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Zero-divisor Graphs of Small Upper Irredundance Number

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ABSTRACT: In this paper, we classify finite rings with upper irredundance number less than or equal to two. We note that, for such zero-divisor graphs, the upper irredundance number coincides with the independence number.

Key Words: Zero-divisors, Zero-divisor graph, Upper irredundance number, Independence number.

Contents

1. Introduction

By a graph $\Gamma = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [9]. One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching. An excellent treatment of fundamentals of domination in graphs is given in the book by Haynes et al. [13]. For any vertex $u \in V$, $N(u)$ is the open neighborhood of the vertex u, i.e. the set of vertices that are adjacent to u in Γ, and the closed neighborhood of u is the set $N[u] = N(u) \cup u$. Let S be a subset of V and let $u \in S$. A vertex v is called a private neighbor of u with respect to S if $N[v] \cap S = \{u\}$. The private neighbor set of u with respect to S is defined as $\mathrm{pn}[u, s] = \{v \mid N[v] \cap S = \{u\}\}$. The set S is called an irredundant set if for every $u \in S$, $\mathrm{pn}[u, s] \neq \emptyset$. The maximum cardinality of an irredundant set in Γ is called the upper irredundance number of Γ and is denoted by $IR(\Gamma)$. The subset S of V is said to be independent if no two vertices in S are adjacent. The maximum cardinality of an independent set in Γ is called the independence number of Γ and is denoted by $\beta_0(\Gamma)$. These two parameters concerning independence and irredundance satisfy the inequality $\beta_0(\Gamma) \leq IR(\Gamma)$ (see [5]). Note that this inequality may be strict and that many research papers have treated some particular cases of graphs Γ over which we have $\beta_0(\Gamma) = IR(\Gamma)$ (see for example [10,11,12,14]).

Throughout the paper R will denote a commutative ring with identity $1 \neq 0$. If X is either an element or a subset of the ring R, then $\mathrm{ann}_R(X)$ denotes the annihilator of X in R. If X is any subset of a ring, then $X^* = X\backslash\{0\}$. The concept of a zero-divisor graph was first introduced by Beck in 1988 for his study of the coloring of a commutative ring [7]. In his work, all elements of the ring were vertices of the graph. In [4], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors. Let R be a ring and let $Z(R)$ denote the set of zero-divisors of R. The zero divisors graph of a ring R, denoted by $\Gamma(R)$, is the simple graph whose vertices are the elements of the set $Z(R)^*$ and, for distinct $x, y \in Z(R)^*$, there is an edge connecting x and y if and only if $xy = 0$.
Let Γ be a graph. We say that Γ is connected if there is a path between any two distinct vertices of Γ. For distinct vertices x and y of Γ, let $d(x, y)$ be the length of the shortest path between x and y ($d(x, y) = \infty$ if there is no such path). The diameter of Γ is $\mathrm{diam}(\Gamma) := \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } \Gamma\}$. The girth of Γ, denoted by $\mathrm{gr}(\Gamma)$, is defined as the length of the shortest cycle in Γ ($\mathrm{gr}(\Gamma) = \infty$ if Γ contains no cycles).
It is proved that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ ([4, Theorem 2.3]) and $\mathrm{gr}(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle ([15, (1.4)]). Thus, $\mathrm{diam}(\Gamma(R)) = 0$, 1, 2 or 3 and $\mathrm{gr}(\Gamma(R)) = 3$, 4 or ∞ (examples of different cases can be found in [2]). The zero-divisor graphs of commutative rings have attracted the

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attention of several researchers (see, for instance, [1,3,6,16]). For a survey and recent results concerning zero-divisor graphs of commutative rings, we refer the reader to [2].

2. Zero-divisor graphs of small upper irredundance number

We start with the following easy remarks.

Remark 2.1. *(1) A graph $\Gamma = (V, E)$ is said a complete bipartite graph if V can be decomposed into two disjoint independent sets such that every pair of graph vertices in the two sets are adjacent. If there are n and m vertices in the two sets, the complete bipartite graph is denoted $K^{(n,m)}$. It is easily seen that $\beta_0(K^{(n,m)}) = \max\{n, m\}$. On the other hand, we have $\beta_0(K^{(n,m)}) = IR(K^{(n,m)})$ ([11]). Now, let F and K be finite fields with distinct orders n and m, respectively. By [4, Example 3.4], $\Gamma(F \times K)$ is the complete bipartite graph $K^{(n-1,m-1)}$. Hence, $\beta_0(\Gamma(F \times K)) = IR(\Gamma(F \times K)) = \max\{n-1, m-1\}$. In particular, for any finite field F, we have $\beta_0(\Gamma(\mathbb{Z}_2 \times F)) = IR(\Gamma(\mathbb{Z}_2 \times F)) = |F| - 1$.*
(2) Note that zero-divisor graphs of non isomorphic rings may have the same upper irredundance number and the same independence number. For example, take $R_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $R_2 = \mathbb{Z}_3 \times \mathbb{Z}_3$. From (1), $\beta_0(\Gamma(R_1)) = IR(\Gamma(R_1)) = \beta_0(\Gamma(R_2)) = IR(\Gamma(R_2)) = 2$.

Proposition 2.2. *Let R be a ring. Then, the following are equivalent:*

1. *$IR(\Gamma(R)) = 0$.*
2. *$\beta_0(\Gamma(R)) = 0$.*
3. *R is a domain.*

Proof. The implication (1) ⇒ (2) follows from the inequality $\beta_0(\Gamma(R)) \leq IR(\Gamma(R))$.
(2) ⇒ (3) If R is not a domain then $Z(R)^* \neq \emptyset$. Hence, for any $x \in Z(R)^*$ is an independent set, a contradiction.
(3) ⇒ (1) By hypothesis, $Z(R)^* = \emptyset$. Thus, $IR(\Gamma(R)) = IR(\emptyset) = 0$. □

Proposition 2.3. *Let R be a ring which is not a domain. Then, the following are equivalent:*

1. *$IR(\Gamma(R)) = 1$.*
2. *$\beta_0(\Gamma(R)) = 1$.*
3. *$\Gamma(R)$ is a complete graph.*
4. *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

Proof. (1) ⇒ (2) follows from the inequality $\beta_0(\Gamma(R)) \leq IR(\Gamma(R))$ and Proposition 2.2.
(2) ⇒ (3) Suppose that $Z(R)^*$ contains at least two different elements x and y. Then, by hypothesis, $\{x, y\}$ is not independent. Hence, $xy = 0$. Hence, $\Gamma(R)$ is a complete graph.
(3) ⇒ (4) Follows from [2, Theorem 2.8].
(4) ⇒ (1) By remark 2.1, $IR(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$. So, suppose that $xy = 0$ for all $x, y \in Z(R)$. Let S be an irredundant set with maximum cardinality. Clearly, $|S| \geq 1$, otherwise $IR(\Gamma(R)) = 0$ and so R is domain. Let $u \in S$. We have $\mathrm{pn}[u, s] \neq \emptyset$, and so there exists $v \in Z(R)^*$ such that $N[v] \cap S = \{u\}$. Now, by hypothesis $N[v] = Z(R)^*$. Thus, $\{u\} = S$. Hence, $|S| = 1$, and then $IR(\Gamma(R)) = 1$. □

Example 2.4.
1. *An easy example of a local finite ring R with $IR(\Gamma(R)) = 1$ is $\mathbb{Z}_{25}$.*
2. *Set $R := \mathbb{R}[x]/(x^2)$. It is easy to see that R is an infinite local ring with maximal ideal $M = (\overline{x})$. Moreover, $M^2 = (0)$. Hence, $IR(\Gamma(R)) = 1$.*

The main goal of this paper is to characterize rings R (local or not) with $IR(\Gamma(R)) = 2$. To do so, we need the following lemma.

Lemma 2.5. *Let R be a local ring with $\beta_0(\Gamma(R)) = 2$. Then, for all $a, b \in Z(R)^*$ such that $ab \neq 0$ the ideals $\operatorname{ann}_R(a)$ and $\operatorname{ann}_R(b)$ are comparable; that is $\operatorname{ann}_R(a) \subseteq \operatorname{ann}_R(b)$ or $\operatorname{ann}_R(b) \subseteq \operatorname{ann}_R(a)$.*

Proof. Since $\beta_0(\Gamma(R)) = 2$, we can always find $a, b \in Z(R)^*$ such that $ab \neq 0$. Suppose that $\operatorname{ann}_R(a) \nsubseteq \operatorname{ann}_R(b)$ and that $\operatorname{ann}_R(b) \nsubseteq \operatorname{ann}_R(a)$. Implicitly, we have $a \neq b$.
Assume first that $\mid \operatorname{ann}_R(a) \backslash \operatorname{ann}_R(b) \mid > 2$ and let $x, y, z \in \operatorname{ann}_R(a) \backslash \operatorname{ann}_R(b)$ distinct elements. Consider also $\alpha \in \operatorname{ann}_R(b) \backslash \operatorname{ann}_R(a)$. Trivially, $x + \alpha$, $y + \alpha$ and $z + \alpha$ are distinct. Hence, we can suppose that $\alpha + x \neq a$ and that $\alpha + x \neq b$. Then, $(\alpha + x)a = \alpha a + xa = \alpha a \neq 0$ and $(\alpha + x)b = \alpha b + xb = xb \neq 0$. Hence, $\{a, b, \alpha + x\}$ is an independent set, a contradiction since $\beta_0(R) = 2$. Thus, $\mid \operatorname{ann}_R(a) \backslash \operatorname{ann}_R(b) \mid \leq 2$. Similarly, we get $\mid \operatorname{ann}_R(b) \backslash \operatorname{ann}_R(a) \mid \leq 2$.
Case 1: Assume that $\operatorname{ann}_R(b) \backslash \operatorname{ann}_R(a) = \{x\}$ for some $x \in \operatorname{ann}_R(b)$.
We have $\operatorname{ann}_R(b) \subseteq \operatorname{ann}_R(a) \cup \{x\}$. Consider $\alpha \in \operatorname{ann}_R(b) \backslash \{0, x\} \subseteq \operatorname{ann}_R(a)$. Then, $\alpha + x \in \operatorname{ann}_R(b) \backslash \{x\}$. Thus, $\alpha + x \in \operatorname{ann}_R(a)$. Thus, $x \in \operatorname{ann}_R(a)$ since $\alpha \in \operatorname{ann}_R(a)$, a contradiction. Consequently, $\operatorname{ann}_R(b) = \{0, x\}$. But $Z(R) = \operatorname{ann}_R(z)$ for some non zero element z. We must have $zb = 0$, and so $z = x$. Hence, x is adjacent to every other element. Hence, $x \in \operatorname{ann}_R(a)$. Then, $\operatorname{ann}_R(b) \subseteq \operatorname{ann}_R(a)$, a contradiction.
Case 2: Assume that $\operatorname{ann}_R(b) \backslash \operatorname{ann}_R(a) = \{x, y\}$ for some $x, y \in \operatorname{ann}_R(b)$.
Case 2.1: Assume that $\operatorname{ann}_R(a) \backslash \operatorname{ann}_R(b) = \{z\}$ for some $z \in \operatorname{ann}_R(b)$. This case is analogous to Case 1, and so we will obtain a contradiction.
Case 2.2: Assume that $\operatorname{ann}_R(a) \backslash \operatorname{ann}_R(b) = \{z, t\}$ for some $y, z \in \operatorname{ann}_R(a)$. As in Case 1.2, we obtain $\operatorname{ann}_R(a) = \{0, z, t\}$ or $\operatorname{ann}_R(a) = \{0, z, t, \alpha\}$ for some α. Moreover, we have $\operatorname{ann}_R(b) \subseteq \operatorname{ann}_R(a) \cup \{x, y\}$ and $Z(R) = \operatorname{ann}_R(a) \cup \operatorname{ann}_R(b) \cup \{a, b\}$. Thus, $Z(R) = \{0, z, t, x, y, a, b\}$ or $Z(R) = \{0, z, t, x, y, a, b, \alpha\}$. Note that x, y, z and t are all different. Then, $5 \leq |Z(R)| \leq 8$. Moreover, the fact that $|\operatorname{ann}_R(a)| = 3,$ or 4 implies that $|Z(R)| = 8$. On the other hand, R is local and so $Z(R)$ is an annihilator ideal. Hence, there is an element of $Z(R)^*$ which is adjacent to every element in $Z(R)$. The only possibility for this element is to be equal to α. Hence, $Z(R) = \operatorname{ann}_R(\alpha)$. Hence, $\operatorname{ann}_R(a) = \{0, z, t, \alpha\}$ and $\operatorname{ann}_R(b) = \{0, x, y, \alpha\}$. It is clear that $a + b \neq 0$, otherwise $\operatorname{ann}_R(a) = \operatorname{ann}_R(b)$. Also, $a + b \neq \alpha$, otherwise $\operatorname{ann}_R(a) = \operatorname{ann}_R(b)$ again. Hence, $a + b \in \{x, y, z, t\}$. For example, suppose that $a + b = x$. The set $\{a, x, y\}$ is not independent. Then, $xy = 0$ since $ax \neq 0$ and $ay \neq 0$. Hence, $0 = xy = (a + b)y = ay$. Hence, $y \in \operatorname{ann}_R(a)$, a contradiction.
All cases are impossible. Then, our hypothesis is false, and so $\operatorname{ann}_R(a) \subseteq \operatorname{ann}_R(b)$ or $\operatorname{ann}_R(b) \subseteq \operatorname{ann}_R(a)$. □

Theorem 2.6. *Let R be a finite local ring. Then $\beta_0(\Gamma(R)) = 2$ if and only if R is isomorphic to one of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, and $\mathbb{Z}_4[x]/(2x, x^2 - 2)$.*

Proof. Suppose that $\beta_0(\Gamma(R)) = 2$. Then, necessarily $|Z(R)| \geq 3$. If $|Z(R)| = 3$ then $Z(R) = \{0, a, -a\}$ for some $a \in R$. Hence, we have immediately $a^2 = 0$, and so $\beta_0(\Gamma(R)) = 1$, a contradiction. Hence, we must have $|Z(R)| \geq 4$. Consider $x, y \in Z(R)^*$ such that $xy \neq 0$. Such elements exist since $\beta(\Gamma(R)) = 2$. Using Lemma 2.5, $\operatorname{ann}_R(x) \subseteq \operatorname{ann}_R(y)$ or $\operatorname{ann}_R(y) \subseteq \operatorname{ann}_R(x)$. Take for example the first case. The hypothesis $\beta(\Gamma(R)) = 2$ implies that $Z(R) = \operatorname{ann}_R(x) \cup \operatorname{ann}_R(y) \cup \{x, y\} = \operatorname{ann}_R(y) \cup \{x, y\}$. Hence, $|Z(R)| = |\operatorname{ann}_R(y)| + 1$ if $y^2 = 0$ and $|Z(R)| = |\operatorname{ann}_R(y)| + 2$ if $y^2 \neq 0$. Since, R is local, $|Z(R)| = p^n$ for some p prime and a positive integer n. Set $|\operatorname{ann}_R(y)| = p^k$ for $1 \leq k < n$. Then $p^n = p^k + 1$ if $y^2 = 0$ and $p^n = p^k + 2$ if $y^2 \neq 0$. The first case (when $y^2 = 0$) is impossible, and the unique possibility in the second case is $p = 2$, $k = 1$, and $n = 2$. Hence, $|Z(R)| = 4 = 2^2$. Following [8, Corollary 2], R is isomorphic to one of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x, y]/(x, y)^2$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_4[x]/(2x, x^2)$, the Galois ring $GR(16, 4)$ ($\cong \mathbb{Z}_4[x]/(x^2 + x + 1)$), and $\mathbb{F}_4[x]/(x^2)$. By drawing the zero divisors graphs of these rings, we can see that those of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$ are isomorphic to $K^{(1,2)}$ and those of the rings $\mathbb{Z}_2[x, y]/(x, y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\cong \mathbb{Z}_4[x]/(x^2 + x + 1)$, and $\mathbb{F}_4[x]/(x^2)$ are isomorphic to K^3.
Since $\beta_0(\Gamma(R)) = 2$, $\Gamma(R)$ is not complete. Thus, $\Gamma(R)$ must be isomorphic to $K^{(1,2)}$. Conversely, $\beta_0(K^{(1,2)}) = 2$. Hence, we obtain the desired equivalence. □

Theorem 2.7. *Let R be a non local finite ring. Then, $\beta_0(\Gamma(R)) = 2$ if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$ or to $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. Without loss of generality, set $R = R_1 \times R_2 \times ... \times R_n$ where $n \geq 2$ and R_i are local rings. Suppose that $n \geq 3$. Clearly, the set $\{(1,1,0,...,0),(0,1,1,0,...,0),(1,0,1,0,...,0)\}$ is an independent set with cardinality ≥ 3. Hence, $\beta_0(R) \geq 3$, a contradiction. Thus, $n = 2$.
Assume that $|R_1| \geq 4$ and let $x \in R_1 \backslash \{0,1\}$. We have $(x,0)(1,0) = (x,0) \neq (0,0)$. Then, for any $y \in R_1 \backslash \{1,x\}$, we have $(y,0)(1,0) = (0,0)$ or $(y,0)(x,0) = (0,0)$. Then $y = 0$ or $xy = 0$. So, $Z(R_1) = R_1 \backslash \{1\}$. Set $|R_1| = p^n$ for some prime p and integer $n \geq 1$. Hence, $Z(R_1) = p^n - 1 = p^k$ for some integer $k \geq 0$. Thus, $p^k(p^{n-k} - 1) = 1$. Hence, $p = 2$, $k = 0$, and $n = 1$. Thus, $|R_1| = 2$, a contradiction since $|R_1| \geq 4$. Hence, we conclude that $R_1 \cong \mathbb{Z}_2$ or $R_1 \cong \mathbb{Z}_3$. Similarly, $R_2 \cong \mathbb{Z}_2$ or $R_2 \cong \mathbb{Z}_3$. To finish, it suffices to remind that $\beta_0(\mathbb{Z}_2 \times \mathbb{Z}_2) = 1$ and $\beta_2(\mathbb{Z}_2 \times \mathbb{Z}_3) = \beta_2(\mathbb{Z}_3 \times \mathbb{Z}_3) = 2$. □

Corollary 2.8. *Let R be a finite ring. Then, the following are equivalent:*

1. $IR(\Gamma(R)) = 2$.
2. $\beta_0(\Gamma(R)) = 2$.
3. *R is isomorphic to one of the rings*

$$\mathbb{Z}_8,\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_4[x]/(2x, x^2 - 2),\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Proof. (1) ⇒ (2) Follows from the inequality $\beta_0(\Gamma(R)) \leq IR(\Gamma(R))$ and Propositions 2.2 and 2.3.
(2) ⇒ (3) Follows from Theorems 2.6 and 2.7. (3) ⇒ (3) Note that the zero divisors graphs of $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ are isomorphic to $K^{(1,2)}$ and the zero divisors graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is isomorphic to $K^{(2,2)}$. In the both cases, the upper irredundance number is two. □

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