f(\widetilde{x},\theta))} e^{\frac{-\left(T(x,\theta)+T(\widetilde{x},\theta)\right)}{h}}\, a\,(x,\theta)\, \overline{a}\,(\widetilde{x},\theta)\, v(\widetilde{x})d\widetilde{x}\widehat{d\theta}, \tag{4.4}$$

for all $v \in \mathcal{S}(\mathbb{R}^n)$. The main idea to show that FF^* is a $h-$pseudodifferential operator, is to use the fact that $f\,(x,\theta) - f\,(\widetilde{x},\theta)$ can be expressed by the scalar product $< x - \widetilde{x}, \xi\,(x,\widetilde{x},\theta) >$ after considering the change of variables

$$(x,\widetilde{x},\theta) \to (x,\widetilde{x},\xi = \xi\,(x,\widetilde{x},\theta))\,.$$

The distribution kernel of FF^* is

$$K\,(x,\tilde{x}) = \int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(f(x,\theta)-f(\tilde{x},\theta))} e^{\frac{-\left(T(x,\theta)+T(\widetilde{x},\theta)\right)}{h}}\, a(x,\theta)\overline{a}\,(\tilde{x},\theta)\, \widehat{d\theta}.$$

We obtain from(3.2)that if

$$|x - \widetilde{x}| \geq \frac{\varepsilon}{2}\lambda\,(x,\widetilde{x},\theta) \ \ (\text{where } \varepsilon > 0 \text{ is } \ \text{sufficiently small})$$

Then

$$|(\partial_\theta f)\,(x,\theta) - (\partial_\theta f)\,(\widetilde{x},\theta)| \geq \frac{\varepsilon}{2C_2}\lambda\,(x,\widetilde{x},\theta)\,. \tag{4.5}$$

Choosing $C^\infty\,(\mathbb{R})$ such that

$$\begin{cases} \omega\,(x) \geq 0, & \forall x \in \mathbb{R} \\ \omega\,(x) = 1 & \text{si} \quad x \in \left[-\frac{1}{2},\frac{1}{2}\right] \\ \text{supp}\omega \subset & \left]-1,1\right[\end{cases}$$

and setting

$$\begin{cases} b\,(x,\tilde{x},\theta) := e^{\frac{-(T(x,\theta)+T(\tilde{x},\theta))}{h}} a(x,\theta)\overline{a}\,(\tilde{x},\theta) = b_{1,\varepsilon}\,(x,\tilde{x},\theta) + b_{2,\varepsilon}\,(x,\tilde{x},\theta) \\ b_{1,\varepsilon}\,(x,\tilde{x},\theta) = \omega\left(\frac{|x-\tilde{x}|}{\varepsilon\lambda(x,\tilde{x},\theta)}\right) b\,(x,\tilde{x},\theta) \\ b_{2,\varepsilon}\,(x,\tilde{x},\theta) = \left[1 - \omega\left(\frac{|x-\tilde{x}|}{\varepsilon\lambda(x,\tilde{x},\theta)}\right)\right] b\,(x,\tilde{x},\theta)\,. \end{cases}$$

We have

$$K\left(x,\widetilde{x}\right)=K_{1,\varepsilon}\left(x,\widetilde{x}\right)+K_{2,\varepsilon}\left(x,\widetilde{x}\right),$$

where

$$K_{j,\varepsilon}\left(x,\tilde{x}\right)=\int\limits_{\mathbb{R}^n}e^{\frac{i}{h}\left(f(x,\theta)-f(\tilde{x},\theta)\right)}b_{j,\varepsilon}\left(x,\tilde{x},\theta\right)\widehat{d\theta},\ j=1,2.$$

We will study separately the kernels $K_{1,\varepsilon}$ and $K_{2,\varepsilon}$.

The study of $K_{2,\varepsilon}$. We shall show that for all h, we have

$$K_{2,\varepsilon}\left(x,\widetilde{x}\right)\in\mathcal{S}\left(\mathbb{R}^n\times\mathbb{R}^n\right).$$

Indeed, let

Then L is a linear partial differential operator L of order 1 such that

$$L\left(e^{\frac{i}{h}\left(f(x,\theta)-f(\tilde{x},\theta)\right)}\right)=e^{\frac{i}{h}\left(f(x,\theta)-f(\tilde{x},\theta)\right)},$$

$$\text{where } L=-ih\left|\left(\partial_\theta f\right)\left(x,\theta\right)-\left(\partial_\theta f\right)\left(\widetilde{x},\theta\right)\right|^{-2}\sum_{l=1}^{n}\left[\left(\partial_{\theta_l}f\right)\left(x,\theta\right)-\left(\partial_{\theta_l}f\right)\left(\widetilde{x},\theta\right)\right]\partial_{\theta_l}.$$

The transpose operator of L is

$${}^tL=\sum_{l=1}^{n}F_l\left(x,\widetilde{x},\theta\right)\partial_{\theta_l}+G\left(x,\widetilde{x},\theta\right)$$

where $F_l\left(x,\widetilde{x},\theta\right)\in\Gamma_0^{-1}\left(\Omega_\varepsilon\right)$, $G\left(x,\widetilde{x},\theta\right)\in\Gamma_0^{-2}\left(\Omega_\varepsilon\right)$:

$$F_l\left(x,\widetilde{x},\theta\right)=ih\left|\left(\partial_\theta f\right)\left(x,\theta\right)-\left(\partial_\theta f\right)\left(\widetilde{x},\theta\right)\right|^{-2}\left(\left(\partial_{\theta_l}f\right)\left(x,\theta\right)-\left(\partial_{\theta_l}f\right)\left(\widetilde{x},\theta\right)\right)$$

$$G\left(x,\widetilde{x},\theta\right)=ih\sum_{l=1}^{n}\partial_{\theta_l}\left[\left|\left(\partial_\theta f\right)\left(x,\theta\right)-\left(\partial_\theta f\right)\left(\widetilde{x},\theta\right)\right|^{-2}\left(\left(\partial_{\theta_l}f\right)\left(x,\theta\right)-\left(\partial_{\theta_l}f\right)\left(\widetilde{x},\theta\right)\right)\right]$$

$$\Omega_\varepsilon=\left\{\left(x,\tilde{x},\theta\right)\in\mathbb{R}^{3n};\ \left|\partial_\theta f(x,\theta)-\partial_\theta f\left(\tilde{x},\theta\right)\right|>\frac{\varepsilon}{2C_2}\lambda\left(x,\tilde{x},\theta\right)\right\}.$$

On the other hand we prove by induction on q that

$$\left({}^tL\right)^q b_{2,\varepsilon}\left(x,\tilde{x},\theta\right)=\sum_{\substack{|\gamma|\leq q\\ \gamma\in\mathbb{N}^n}}g_\gamma^{(q)}\left(x,\tilde{x},\theta\right)\partial_\theta^\gamma b_{2,\varepsilon}\left(x,\tilde{x},\theta\right),\ g_\gamma^{(q)}\in\Gamma_0^{-q},\ \forall q\in\mathbb{N},$$

and so,

$$K_{2,\varepsilon}\left(x,\tilde{x}\right)=\int\limits_{\mathbb{R}^n}e^{\frac{i}{h}\left(f(x,\theta)-f(\tilde{x},\theta)\right)}\left({}^tL\right)^q b_{2,\varepsilon}\left(x,\tilde{x},\theta\right)\widehat{d\theta}$$

Using Leibnitz's formula, $(G2)$ and the form $\left({}^tL\right)^q$, we can choose q large enough such that

$$\forall\alpha,\alpha',\beta,\beta'\in\mathbb{N}^n,\exists\, C_{\alpha,\alpha',\beta,\beta'}>0;\ \sup_{x,\widetilde{x}\in\mathbb{R}^n}\left|x^\alpha\widetilde{x}^{\alpha'}\partial_x^\beta\partial_{\widetilde{x}}^{\beta'}K_{2,\varepsilon}\left(x,\widetilde{x}\right)\right|\leq C_{\alpha,\alpha',\beta,\beta'}.$$

Next, we study $K_{1,\varepsilon}$.This is more difficult and depends on the choice of the parameter ε. It follows from Taylor's formula that

$$f\left(x,\theta\right)-f\left(\widetilde{x},\theta\right)=<x-\widetilde{x},\xi\left(x,\widetilde{x},\theta\right)>_{\mathbb{R}^n}$$

$$\xi\left(x,\widetilde{x},\theta\right)=\int\limits_0^1\left(\partial_x f\right)\left(\widetilde{x}+t\left(x-\widetilde{x}\right),\theta\right)dt.$$

We define the vectorial function:

$$\widetilde{\xi}_{\varepsilon}\left(x,\widetilde{x},\theta\right)=\omega\left(\frac{\left|x-\widetilde{x}\right|}{2\varepsilon\lambda\left(x,\tilde{x},\theta\right)}\right)\xi\left(x,\widetilde{x},\theta\right)+\left(1-\omega\left(\frac{\left|x-\widetilde{x}\right|}{2\varepsilon\lambda\left(x,\tilde{x},\theta\right)}\right)\right)\left(\partial_{x}f\right)\left(\widetilde{x},\theta\right).$$

We have

$$\widetilde{\xi}_{\varepsilon}\left(x,\widetilde{x},\theta\right)=\xi\left(x,\widetilde{x},\theta\right)\ \text{ on } \operatorname{supp}b_{1,\varepsilon},$$

Moreover, for ε sufficiently small,

$$\lambda\left(x,\theta\right)\simeq\lambda\left(\widetilde{x},\theta\right)\simeq\lambda\left(x,\widetilde{x},\theta\right)\ \text{ on } \operatorname{supp}b_{1,\varepsilon}. \tag{4.6}$$

Let us consider the mapping

$$\mathbb{R}^{3n}\ni\left(x,\widetilde{x},\theta\right)\to\left(x,\widetilde{x},\widetilde{\xi}_{\varepsilon}\left(x,\widetilde{x},\theta\right)\right); \tag{4.7}$$

for which Jacobian matrix is

$$\begin{pmatrix} I_n & 0 & 0\\ 0 & I_n & 0\\ \partial_x\widetilde{\xi}_{\varepsilon} & \partial_{\widetilde{x}}\widetilde{\xi}_{\varepsilon} & \partial_{\theta}\widetilde{\xi}_{\varepsilon}\end{pmatrix}.$$

We have

$$\begin{aligned}\frac{\partial\widetilde{\xi}_{\varepsilon,j}}{\partial\theta_i}\left(x,\widetilde{x},\theta\right)=&\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x},\theta\right)+\omega\left(\frac{\left|x-\tilde{x}\right|}{2\varepsilon\lambda\left(x,\tilde{x},\theta\right)}\right)\left(\frac{\partial\xi_j}{\partial\theta_i}\left(x,\widetilde{x},\theta\right)-\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x},\theta\right)\right)\\ &-\frac{\left|x-\tilde{x}\right|}{2\varepsilon\lambda\left(x,\tilde{x},\theta\right)}\frac{\partial\lambda}{\partial\theta_i}\left(x,\tilde{x},\theta\right)\lambda^{-1}\left(x,\tilde{x},\theta\right)\\ &\times\omega'\left(\frac{\left|x-\widetilde{x}\right|}{2\varepsilon\lambda\left(x,\tilde{x},\theta\right)}\right)\left(\xi_j\left(x,\widetilde{x},\theta\right)-\frac{\partial f}{\partial x_j}\left(\widetilde{x},\theta\right)\right).\end{aligned}$$

Thus, using that supp $\omega'\subset$ supp $\omega\subset\left]-1,1\right[$ and $\frac{\partial\lambda}{\partial\theta_i}\left(x,\tilde{x},\theta\right)\leq1$, we obtain

$$\begin{aligned}\left|\frac{\partial\widetilde{\xi}_{\varepsilon,j}}{\partial\theta_i}\left(x,\widetilde{x},\theta\right)-\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x},\theta\right)\right|\leq&\left|\omega\left(\frac{\left|x-\tilde{x}\right|}{2\varepsilon\lambda\left(x,\tilde{x},\theta\right)}\right)\right|\left|\frac{\partial\xi_j}{\partial\theta_i}\left(x,\widetilde{x},\theta\right)-\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x},\theta\right)\right|\\ &+\lambda^{-1}\left(x,\tilde{x},\theta\right)\\ &\times\left|\omega'\left(\frac{\left|x-\tilde{x}\right|}{2\varepsilon\lambda\left(x,\tilde{x},\theta\right)}\right)\right|\left|\xi_j\left(x,\widetilde{x},\theta\right)-\frac{\partial f}{\partial x_j}\left(\widetilde{x},\theta\right)\right|.\end{aligned}$$

Now it follows from $(G2)$, (4.6) and Taylor's formula that

$$\begin{aligned}\left|\frac{\partial\xi_j}{\partial\theta_i}\left(x,\widetilde{x},\theta\right)-\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x},\theta\right)\right|\leq&\int_0^1\left|\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x}+t\left(x-\widetilde{x}\right),\theta\right)-\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x},\theta\right)\right|dt\\ \leq&C_5\left|x-\widetilde{x}\right|\lambda^{-1}\left(x,\tilde{x},\theta\right),\ C_5>0\end{aligned} \tag{4.8}$$

$$\begin{aligned}\left|\xi_j\left(x,\widetilde{x},\theta\right)-\frac{\partial f}{\partial x_j}\left(\widetilde{x},\theta\right)\right|&\leq\int_0^1\left|\frac{\partial f}{\partial x_j}\left(\widetilde{x}+t\left(x-\widetilde{x}\right),\theta\right)-\frac{\partial f}{\partial x_j}\left(\widetilde{x},\theta\right)\right|dt\\ &\leq C_6\left|x-\widetilde{x}\right|,\ C_6>0\ .\end{aligned} \tag{4.9}$$

From (4.8) and (4.9), there exists a positive constant $C_7>0$, such that

$$\left|\frac{\partial\widetilde{\xi}_{\varepsilon,j}}{\partial\theta_i}\left(x,\widetilde{x},\theta\right)-\frac{\partial^2 f}{\partial\theta_i\partial x_j}\left(\widetilde{x},\theta\right)\right|\leq C_7\varepsilon,\ \forall i,j\in\{1,...,n\}. \tag{4.10}$$

If $\varepsilon < \frac{\delta_0}{2\widetilde{C}}$, then(4.10) *and* $(G4)$ yields the estimate

$$\delta_0/2 \leq -\widetilde{C}\varepsilon + \delta_0 \leq -\widetilde{C}\varepsilon + \det \frac{\partial^2 f}{\partial x \partial \theta}(x,\theta) \leq \det \partial_\theta \widetilde{\xi}_\varepsilon (x, \widetilde{x}, \theta), \text{ with } \widetilde{C} > 0. \tag{4.11}$$

If ε is such that (4.6) and (4.11) are true, then the mapping given in (4.7) is a global diffeomorphism of $\mathbb{R}^{3n}$. Hence there exists a mapping

$$\theta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (x, \widetilde{x}, \xi) \to \theta (x, \widetilde{x}, \xi) \in \mathbb{R}^n$$

such that

$$\begin{cases} \widetilde{\xi}_\varepsilon (x, \widetilde{x}, \theta (x, \widetilde{x}, \xi)) = & \xi \\ \theta \left(x, \widetilde{x}, \widetilde{\xi}_\varepsilon (x, \widetilde{x}, \theta)\right) = & x \\ \partial^\alpha \theta (x, \widetilde{x}, \xi) = \mathcal{O}(1), \quad \forall \alpha \in \mathbb{N}^{3n} \backslash \{0\}. \end{cases} \tag{4.12}$$

If we change the variable ξ by $\theta (x, \widetilde{x}, \xi)$ in $K_{1,\varepsilon} (x, \widetilde{x})$ we obtain

$$K_{1,\varepsilon} (x, \widetilde{x}) = \int_{\mathbb{R}^n} e^{\frac{i}{h} <x - \tilde{x}, \xi>} b_{1,\varepsilon} (x, \tilde{x}, \theta (x, \widetilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi} (x, \widetilde{x}, \xi) \right| \widehat{d\xi} \tag{4.13}$$

From (4.12) we have, for $k = 0,1$, that $b_{1,\varepsilon} (x, \tilde{x}, \theta (x, \widetilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi} (x, \widetilde{x}, \xi) \right|$ belongs to $\Gamma_k^{2m} \left(\mathbb{R}^{3n}\right)$ if $a \in \Gamma_k^m \left(\mathbb{R}^{2n}\right)$.

Applying the stationary phase theorem (cf. [20,17]) to (4.13), we obtain the expression of the symbol of the $h-$pseudodifferential operatorFF^*:

$$\sigma(FF^*) = b_{1,\varepsilon} (x, \tilde{x}, \theta (x, \widetilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi} (x, \widetilde{x}, \xi) \right|_{|\widetilde{x} = x} + R(x, \xi),$$

where $R(x,\xi) \in \Gamma_k^{2m-2} \left(\mathbb{R}^{2n}\right)$ if $a \in \Gamma_k^m \left(\mathbb{R}^{2n}\right), k = 0,1$.

For $\tilde{x} = x$, we have

$$b_{1,\varepsilon} (x, \tilde{x}, \theta (x, \widetilde{x}, \xi)) = e^{\frac{-2T(x,\theta)}{h}} \left| a (x, \theta (x, x, \xi)) \right|^2,$$

where $\theta (x, x, \xi)$ is the inverse of the mapping $\theta \to \partial_x f (x, \theta) = \xi$. Thus

$$\sigma(FF^*) (x, \partial_x f (x, \theta)) \equiv e^{\frac{-2T(x,\theta)}{h}} \left| a (x, \theta) \right|^2 \left| \det \frac{\partial^2 f}{\partial \theta \partial x} (x, \theta) \right|^{-1}.$$

By (4.2) and (4.3) we have:

$$\begin{aligned} \left(\mathcal{F}(F^*F)\mathcal{F}^{-1}\right) v (\theta) &= \int_{\mathbb{R}^n} e^{-\frac{i}{h} f(x,\theta) - \frac{T(x,\theta)}{h}} \overline{a} (x, \theta) \left(F(\mathcal{F}^{-1} v)\right)(x) dx \\ &= \int_{\mathbb{R}^n} e^{-\frac{i}{h} S(x,\theta) - \frac{T(x,\theta)}{h}} \overline{a}(x, \theta) \\ &\quad \times (\int_{\mathbb{R}^n} e^{\frac{i}{h} f\left(x, \widetilde{\theta}\right) - \frac{T\left(x, \widetilde{\theta}\right)}{h}} a \left(x, \widetilde{\theta}\right) \left(\mathcal{F}(\mathcal{F}^{-1} v)\right) \left(\widetilde{\theta}\right) \widehat{d\theta} dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{i}{h}\left(f(x,\theta) - f\left(x, \widetilde{\theta}\right)\right)} e^{-(\frac{T(x,\theta) + T\left(x, \widetilde{\theta}\right)}{h})} \\ &\quad \times \overline{a}(x, \theta)\, a \left(x, \widetilde{\theta}\right) v \left(\widetilde{\theta}\right) \widehat{d\theta} dx, \ \forall v \in \mathcal{S}\left(\mathbb{R}^n\right). \end{aligned}$$

The distribution kernel of the integral operator $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is

$$\widetilde{K}(\theta,\widetilde{\theta}) = \int\limits_{\mathbb{R}^n} e^{-\frac{i}{h}\left(f(x,\theta)-f\left(x,\tilde{\theta}\right)\right)}\, e^{-(\frac{T(x,\theta)+T\left(x,\tilde{\theta}\right)}{h})}\overline{a}(x,\theta)\; a\left(x,\tilde{\theta}\right)\widehat{dx}.$$

Observe that we can deduce $K(x,\widetilde{x})$ from $\widetilde{K}(\theta,\widetilde{\theta})$ by replacing x by θ. On the other hand, all assumptions used here are symmetrical on x and θ therefore $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is a nice $h-$pseudodifferential operator with symbol

$$\sigma(\mathcal{F}(F^*F)\mathcal{F}^{-1})\,(\theta,-\partial_\theta f(x,\theta)) \equiv e^{-\frac{2T(x,\theta)}{h}}\,|a(x,\theta)|^2\left|\det\frac{\partial^2 f}{\partial x\partial\theta}(x,\theta)\right|^{-1}.$$

Thus the symbol of F^*F is given by (cf. [14])

$$\sigma(F^*F)(\partial_\theta f(x,\theta),\theta) \equiv e^{\frac{-2T(x,\theta)}{h}}\,|a(x,\theta)|^2\left|\det\frac{\partial^2 f}{\partial x\partial\theta}(x,\theta)\right|^{-1}.$$

□

Corollary 4.2. *Let F_h be the integral operator with the distribution kernel*

$$K(x,y) = \int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(f(x,\theta)+iT(x,\theta))}a(x,\theta)\widehat{d\theta}$$

where $a\in\Gamma_0^m(\mathbb{R}^{2n}_{x,\theta})$ and S satisfies $(G1)$, $(G2)$, $(G3)$ and $(G4)$.

Then, we have:

1) If $m\le 0$, F_h can be extended to a bounded linear mapping on $L^2(\mathbb{R}^n)$.

2) If $m<0$, F_h can be extended to a compact operator on $L^2(\mathbb{R}^n)$.

Proof. It follows from Theorem 4.1 that $F_h^*F_h$ is a h-pseudodifferential operator with symbol in $\Gamma_0^{2m}\left(\mathbb{R}^{2n}\right)$.

1) If $m\le 0$, the weight $\lambda^{2m}(x,\theta)$ is bounded, so we can apply the Caldéron-Vaillancourt theorem (cf. [3,17,19]) for $F_h^*F_h$ and obtain the existence of a positive constant $\gamma(n)$

and a integer $k(n)$ such that

$$\|(F_h^*F_h)\,u\|_{L^2(\mathbb{R}^n)} \le \gamma(n)\,Q_{k(n)}\left(\sigma(F_h^*F_h)\right)\|u\|_{L^2(\mathbb{R}^n)},\ \forall u\in\mathcal{S}(\mathbb{R}^n),$$

where

$$Q_{k(n)}\left(\sigma(F_h^*F_h)\right) = \sum_{|\alpha|+|\beta|\le k(n)}\ \sup_{(x,\theta)\in\mathbb{R}^{2n}}\left|\partial_x^\alpha\partial_\theta^\beta\sigma(F_h^*F_h)(\partial_\theta f(x,\theta),\theta)\right|.$$

Hence, we have for all $u\in\mathcal{S}(\mathbb{R}^n)$

$$\|F_h u\|_{L^2(\mathbb{R}^n)} \le \|F_h^*F_h\|^{1/2}_{\mathcal{L}(L^2(\mathbb{R}^n))}\|u\|_{L^2(\mathbb{R}^n)} \le \left(\gamma(n)\,Q_{k(n)}\left(\sigma(F_h^*F_h)\right)\right)^{1/2}\|u\|_{L^2(\mathbb{R}^n)}.$$

Thus F_h is also a bounded linear operator on $L^2(\mathbb{R}^n)$.

2) If $m<0$, $\lim\limits_{|x|+|\theta|\to+\infty}\lambda^m(x,\theta)=0$, and the compactness theorem (see. [17,19]) show that the operator $F_h^*F_h$ can be extended to a compact operator on $L^2(\mathbb{R}^n)$. Thus, the Fourier integral operator F_h is compact on $L^2(\mathbb{R}^n)$. Indeed, let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$\left\|F_h^*F_h - \sum_{j=1}^{n} <\varphi_j,.>F_h^*F_h\varphi_j\right\| \underset{n\to+\infty}{\longrightarrow} 0.$$

Since F_h is bounded, we have for all $l \in L^2(\mathbb{R}^n)$

$$\left\| F_h l - \sum_{j=1}^{n} < \varphi_j, l > F_h \varphi_j \right\|^2 \leq$$
$$\left\| F_h^* F_h l - \sum_{j=1}^{n} < \varphi_j, l > F_h^* F_h \varphi_j \right\| \left\| l - \sum_{j=1}^{n} < \varphi_j, l > \varphi_j \right\|.$$

Hence

$$\left\| F_h - \sum_{j=1}^{n} < \varphi_j, . > F_h \varphi_j \right\| \underset{n \to +\infty}{\longrightarrow} 0.$$

□

References

1. K. Asada; D. Fujiwara. *On some oscillatory transformation in $L^2(\mathbb{R}^n)$. Japan J. Math. vol 4, (2), 1978, p299-361.*
2. S. Bekkara, B. Messirdi, A. Senoussaoui, *A class of generalized integral operators, Electron J. Differential Equations 88 (2009), 1–7.*
3. A. P. Calderon; R. Vaillancourt. *On the boundness of pseudo-differential operators. J. Math. Soc. Japan 23, 1972, p374-378bolicité*. J. *Math. Pures et Appl. 51 (1972), 231-256.*
4. P. Drabek and J. Milota, *"Lectures on nonlinear analysis", Plezeň, Czech Republic, 2004.*
5. J.J. Duistermaat. *Applications of Fourier Integral. Séminaire Goulaouic-Schwartz, 197, (1972).*
6. J.J. Duistermaat. *Fourier integral operators. Courant Institute Lecture Notes, New-York 1973.*
7. M. Hasanov. *A class of unbounded Fourier integral operators. Journal of Mathematical Analysis and Applications 225 (1998), 641-651.*
8. C. Harrat and A. Senoussaoui, *On a class of h-Fourier integral operators, Demonstratio Mathematica, Vol. XLVII, No. 3, 2014.*
9. B. Helffer. *Théorie spectrale pour des opérateurs globalement elliptiques. Société Mathématiques de France, Astérisque 112, 1984.*
10. B. Helffer; J. Sjöstrand. *Multipes wells in the semiclassical limit I. Comm. P.D.E. vol 9 (4), 1984, p337-408.*
11. L. Hörmande. *Linear partial differential operators. Springer-Verlag. Berlin, (1964).*
12. L. Hörmander. *Fourier integral operators I. Acta Math. vol 127, 1971, p33-57.*
13. L. Hörmander. *The spectral function of an elliptic operator. Acta. Math., vol 121 (1968), p173-218.*
14. L. Hörmander. *The Weyl calculus of pseudodifferential operators. Comm. Pure. Appl. Math. 32 (1979), p359-443.*
15. Melin, A and Sjöstrand, J. *Fourier intégraux à phases complexes. Séminaire Équations aux dérivées partielles (Polytechnique) (1973-1974)p 1-10.*
16. B. Messirdi; A. Senoussaoui. *On L^2-boundness and L^2-compactness of a class of Fourier integral operators, Electron J. Differential Equations 26 (2006), 1–12.*
17. D. Robert. *Autour de l'approximation semi-classique. Birkäuser, 1987.*
18. L. Rodino. *Global hypoellipticity and spectral theory. Mathematical Researche vol. 92. Berlin, Akademic Verlag, 1996.*
19. A. Senoussaoui, *Opérateurs h-admissibles matriciels à symbole opérateur, African Diaspora J. Math. 4(1) (2007), 7–26.*
20. J. Sjöstrand. *Singularités analytiques microlocales. Astérisque 95, 1982.*

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