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Investigation Approach for a Nonlinear Singular Fredholm Integro-differential Equation

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ABSTRACT: In this paper, we examine the existence and uniqueness of the solution of nonlinear integro-differential Fredholm equation with a weakly singular kernel. Then, we develop an iterative scheme to approach this solution using the product integration method. Finally, we conclude with a numerical test to show the effectiveness of the proposed method.

Key Words: Fredholm integro-differential equation, Singular kernel, Fixed point, Nonlinear equation, Product integration method.

Contents

1. Introduction

Lately, the Singular integral and integro-differential equation represent a great interest in famous domains, especially in the study of a problem concerning stress analysis, fracture mechanics, heat conduction and radiation, in elastic contact see [8,10].

Several articles have been published in the approximation of integro-differential equations, such as using Haar wavelet bases [3], with RH wavelet method [4,7], or using cubic B-spline finite element method [6].

So far, there are no publications in the field of Fredholm integro-differential equations, when the derivative of the unknown appears inside the integral.

In a recent paper, Ghiat and Guebbai [9] studied the numerical and analytical analysis of an integro-differential nonlinear Volterra equation with a weakly singular kernel. Analogously, we will study a similar equation that the proposed in [9] but for Fredholm equation type, so consider our equation:

$$u(t) = f(t) + \int_a^b p(\mid t-s \mid) G(t,s,u(s),u'(s))\, ds, \tag{1.1}$$

where, the unknown is $u \in C^1([a,b])$ and f is given function in same space.

As an advantage of our equation compared with equations available in the literature, the kernel of equation has a convolution term, and the derivative of the unknown is inside of the non linear kernel. In addition, we need to join the equation (1.1) with another equation contains more information about the solution u, so, if we derive both sides of equation (1.1) we get the following equation:

$$u'(t) = f'(t) + \int_a^b sign(t-s)\, p'(\mid t-s \mid) G(t,s,u(s),u'(s))\, ds + \int_a^b p(\mid t-s \mid) \frac{\partial G}{\partial t}(t,s,u(s),u'(s))\, ds. \tag{1.2}$$

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Where $sign(t-s)$ represents the signum function of the $(t-s)$, defined as follow:

$$sign(t-s) = \begin{cases} 1 & t > s, \\ -1 & t < s, \\ 0 & t = s. \end{cases}$$

We want to point out from [9], that the singularity came from the derivative of $p(s)$ as the following form:

$$(H1) \quad \left\| \begin{array}{ll} (1) & p(s) \in W^{1,1}(0, b-a), \\ (2) & \lim_{s \to 0^+} p'(s) = +\infty, \end{array} \right.$$

where,

$$W^{1,1}(0, b-a) = p \in L^1(0, b-a) : p' \in L^1(0, b-a)$$

is a Banach space with the following norm

$$\|p\|_{W^{1,1}(0,b-a)} = \|p\|_{L^1(0,b-a)} + \|p'\|_{L^1(0,b-a)}. \tag{1.3}$$

2. Analytical study

We suppose that G fulfill the following hypotheses :

$$(H2) \quad \left\| \begin{array}{ll} (1) & f \in C^1([a,b], \mathbb{R}), \quad \dfrac{\partial G}{\partial t} \in C^0([a,b]^2 \times \mathbb{R}^2) \\ & \exists A,\, B,\, \bar{A},\, \bar{B} > 0,\ \forall t, s \in [a,b],\ \forall x,\, y,\, \bar{x},\, \bar{y} \in \mathbb{R}, \text{ such that :} \\ (3) & |G(t,s,x,y) - G(t,s,\bar{x},\bar{y})| \le A|x-y| + B|\bar{x}-\bar{y}|, \\ & \left| \dfrac{\partial G}{\partial t}(t,s,x,y) - \dfrac{\partial G}{\partial t}(t,s,\bar{x},\bar{y}) \right| \le \bar{A}|x-y| + \bar{B}|\bar{x}-\bar{y}|. \end{array} \right.$$

Now, to study the existence and uniqueness of the solution of (1.1), we define the functional T_f by:

$$T_f(\xi)(t) = f(t) + \int_a^b p(|\, t-s\,|) G(t,s,\xi(s),\xi'(s))\, ds,$$

where, $\xi \in C^1([a,b])$ with norm $\|\xi\|_{C^1([a,b])} = \|\xi\|_\infty + \|\xi'\|_\infty$ such $\|\xi\|_\infty = \sup_{t\in[a,b]} \{|\xi(t)|\}$.

Differentiating both sides of the equation above with respect to t, we get

$$\begin{aligned} T'_f(\xi)(t) = f'(t) \quad &+ \quad \int_a^b sign(t-s)\, p'(|\, t-s\,|) G(t,s,\xi(s),\xi'(s))\, ds \\ &+ \quad \int_a^b p(|\, t-s\,|) \frac{\partial G}{\partial t}(t,s,\xi(s),\xi'(s))\, ds. \end{aligned}$$

For all $f \in C^1([a,b])$, the functional T_f is continuous from $C^1([a,b])$ into itself (well defined), because it is the sum of two well defined Volterra operators (in [9] the authors proved that the Volterra operator is well defined).

Theorem 2.1. *Let assumptions $(H1)$ and $(H2)$ be satisfied and suppose that there exist $\lambda > 0$ such that, for all $t \in [a,b]$*

$$\max\{A, B, \bar{A}, \bar{B}\} \int_a^b |p(|\, t-s\,|)|\, ds \le \lambda < \frac{1}{3},$$

$$\max\{A, B\} \int_a^b |p'(|\, t-s\,|)|\, ds \le \lambda < \frac{1}{3},$$

then (1.1) has a unique solution in $C^1([a,b])$.

Proof. By utilizing the Banach fixed point theorem on the functional T_f in the space $C^1([a,b])$ endowed with the norm defined above, we have

$$\begin{aligned}
|T_f(u_1(t)) - T_f(u_2(t))| &\leq \int_a^b |p(|\, t-s\,|)||G(t,s,u_1(s),u_1'(s)) - G(t,s,u_2(s),u_2'(s))|\,ds \\
&\leq A\int_a^b |p(|\, t-s\,|)||u_1(s)-u_2(s)|\,ds + B\int_a^b |p(|\, t-s\,|)||u_1'(s)-u_2'(s)|\,ds \\
&\leq A\int_a^b |p(|\, t-s\,|)|\,ds\|u_1-u_2\|_\infty + B\int_a^b |p(|t-s|)|\,ds\|u_1'-u_2'\|_\infty \\
&\leq \left(\max\{A,B,\bar{A},\bar{B}\}\int_a^b |p(|t-s|)|\,ds\right)\|u_1-u_2\|_{C^1([a,b])},
\end{aligned}$$

then

$$\|T_f(u_1) - T_f(u_2)\|_\infty \quad\leq\quad \lambda\,\|u_1-u_2\|_{C^1([a,b])}. \tag{2.1}$$

In the same way

$$\begin{aligned}
|T_f'(u_1(t)) - T_f'(u_2(t))| &\leq \int_a^b |p'(|\, t-s\,|)|\,|G(t,s,u_1(s),u_1'(t)) - G(t,s,u_2(s),u_2'(t))|\,ds \\
&+ \int_a^b |p(|\, t-s\,|)|\left|\frac{\partial G}{\partial t}(t,s,u_1(s),u_1'(t)) - \frac{\partial G}{\partial t}(t,s,u_2(s),u_2'(t))\right|\,ds \\
&\leq A\int_a^b |p'(|\, t-s\,|)|\,|u_1(s)-u_2(s)|\,ds + B\int_a^b |p'(|\, t-s\,|)|\,|u_1'(s)-u_2'(s)|\,ds \\
&+ \bar{A}\int_a^b |p(|\, t-s\,|)||u_1(s)-u_2(s)|\,ds + \bar{B}\int_a^b |p(|\, t-s\,|)||u_1'(s)-u_2'(s)|\,ds \\
&\leq \left(\max\{A,B\}\int_a^b |p'(|\, t-s\,|)|\,ds\right)\|u_1-u_2\|_{C^1([a,b])} \\
&+ \left(\max\{A,B,\bar{A},\bar{B}\}\int_a^b |p(|\, t-s\,|)|\,ds\right)\|u_1-u_2\|_{C^1([a,b])},
\end{aligned}$$

then

$$\|T_f'(u_1) - T_f'(u_2)\|_\infty \leq 2\lambda\|u_1-u_2\|_{C^1([a,b])}. \tag{2.2}$$

Therefore, from (2.1) and (2.2) we deduce that

$$\|T_f(u_1) - T_f(u_2)\|_{C^1([a,b])} \leq 3\lambda\|u_1-u_2\|_{C^1([a,b])}.$$

Consequently, there exists only one fixed point for the functional T_f, This completes the proof. ☐

Remark 2.2. *In [11], we can see that, the author studied by using a similar idea, the existence and uniqueness of the solution of the version of Fredholm integro-differential equation with high derivative order, defined as:*

$$x(t) = f(t) + \int_a^b G(t,s,x(s),x^{'}(s),\ldots,x^{n-1}(s))\,ds, \qquad a\leq t\leq b.$$

3. Main numerical results

Generally, there are various numerical methods to obtain approximate solution of the equations. In our work, we interest on the method described in [2] by Borzabadi and Fard to study the following nonlinear Fredholm integral equation:

$$x(t) = f(t) + \int_a^b G(t, s, x(s))\, ds, \quad a \le t \le b, G \in C^1([a, b]),$$

the method consists to estimate the integral by Newton-Cotes method then neglects the truncation error to get a discretized form of the equation, after that, with the same suitable condition authors proved the converge of the method.

Similarly, using the same previous method the author of [13] studied the mixed nonlinear Volterra-Fredholm integral equation defined by:

$$x(t) = f(t) + \lambda_1 \int_a^t G_1(t, s, x(s))\, ds + \lambda_2 \int_a^b G_2(t, s, x(s))\, ds, \quad a \le t \le b, \lambda_1, \lambda_1 \in \mathbb{R}, G_1, G_2 \in C^1([a, b]).$$

Throughout this section, we will apply a similar idea that's mentioned above, to approach the equations (1.1) and (1.2).
First, let $N \in \mathbb{N}^*$, and considering the equidistance subdivision Δ_N defined by:

$$\Delta_N = \left\{ t_i = a + ih,\, h = \frac{b-a}{N},\, i = 0, \dots, N \right\}.$$

Therefore, by taking equidistance subdivision Δ_N, as above, and if we denote $u(t_i) = u_i$, $u'(t_i) = u'_i$, $f(t_i) = f_i$ and $f'(t_i) = f'_i$ for all $i = 0, \dots N$, then, the equations (1.1) and (1.2) are given as follow:

$$u_i = f_i + \int_a^b p(|t_i - s|) G(t_i, s, u(s), u'(s))\, ds, \tag{3.1}$$

$$u'_i = f'_i + \int_a^b sign(t_i - s)\, p'(|t_i - s|) G(t_i, s, u(s), u'(s))\, ds + \int_a^b p(|t_i - s|) \frac{\partial G}{\partial t}(t_i, s, u(s), u'(s))\, ds. \tag{3.2}$$

In our case, as $p \in W^{1,1}(0, b-a)$, to estimate the integral terms of (3.1) and (3.2), we must use the product integration method see [1,12]; this method consists to interpolate the regular terms G and $\dfrac{\partial G}{\partial t}$ on Δ_N using the piecewise linear functions in every subinterval $[t_j, t_{j+1}]$, $j = 0, \dots, N$, we have

$$\begin{aligned} P_{n,1}[G](t_i, s, u(s), u'(s)) \;&\simeq\; \left(\frac{s - t_j}{h}\right) G(t_i, t_{j+1}, u_{j+1}, u'_{j+1}) \\ &+\; \left(\frac{t_{j+1} - s}{h}\right) G(t_i, t_j, u_j, u'_j), \quad s \in [t_j, t_{j+1}], \end{aligned}$$

$$\begin{aligned} P_{n,1}\left[\frac{\partial G}{\partial t}\right](t_i, s, u(s), u'(s)) \;&\simeq\; \left(\frac{s - t_j}{h}\right) \frac{\partial G}{\partial t}(t_i, t_{j+1}, u_{j+1}, u'_{j+1}) \\ &+\; \left(\frac{t_{j+1} - s}{h}\right) \frac{\partial G}{\partial t}(t_i, t_j, u_j, u'_j), \quad s \in [t_j, t_{j+1}]. \end{aligned}$$

Then, for all $i = 0, 1, \dots, N$, equations (3.1) and(3.2) can be written as follow:

$$u_i = f_i + \sum_{j=0}^{N} \alpha_j G(t_i, t_j, u_j, u'_j) + \varepsilon_1, \tag{3.3}$$

$$u'_i = f'_i + \sum_{j=0}^{N} \beta_j G(t_i, t_j, u_j, u'_j) + \alpha_j \frac{\partial G}{\partial t}(t_i, t_j, u_j, u'_j) + \varepsilon_2, \tag{3.4}$$

where, ε_1, ε_2 are the convergence orders of the product integration rule, and α_j, β_j are given by:

$$\begin{aligned}
\alpha_0 &= \frac{1}{h}\int_a^{t_1} (t_1 - s)p(|t_i - s|)\, ds,\\
\alpha_j &= \frac{1}{h}\left(\int_{t_{j-1}}^{t_j} (s - t_{j-1})p(|t_i - s|)\, ds + \int_{t_j}^{t_{j+1}} (t_{j+1} - s)p(|t_i - s|)\, ds\right), \quad j = 1, \dots, N-1,\\
\alpha_N &= \frac{1}{h}\int_{t_{N-1}}^{b} (s - t_{N-1})p(|t_i - s|)\, ds,\\
\beta_0 &= \frac{1}{h}\int_a^{t_1} (t_1 - s)sign(t_i - s)\, p'(|t_i - s|)\, ds,\\
\beta_j &= \frac{1}{h}\left(\int_{t_{j-1}}^{t_j} (s - t_{j-1})sign(t_i - s)\, p'(|t_i - s|)\, ds\right.\\
&+ \left.\int_{t_j}^{t_{j+1}} (t_{j+1} - s)sign(t_i - s)\, p'(|t_i - s|)\, ds\right), \quad j = 1, \dots, N-1,\\
\beta_N &= \frac{1}{h}\int_{t_{N-1}}^{b} (s - t_{N-1})sign(t_i - s)\, p'(|t_i - s|)\, ds.
\end{aligned}$$

Proposition 3.1. *ε_1 and ε_2 considered in equation (3.1) and (3.2) respectively, satisfies: $|\varepsilon_1| \le \mu_1(h)$, $|\varepsilon_2| \le \mu_2(h)$ for some $\mu_1(h), \mu_1(h) > 0$ with $\mu_1(h) \to 0$ and $\mu_1(h) \to 0$ as $h \to 0$.*

Proof.

$$\begin{aligned}
|\varepsilon_1| &\le \int_a^b |p(|\, t_i - s\,|) \max_{|s-t|<h} |(G(t, s, u(s), u'(s)) - P_{n1}[G](t, \tau, u(\tau), u'(\tau))|\\
&\le \int_a^b |p(|\, t_i - s\,|)\, ds| \,(|G(t_i, \tau, u(\tau), u'(\tau)) - G(t_i, \tau, u(\theta), u'(\theta))|\\
&+ |G(t_i, \tau, u(\theta), u'(\theta)) - G(t_i, \theta, u(\theta), u'(\theta))|)\\
&\le \int_a^b |p(|\, t - s\,|)|\, ds \left(\max_{|\tau-\theta|<h} |G(t, \tau, u(\theta), u'(\theta)) - G(t, \theta, u(\theta), u'(\theta))|\right)\\
&+ \max\{A, B, \overline{A}, \overline{B}\} \int_a^b |p(|\, t - s\,|)|\, ds \max_{|\tau-\theta|<h} |u(\tau) - u(\theta)| + \max_{|\tau-\theta|<h} |u'(\tau) - u'(\theta)|,
\end{aligned}$$

using equation (1.1), (1.2) and the following notations

$$H(t) = \int_a^b p(|\, t - s\,|)G(t, s, u(s), u'(s))\, ds,$$

$$w_0(h, \varphi) = \max_{|s-t|<h} |\varphi(t) - \varphi(\theta)|,$$

$$w_1(h, \varphi) = w_0(h, \varphi) + w_0(h, \varphi'),$$

we get

$$\begin{aligned}
|\varepsilon_1| &\le \int_a^b |p(|\, t - s\,|)|\, ds \left(\max_{|\tau-\theta|<h} |G(t, \tau, u(\theta), u'(\theta)) - G(t_i, \theta, u(\theta), u'(\theta))|\right)\\
&+ \max\{A, B, \overline{A}, \overline{B}\} \int_a^b |p(|\, t - s\,|)|\, ds(\max_{|\tau-\theta|<h} |f(\tau) - f(\theta)| + \max_{|\tau-\theta|<h} |f'(\tau) - f'(\theta)|\\
&+ \max_{|\tau-\theta|<h} |H(\tau) - H(\theta)| + \max_{|\tau-\theta|<h} |H'(\tau) - h'(\theta)|),
\end{aligned}$$

$$|\varepsilon_1| \leq \lambda \left(w_1(h,f) + w_1(h,H) + \frac{\max\limits_{a\leq t\leq b; x,y\in\mathbb{R}} w_0(h, G(t,.,x,y))}{\max\{A,B,\overline{A},\overline{B}\}} \right).$$

Similar steps,

$$\begin{aligned}|\varepsilon_2| &\leq \lambda \left(2w_1(h,f) + 2w_1(h,H) + \frac{\max\limits_{a\leq t\leq b; x,y\in\mathbb{R}} (w_0(h, G(t,.,x,y))}{\max\{A,B\}} \right.\\ &+ \left. \frac{\max\limits_{a\leq t\leq b; x,y\in\mathbb{R}} w_0(h, \frac{\partial K}{\partial t}(t,.,x,y))}{\max\{A,B,\overline{A},\overline{B}\}} \right).\end{aligned}$$

Since $f \in C^1([a,b])$ and from condition (1) of $(H2)$, we get the result. □

Now, we consider the following nonlinear approached equations by neglecting the error terms ε_1 and ε_2 in (3.3) and (3.4) respectively,

$$v_i = f_i + \sum_{j=0}^{N} \alpha_j G(t_i, t_j, v_j, w_j), \tag{3.5}$$

$$w_i = f_i' + \sum_{j=0}^{N} \beta_j G(t_i, t_j, v_j, w_j) + \alpha_j \frac{\partial G}{\partial t}(t_i, t_j, v_j, w_j), \tag{3.6}$$

where $(v,w) = (v_0, \ldots, v_N, w_0, \ldots, w_N)$ are the solution of (3.5) and (3.6) successively.

Before to study our system (3.5) and (3.6), we mention that our numerical method remains very effective, because it is simple, easy to structure and rapid in the execution. In addition, we don't need to add a new conditions in order to confirm the convergence of our numerical process, but we have just taken the same analytical conditions proposed in the first.

3.1. System study

Theorem 3.2. *The system ((3.5)-(3.6)) has a unique solution, for* $\lambda < \frac{1}{3}$.

Proof. By using a functional denoted by $\Psi(V)$ in order to equip the system ((3.5)-(3.6)) as $V = \Psi(V)$, where, $V = (v_0, \ldots, v_N, w_0, \ldots, w_N)$ is the unknown vector in $\mathbb{R}^{2N+2}$, with the following norm

$$\|V\|_{\mathbb{R}^{2N+2}} = \max_{0\leq i\leq N}\{|v_i|\} + \max_{0\leq i\leq N}\{|w_i|\}.$$

It is clear that:

$$\begin{aligned}|\Psi(V) - \Psi(\bar{V})| &\leq \sum_{j=0}^{N} |\alpha_j|(A|v_j - \bar{v}_j| + B|w_j - \bar{w}_j|)\\ &+ \sum_{j=0}^{N} |\alpha_j|(\overline{A}|v_j - \bar{v}_j| + \overline{B}|w_j - \bar{w}_j|)\\ &+ \sum_{j=0}^{N} |\beta_j|(A|v_j - \bar{v}_j| + B|w_j - \bar{w}_j|),\\ &\leq 2\max\{A,B,\overline{A},\overline{B}\} \int_a^b |p(|t_i - s|)| ds \|V - \bar{V}\|_{\mathbb{R}^{2N+2}}\\ &+ \max\{A,B\} \int_a^b |p'(|t_i - s|)|\, ds \|V - \bar{V}\|_{\mathbb{R}^{2N+2}},\ i = 0, \ldots N.\end{aligned}$$

Finally, we obtain

$$\|\Psi(V) - \Psi(\bar{V})\|_{\mathbb{R}^{2n+2}} \leq 3\lambda\|V - \bar{V}\|_{\mathbb{R}^{2n+2}},$$

Using Banach's fixed point theorem, we get the result. □

3.2. Convergence analysis

In the following proposition, we seek the conditions of vanishing of:

$$\|u - v\|_\infty + \|u' - w\|_\infty$$

.

Proposition 3.3. *Under the assumptions (H1), (H2) and for* $0 < \lambda < \frac{1}{3}$*, we have:*

$$\|u - v\|_\infty + \|u' - w\|_\infty \leq \frac{\varepsilon}{1 - 3\lambda},$$

with $\varepsilon = \varepsilon_1 + \varepsilon_2$.

Proof. Firstly, it is easy to observe that there exists $q, r \in \{0, \dots, N\}$ such that

$$\|u - v\|_\infty + \|u' - w\|_\infty = |u_q - v_q| + |u'_r - w_r|,$$

thus we have:

$$\begin{aligned}
|u_q - v_q| + |u'_r - w_r| &\leq \sum_{j=0}^{N} |\alpha_j||G(t_q, t_j, u_j, u'_j) - G(t_q, t_j, v_j, w_j)| \\
&+ |\beta_j||G(t_r, t_j, u_j, u'_j) - G(t_r, t_j, v_j, w_j)| \\
&+ |\alpha_j|\left|\frac{\partial G}{\partial t}(t_r, t_j, u_j, u'_j) - \frac{\partial G}{\partial t}(t_r, t_j, v_j, w_j)\right| \\
&+ \varepsilon_1 + \varepsilon_2, \\
&\leq \sum_{j=0}^{N} (|\alpha_j| + |\beta_j|)(A|u_j - v_j| + B|u'_j - w_j|) \\
&+ |\alpha_j|(\bar{A}|u_j - v_j| + \bar{B}|u'_j - w_j|) + \varepsilon_1 + \varepsilon_2, \\
&\leq \left\{\max\{A, B\}\left(\int_a^b |p(|t_q - s|)|\, ds + \int_a^b |p'(|t_q - s|)|\, ds\right)\right. \\
&+ \left.\max\{\bar{A}, \bar{B}\}\int_a^b |p(|t_r - s|)|\, ds\right\}(|u_q - v_q| + |u'_r - w_r|) + \varepsilon_1 + \varepsilon_2, \\
&\leq 3\lambda(|u_q - v_q| + |u'_r - w_r|) + \varepsilon_1 + \varepsilon_2.
\end{aligned}$$

Hence, we have

$$\|u - v\|_\infty + \|u' - w\|_\infty \leq \frac{\varepsilon}{1 - 3\lambda}.$$

Therefore, when the value of $h \to 0$, we have (v, w) tends to (u, u'). □

The solution of nonlinear equations systems (3.3) and (3.4) can be obtained by using iterative process, which leads to the following system: for all $i = 0 \dots N$,

$$v_i^{k+1} = f_i + \sum_{j=0}^{N} \alpha_j G(t_i, t_j, v_j^k, w_j^k), \tag{3.7}$$

$$w_i^{k+1} = f_i' + \sum_{j=0}^{N} \beta_j G(t_i, t_j, v_j^k, w_j^k) + \alpha_j \frac{\partial K}{\partial t}(t_i, t_j, v_j^k, w_j^k). \tag{3.8}$$

Next, we shall prove the following theorem concerning the Convergence of the iterative solution (v^{k+1}, w^{k+1}) of (3.7) and (3.8) to (v, w) when $k \to \infty$.

Theorem 3.4. *According to the hypotheses (H1), (H2) and for any arbitrary initial vector (v^0, w^0), then we have:*

$$\|v^{k+1} - v\|_\infty + \|w^{k+1} - w\|_\infty \quad \leq \quad (3\lambda)^k (\|v^0 - v\|_\infty + \|w^0 - w\|_\infty).$$

Proof. By using (3.5), (3.6), (3.7) and (3.8) we have for all $i = 0, \dots, N$:

$$\begin{aligned} v_i^{k+1} - v_i &= \sum_{j=0}^{N} \alpha_j (G(t_i, t_j, v_j^k, w_j^k) - G(t_i, t_j, v_j, w_j)), \\ w_i^{k+1} - w_i &= \sum_{j=0}^{N} \beta_j (G(t_i, t_j, v_j^k, w_j^k) - G(t_i, t_j, v_j, w_j)) \\ &+ \alpha_j (\frac{\partial G}{\partial t}(t_r, t_j, v_j^k, w_j^k) - \frac{\partial G}{\partial t}(t_r, t_j, v_j, w_j)). \end{aligned}$$

Therefore,

$$\begin{aligned} | \, v_i^{k+1} - v_i \, | &\leq \sum_{j=0}^{N} | \, \alpha_j \, || \, G(t_i, t_j, v_j^k, w_j^k) - G(t_i, t_j, v_j, w_j) \, |, \\ &\leq \sum_{j=0}^{N} |\alpha_j| (A|v_j^k - v_j| + B|w_j^k - w_j|), \\ &\leq \max\{A, B\} \int_a^b |p(|t_i - s|)| \, ds (\|v^k - v\|_\infty + \|w^k - w|\|_\infty), \end{aligned}$$

thus,

$$\|v^{k+1} - v\|_\infty \leq \lambda(\|v^k - v\|_\infty + \|w^k - w|\|_\infty).$$

In the same way

$$\|w^{k+1} - w\|_\infty \leq 2\lambda(\|v^k - v\|_\infty + \|w^k - w\|_\infty).$$

Finally, we obtain

$$\|v^{k+1} - v\|_\infty + \|w^{k+1} - w\|_\infty \leq 3\lambda(\|v^k - v\|_\infty + \|w^k - w\|_\infty).$$

Repeating the last inequality k-times, we get

$$\|v^{k+1} - v\|_\infty + \|w^{k+1} - w\|_\infty \leq (3\lambda)^k (\|v^0 - v\|_\infty + \|w^0 - w\|_\infty).$$

Since $0 < \lambda < \frac{1}{3}$, $k \to \infty$ implies that $\|v^{k+1} - v\|_\infty + \|w^{k+1} - w\|_\infty \to 0$ $\square$

4. Numerical example

Consider the following integro-differential equation:

$$u(t) = \frac{1}{20}\int_0^1 \sqrt{|t-s|}\sin\left(e^s + \arcsin\left(\frac{s+t}{3}\right) + u(s) - u'(s)\right) ds + f(t), \quad t \in [0,1],$$

where,

$$f(t) = te^t - \frac{(7t+3)(1-t)^{3/2} + 7t^{5/2}}{450},$$

then the exact solution u of this equation is given by:

$$u(t) = te^t.$$

We can see that the kernel $G(s,t,x,y) = \sin\left(e^s + \arcsin\left(\frac{s+t}{3}\right) + x - y\right)$ satisfies the hypothesis (H_2) with:

$$A = B = \frac{1}{20}, \overline{A} = \overline{B} = \frac{1}{20\sqrt{5}},$$

and the function $p = \sqrt{|t-s|}$ satisfies H_1, also the parameter $\lambda < \frac{1}{3}$.

Now, we try to establish our numerical method to find the solution v^{k+1} and w^{k+1} according schemes (3.7) and (3.8) respectively, where the arbitrary initial vector $v^0 = w^0 = 0$ and the stopping condition on the parameter k is taken as:

$$|v^{k+1} - v^k| + |w^{k+1} - w^k| \leq 10^{-7}.$$

Denote the error function E_N of this method by:

$$E_N = \max_{0\leq i\leq N}\{|u(t_i) - v_i^{k+1}| + |u'(t_i) - w_i^{k+1}|\}.$$

Table (1) shows us the effectiveness of our numerical method by examining the error function E_N, where we have found that E_N converges to 0 when N increases ($E_N \to 0$ when $h \to 0$), as well as, the graph of error function is showed in Figure (1). Also, Figure (2) shows the comparison between exact and numerical solutions for $N = 50$.

On the other hand, the CPU run times of our numerical processes is presented in Table (1), where we used Matlab computation software, with a machine of type Intel Core i5 Duo processor 2.6 GHz and 4 GB RAM, the latter increases according the increment of dimension of our non linear system (3.5) and (3.6), for example with $N = 1000$ we obtain a non linear system with 2002 unknowns.

Table 1: Numerical Results.

N	E_N	CPU time
10	5.43E-4	0.002524s
50	5.29E-5	0.032543s
100	1.91E-5	0.118545s
500	1.76E-6	3.290755s
1000	6.25E-7	15.675546s

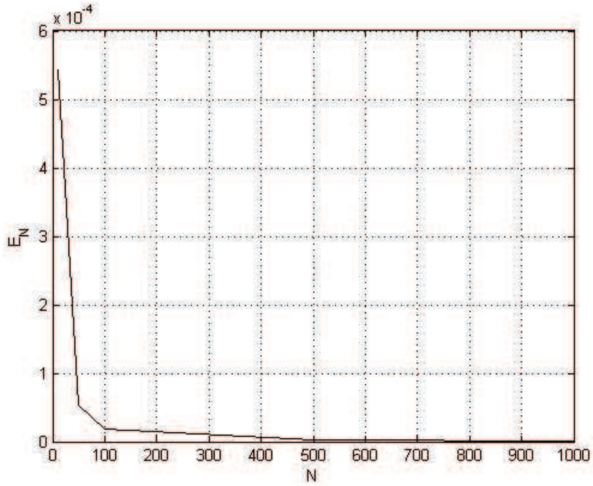

Figure 1: The error function E_N

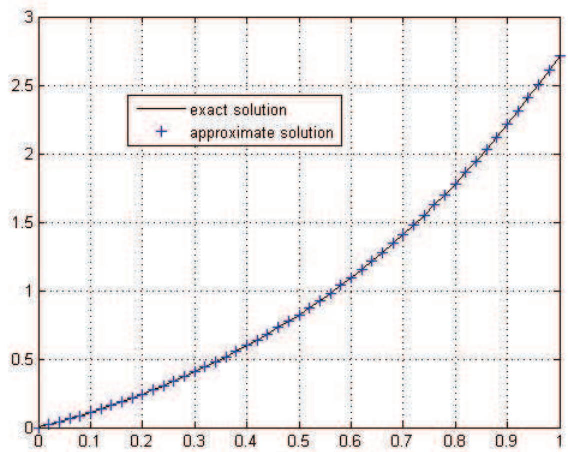

Figure 2: Exact solution vs approximate solution for $N = 50$

5. Conclusion

In this work, we have presented a simple numerical method for solving a weakly singular Fredholm integro-differential equation, where the numerical test shows its effectiveness. As a perspective, this method can be extended to approach the Fredholm integro-differential equation with a high derivative order presented in [11]. Also, other equation's type as fuzzy or fractional Fredholm integro-differential equations can be investigated using our numerical method. Last, but not least, the author plans to explore more venues: Similar ideas to the ones described in the paper of [5] can be applied to an integro-differential equation.

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