Solution of an Integro-differential Nonlinear Equation of Volterra Arising of Earthquake Model

Selma Salah, Hamza Guebbai, Samir Lemita and Mohamed Zine Aissaoui

ABSTRACT: In this paper, we study a new type of modeling of an Earthquake phenomenon, a mechanical model of the earthquake process in one-dimension using usual mathematical functions, the latter leads to the study of nonlinear integro-differential equation of Volterra. The existence and the uniqueness of the solution are proved. Using Nyström method is builded to approximate the solution. The numerical tests show the effectiveness of this type of modeling.

Key Words: Nonlinear Volterra equation, Integro-differential, Earthquake Machine, Fix point, Nyström method.

Contents

1. Introduction and Motivation

We are interested in modeling the Earthquake phenomenon that results from movement of tectonic plates. During their movements, tectonic plates may collide along faults. While they pursue their movement, the pressure increases. When this pressure reaches a significant degree, a sudden slip of the plate occurs . The resulting shock trains release of energy in the form of seismic waves which cause the sensation of trembling (see [1,2]).

The Incorporated Research Institutions for Seismology (IRIS) (www.iris.edu) presented a demo (”Earthquake Machine: Basic One block and simple graph animated”) to explain the Earthquake model. This demonstration is presented in figure 1
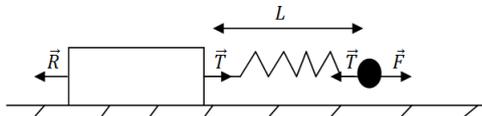

Figure 1: Earthquake Machine in dimension 1

In this work, we will present a model that provid a constant friction. This choice is built on the difficulty in the theoretical and numerical plan. However with this choice our vision still a performance vision.

Thereby, we study this physical experience, we propose a mathematical modeling that describes the movement of earthquake machine, which is presented in the form of a non-linear integro-differential equation of Volterra of the second kind

$$u(t) = \int_0^t (t-s)\Big(-\varrho(\frac{1}{m}+\frac{1}{M})u(s) - \frac{\varrho}{2M}\delta_\xi\big(-\frac{1}{\varrho}u'(s)+ \tag{1.1}$$

$$+\frac{1}{m}\int_0^s (F(\alpha)-R-u(\alpha))d\alpha\big)\big(|u(s)|-u(s)\big)\Big)ds + f(t).$$

This paper represents a physical foundation for recent works ([3,4,5]) on this equation and a generalization of the mathematical assumptions required in these papers to ensure the existence and uniqueness of the solution.

We will construct a numerical method based on the vision of Nyström (see [6,7,8,9]) to approach this unique solution, and show the convergence under the local Lipschitz assumptions, which represents a generalization of those assumptions required in [3,4,5,6].

The numerical tests developed in this paper show our good vision on mathematical modeling, analytical and numerical plans.

In the sense of establishing a coherent system and keeping the fundamental aspects of our phenomenon, and to properly present the mechanism of the earthquake machine, we consider a coupled mass spring oscillator model consists of two masses M and m, linked together by a spring of length L at rest, which are both free to move. The resulting motions can be very intriguing, but for simplicity, we will neglect the effects of friction and external forces (The latter is due to the difficulty of theoretical and numerical studies). The system is placed on a smooth horizontal surface.

We denote by F the force applied to the mass m, R the resistance of the mass M. Because each mass is free to move, we apply Newton's second law (see [10]) to each object. Let $x(t)$ denote the displacement of the mass M from its equilibrium position and similarly, let $y(t)$ denote the corresponding displacement for the mass m which has a force F acting on its right side, applying Hooke's law (see [10]),

$$T(t) = \varrho l(t),$$

where T is the tension of the spring, ϱ is the stiffness constant and $l(t)$ the variation of the length between the two masses at a moment t such that

$$l(t) = (y(t) - x(t)) - L.$$

Applying Newton's second law to these objects, we obtain the system in the state of rest

$$\begin{cases} H(\varrho l(t)-R)(\varrho l(t)-R) = Mx''(t), \\ F-\varrho l(t) = my''(t), \\ l(t) = (y(t)-x(t))-L, \end{cases} \tag{1.2}$$

and the system in the state of motion

$$\begin{cases} \varrho l(t)-R = Mx''(t), \\ F-\varrho l(t) = my''(t), \\ l(t) = (y(t)-x(t))-L, \end{cases} \tag{1.3}$$

where $x''(t)$ and $y''(t)$ are the acceleration of masses M and m respectively, and H is Heaviside function defined by

$$H(\tau) = \begin{cases} 1, & \tau \geq 0, \\ 0, & \tau < 0. \end{cases} \tag{1.4}$$

By using the function δ defined by

$$\delta(\tau) = \begin{cases} 1, & \tau = 0, \\ 0, & \tau \neq 0, \end{cases}$$

the systems (1.2) and (1.3) can be brought together in one system as the following form

$$\begin{cases} (1 - \delta(x'(t)))(\varrho l(t) - R) + \delta(x'(t))H(\varrho l(t) - R)(\varrho l(t) - R) = Mx''(t), \\ F - \varrho l(t) = my''(t), \\ l(t) = (y(t) - x(t)) - L. \end{cases} \quad (1.5)$$

Where $x'(t)$ and $y'(t)$ are the velocity of masses M and m respectively.

As the model represents a simulation of an Earthquake Machine, and in order to be well posed, we put the seismic function u given by

$$u(\tau) = \varrho l(\tau) - R. \quad (1.6)$$

Hence, the previous system (1.5) can be written as the form

$$u(t) + \tfrac{1}{2}\delta(x'(t))(|u(t)| - u(t)) = Mx''(t), \quad (1.7)$$

$$F - R - u(t) = my''(t), \quad (1.8)$$

$$u(t) = \varrho(y(t) - x(t) - L) - R, \quad (1.9)$$

where

$$H(\tau)\tau = \frac{1}{2}(|\tau| + \tau). \quad (1.10)$$

We integrate the equations (1.7) and (1.8), replacing the results in (1.9), under the initial conditions

$$\begin{cases} x(0) = x'(0) = y'(0) = 0, \\ y(0) = L, \end{cases} \quad (1.11)$$

we obtain the equation

$$u(t) + \varrho \int_0^t \int_0^s \left(\left(\frac{1}{m} + \frac{1}{M} \right) u(\theta) + \frac{1}{2M}\delta\left(-\frac{1}{\varrho}u'(\theta) + \right. \right. \quad (1.12)$$

$$+ \frac{1}{m} \int_0^\theta (F(\alpha) - R - u(\alpha))d\alpha \Big)(|u(\theta)| - u(\theta)) \Big) d\theta ds = -R + \frac{\varrho}{m} \int_0^t \int_0^s (F(\theta) - R)d\theta ds.$$

The equation (1.12) represents a discontinuity given by the function δ. Therefore, to solve this problem we will approach the equation (1.12) by another equation through approaching δ using δ_ξ of class $C^1(\mathbb{R})$ defined as

$$\delta_\xi(\tau) = \begin{cases} (1 - (\frac{\tau}{\xi})^2)^2, & \text{if } \tau \in [-\xi, \xi], \\ 0, & \text{if } \tau \in]-\infty, -\xi[\cup]\xi, +\infty[, \end{cases} \quad (1.13)$$

such that

$$\forall \tau \in \mathbb{R}, \lim_{\xi \to 0} \delta_\xi(\tau) = \delta(\tau).$$

Therefore, the equation (1.12) is written as

$$u(t) = \int_0^t (t - s)\left(-\varrho\left(\frac{1}{m} + \frac{1}{M}\right)u(s) - \frac{\varrho}{2M}\delta_\xi\left(-\frac{1}{\varrho}u'(s) \right. \right.$$

$$+ \frac{1}{m} \int_0^s (F(\alpha) - R - u(\alpha))d\alpha \Big)(|u(s)| - u(s)) \Big) ds - R + \frac{\varrho}{m} \int_0^t (t - s)(F(s) - R)ds. (1.14)$$

Which also gives,

$$u'(t) = \int_0^t \Big(-\varrho(\frac{1}{m} + \frac{1}{M})u(s) - \frac{\varrho}{2M}\delta_\xi\big(-\frac{1}{\varrho}u'(s) +$$

$$+ \frac{1}{m}\int_0^s (F(\alpha) - R - u(\alpha))d\alpha\big)(|u(s)| - u(s))\Big)ds + \frac{\varrho}{m}\int_0^t (F(s) - R)ds. \tag{1.15}$$

For $T > 0$ maximum experiment time, the equation (1.14) which is an integro-differential nonlinear equation of Volterra can be written as the following simple and clear formula

$$\forall t \in [0, T], \quad u(t) = \int_0^t (t - s)K(s, u(s), u'(s))ds + f(t), \tag{1.16}$$

which gives

$$\forall t \in [0, T], \quad u'(t) = \int_0^t K(s, u(s), u'(s))ds + f'(t), \tag{1.17}$$

where

$$K(s, u(s), u'(s)) = -\varrho(\frac{1}{m} + \frac{1}{M})u(s) - \frac{\varrho}{2M}\delta_\xi\big(-\frac{1}{\varrho}u'(s) +$$

$$+ \frac{1}{m}\int_0^s (F(\alpha) - R - u(\alpha))d\alpha\big)(|u(s)| - u(s)),$$

and

$$f(t) = -R + \frac{\varrho}{m}\int_0^t (t - s)(F(s) - R)ds.$$

The present paper is organized as follow, in section 2, we study our equation in an analytical sense, we show the existence and the uniqueness of the solution using similar methods to those used in [3,4,5,6]. In section 3, we study this equation in a numerical sense using a Nyström method (cf. [3,4,6,8]) to approximate the solution of this equation. Finally, in section 4 we give an experimental tests which show the effectiveness of our model.

2. Analytical study

We consider the previous equation

$$\forall t \in [0, T], \quad u(t) = \int_0^t (t - s)K(s, u(s), u'(s))ds + f(t). \tag{2.1}$$

It is clear that f and K verify the following properties

(H)
1. $f \in C^1([0, T])$,

2. $\exists \alpha, \beta, \bar{\alpha}, \bar{\beta} \in \mathbb{R}, \ \forall t \in [0, T], \ \alpha < f(t) < \beta, \ \bar{\alpha} < f'(t) < \bar{\beta}$,

3. $\forall t, s \in [0, T], \ \forall u, \bar{u} \in [\alpha, \beta], \ \forall v, \bar{v} \in [\bar{\alpha}, \bar{\beta}], \ \exists M \in \mathbb{R}_+^*, \ |K(s, u, v)| \le M$.

4. $\forall t, s \in [0, T], \ \forall u, \bar{u} \in [\alpha, \beta], \ \forall v, \bar{v} \in [\bar{\alpha}, \bar{\beta}], \ \exists L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}} \in \mathbb{R}_+^*$,

$$|K(s, u, v) - K(s, \bar{u}, \bar{v})| \le L_{\alpha,\beta}|u - \bar{u}| + \bar{L}_{\bar{\alpha},\bar{\beta}}|v - \bar{v}|.$$

The classical method that is used to prove the existence and uniqueness of the solution of the equation (1.16) is based on the construction of two successive sequences $\{u_n(t)\}_{n\in\mathbb{N}}$, $\{\varphi_n(t)\}_{n\in\mathbb{N}}$. This method is called *the Picard method*. These sequences are given by

$$\begin{cases} u_0(t) = f(t), \\ u_n(t) = f(t) + \int_0^t (t - s)K(s, u_{n-1}(s), u'_{n-1}(s))ds, & \forall n \in \mathbb{N}^* \end{cases} \tag{2.2}$$

and

$$\begin{cases} \varphi_0(t) = f(t), \\ \varphi_n(t) = u_n(t) - u_{n-1}(t), & \forall n \in \mathbb{N}^* \end{cases} \tag{2.3}$$

which gives us successively

$$\begin{cases} u_0'(t) = f'(t), \\ u_n'(t) = f'(t) + \int_0^t K(s, u_{n-1}(s), u_{n-1}'(s))ds, & \forall n \in \mathbb{N}^* \end{cases} \tag{2.4}$$

and

$$\begin{cases} \varphi_0'(t) = f'(t), \\ \varphi_n'(t) = u_n'(t) - u_{n-1}'(t). & \forall n \in \mathbb{N}^* \end{cases} \tag{2.5}$$

Also, from (2.3) and (2.5) we obtain

$$\begin{cases} \sum_{i=0}^n \varphi_i(t) = u_n(t), \\ \sum_{i=0}^n \varphi_i'(t) = u_n'(t). \end{cases} \tag{2.6}$$

Theorem 2.1. *According to the properties (H), and knowing that there exist points $0 = \delta_0 < \delta_1 < \delta_2 < ... < \delta_n = T$ such that, for $0 \le i \le n$, and for $t \in [\delta_i, \delta_{i+1}]$, the equation (1.16) has a unique continuous solution in $C^1(0, T)$.*

Proof. First, we establish the existence and the uniqueness in some interval $[0, \delta_1]$, than this solution can be continuous to successive intervals $[\delta_1, \delta_2]$, $[\delta_2, \delta_3]$ and so on. Under suitable conditions, we eventually cover the whole interval $[0, T]$.

Let be $t \in [0, \delta_1]$. Because of 1., 2. and 3. from (H), we can find a constant C such that

$$\int_0^t |K(s, f(s), f'(s))|ds \le Ct, \quad \forall t \in [0, \delta_1], \tag{2.7}$$

From 1., we choose a positive number d such that

$$\forall t \in [0, \delta_1], \quad \alpha \le f(t) - Cde^{Ld} \le f(t) \le f(t) + Cde^{Ld} \le \beta, \tag{2.8}$$

and

$$\forall t \in [0, \delta_1], \quad \bar{\alpha} \le f'(t) - Cde^{Ld} \le f'(t) \le f'(t) + Cde^{Ld} \le \bar{\beta}, \tag{2.9}$$

Let be

$$\delta_1 = \min(d, T).$$

Defining the sequences u_n, φ_n, u_n' and φ_n' as in (2.2)-(2.5), we can now show by induction, for $n \in \mathbb{N}^*$ that

$$(P_n) \begin{cases} (a_n) \ \alpha \le u_n(t) \le \beta, \\ (b_n) \ \bar{\alpha} \le u_n'(t) \le \bar{\beta}, \\ (c_n) \ |\varphi_n(t)| + |\varphi_n'(t)| \le C \frac{\gamma^{n-1} t^n}{n!}, \\ (d_n) \ |u_n(t) - f(t)| + |u_n'(t) - f'(t)| \le C \sum_{i=1}^n \frac{\gamma^{i-1} t^i}{i!}. \end{cases}$$

In fact, for $n = 1$, we have $u_1(t) = f(t)$ and $u_1'(t) = f'(t)$, then

$$\alpha \le u_1(t) \le \beta,$$
$$\bar{\alpha} \le u_1'(t) \le \bar{\beta},$$

so that (a_1) and (b_1) are satisfied. Also,

$$\begin{aligned}
|\varphi_1(t)| + |\varphi_1'(t)| &= |u_1(t) - f(t)| + |u_1'(t) - f'(t)| \\
&\leq \int_0^t |(t-s)K(s, f(s), f'(s))| ds + \int_0^t |K(s, f(s), f'(s))| ds \\
&\leq |(\delta_1 + 1)| \int_0^t |K(s, f(s), f'(s))| ds \\
&\leq Ct, \quad \forall t \in [0, \delta_1].
\end{aligned} \tag{2.10}$$

(c_1) and (d_1) holds.

The property (P_1) is satisfied. Now, assuming that (P_n) is satisfied, we prove the property (P_{n+1}). Then

$$\begin{aligned}
|\varphi_{n+1}(t)| + |\varphi_{n+1}'(t)| &\leq \gamma \int_0^t |\varphi_n(s)| + |\varphi_n'(s)| ds \\
&\leq \gamma \int_0^t C \frac{\gamma^{n-1} s^n}{n!} ds \\
&\leq C \frac{\gamma^n}{n!} \int_0^t s^n ds \\
&\leq C \frac{\gamma^n t^{n+1}}{(n+1)n!}.
\end{aligned} \tag{2.11}$$

So that (c_{n+1}) is satisfied. Also

$$\begin{aligned}
|u_{n+1}(t) - f(t)| + |u_{n+1}'(t) - f'(t)| &= |u_n(t) - f(t) + \varphi_{n+1}(t)| + |u_{n+1}'(t) - f'(t) + \varphi_{n+1}'(t)| \\
&\leq |u_n(t) - f(t)| + |u_{n+1}'(t) - f'(t)| + |\varphi_{n+1}(t)| + |\varphi_{n+1}'(t)| \\
&\leq C \sum_{i=1}^n \frac{\gamma^{i-1} t^i}{i!} + C \frac{\gamma^n t^{n+1}}{(n+1)!},
\end{aligned}$$

although (d_{n+1}) holds.

Finally, for $0 < t < \delta_1$,

$$\sum_{i=1}^n \frac{\gamma^{i-1} t^i}{i!} < d e^{Ld}, \tag{2.12}$$

on the other hand, we have

$$|u_{n+1}(t) - f(t)| \leq C \sum_{i=1}^n \frac{\gamma^{i-1} t^i}{i!}, \tag{2.13}$$

and

$$|u_{n+1}'(t) - f'(t)| \leq C \sum_{i=1}^n \frac{\gamma^{i-1} t^i}{i!}, \tag{2.14}$$

then from (2.8), (2.12) and (2.13) we have

$$\alpha \leq u_{n+1}(t) \leq \beta. \tag{2.15}$$

Also, from (2.9), (2.12) and (2.14) we obtain

$$\bar{\alpha} \leq u_{n+1}'(t) \leq \bar{\beta}, \tag{2.16}$$

this completes the inductive argument.

Since (P_n) is obviously true for all n, this bound makes it obvious that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to $u \in C^1(0, \delta_1)$, and we can write

$$\sum_{i=0}^{\infty} \varphi_i(t) = \lim_{n\to+\infty} u_n(t) = u(t).$$

To prove that u satisfies the original equation (1.16), we put

$$u(t) = u_n(t) + \Delta_n(t),$$

then

$$u'(t) = u'_n(t) + \Delta'_n(t).$$

We have

$$|u(t) - f(t) - \int_0^t (t-s)K(s, u(s), u'(s))ds| =$$

$$= |u_n(t) + \Delta_n(t) - f(t) - \int_0^t (t-s)K(s, u(s), u'(s))ds|$$

$$\leq |\Delta_n(t)| + |\int_0^t (t-s)(K(s, u_{n-1}, u'_{n-1}) - K(s, u(s), u'(s)))ds|$$

$$\leq |\Delta_n(t)| + \delta_1 \int_0^t L_{\alpha,\beta}|u_{n-1}(s) - u(s)| + \bar{L}_{\bar{\alpha},\bar{\beta}}|u'_{n-1}(s) - u'(s)|ds$$

$$\leq |\Delta_n(t)| + \delta_1 \int_0^t L_{\alpha,\beta}|\Delta_{n-1}(s)| + \bar{L}_{\bar{\alpha},\bar{\beta}}|\Delta'_{n-1}(s)|ds$$

$$\leq |\Delta_n(t)| + \delta_1 \max\{L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}}\}\|\Delta_{n-1}(s)\|_{C^1([0,\delta_1])}$$

$$\leq \|\Delta_n\|_{C^1([0,\delta_1])} + \|\Delta_{n-1}\|_{C^1([0,\delta_1])}.$$

But,

$$\lim_{n\to+\infty} \|\Delta_n\|_{C^1([0,\delta_1])} = 0,$$

and thus u is a solution of (1.16).

To show that $u(t)$ is the only continuous solution, suppose that there exists another solution $\tilde{u}(t) \in C^1(0, \delta_1)$ of (1.16). Then, for all $t \in [0, \delta_1]$

$$|u(t) - \tilde{u}(t)| + |u'(t) - \tilde{u}'(t)| = \left|\int_0^t (t-s)(K(s, u(s), u'(s)) - K(s, \tilde{u}(s), \tilde{u}'(s)))ds\right| +$$

$$+ \left|\int_0^t (K(s, u(s), u'(s)) - K(s, \tilde{u}(s), \tilde{u}'(s)))ds\right|,$$

from which it follows that

$$|u(t) - \tilde{u}(t)| + |u'(t) - \tilde{u}'(t)| \leq \delta_1 \int_0^t L_{\alpha,\beta}|u(s) - \tilde{u}(s)| + \bar{L}_{\bar{\alpha},\bar{\beta}}|u'(s) - \tilde{u}'(s)|ds$$

$$+ \int_0^t L_{\alpha,\beta}|u(s) - \tilde{u}(s)| + \bar{L}_{\bar{\alpha},\bar{\beta}}|u'(s) - \tilde{u}'(s)|ds,$$

$$\leq \gamma \int_0^t |u(s) - \tilde{u}(s)| + |u'(s) - \tilde{u}'(s)|ds. \tag{2.17}$$

Since $u(t)$ and $\tilde{u}(t)$ are both continuous in $[0, \delta_1]$, $\exists C > 0$ such that,

$$|u(t) - \tilde{u}(t)| + |u'(t) - \tilde{u}'(t)| \leq C, \qquad \forall t \in [0, \delta_1].$$

substituting this into (2.17)

$$|u(t) - \tilde{u}(t)| + |u'(t) - \tilde{u}'(t)| \leq C\gamma t.$$

By repeating the argument n times, we are led to

$$|u(t) - \tilde{u}(t)| + |u'(t) - \tilde{u}'(t)| \leq C\frac{\gamma^n t^n}{n!},$$

for any n hence, we conclude that $u(t) = \tilde{u}(t)$ for all $t \in [0, \delta_1]$.

Now, for $t \in [\delta_1, \delta_2]$, we write the equation as

$$u_1(t) = F(t) + \int_{\delta_1}^{t} (t - s)K(s, u_1(s), u_1'(s))ds, \quad \forall t \in [\delta_1, \delta_2], \qquad (2.18)$$

where

$$F(t) = f(t) + \int_0^{\delta_1} (t - s)K(s, u_0(s), u_0'(s))ds,$$

and $u_0(s)$ is the solution obtained in the first step, but (2.18) is just the same Volterra equation with an origin shifted from 0 to δ_1. So, we can apply the same basic steps. We define

$$u(t) = \begin{cases} u_0(t), & t \in [0, \delta_1], \\ u_1(t), & t \in [\delta_1, \delta_2]. \end{cases} \qquad (2.19)$$

It is clear that $u \in C^1(0, \delta_2)$ is the unique solution of (1.16) over $[0, \delta_2]$.

This argument can be repeated and since there is only a finite number of subintervals in $[0, T]$, we thereby construct the unique solution in $C^1(0, T)$. $\qquad \square$

3. Numerical study

In this part, we use a numerical method based on the numerical integration to approximate the solution of the equation. This technique is called *Nyström method*, which is studied in [3,4,6]

For $N \in \mathbb{N}^*$, we construct a subdivision of the interval $[0, T]$,

$$h = \frac{T}{N}, \qquad t_i = ih, \qquad 0 \leq i \leq N.$$

We denote $U_n \approx u(t_n)$ and $V_n \approx u'(t_n)$, by applying the Nyström method for the approximation of the integrals that appear in our equation, using the quadratic formula of the numerical integration on the equations (1.16) and (1.17)

$$\int_a^b \xi(t)dt \approx h \sum_{i=0}^{N} w_i \xi(t_i),$$

where, w_i are real, such that it exits $W > 0$, $\forall N \in \mathbb{N}^*$, $\max_{0 \leq j \leq N} |w_j| \leq W$.

We obtain the following system

$$U_0 = f(0), \qquad (3.1)$$

$$V_0 = f'(0), \qquad (3.2)$$

$$U_n = f(t_n) + h\sum_{i=0}^{n-1} w_i(t_n - t_i)K(t_i, U_i, V_i), \qquad 1 \leq n \leq N \qquad (3.3)$$

$$V_n = f'(t_n) + h\sum_{i=0}^{n} w_i K(t_i, U_i, V_i), \qquad 1 \leq n \leq N \qquad (3.4)$$

Lemma 3.1. *Consider the system* (3.1)-(3.4), *for a fixed* $N \in \mathbb{N}^*$, $\exists \alpha_1, \beta_1, \bar{\alpha}_1, \bar{\beta}_1 \in \mathbb{R}$, *for all* $0 \leq n \leq N$, *we have*

$$\alpha_1 \leq \quad U_n \quad \leq \beta_1 \qquad\qquad (3.5)$$
$$\bar{\alpha}_1 \leq \quad V_n \quad \leq \bar{\beta}_1 \qquad\qquad (3.6)$$

where, $\alpha_1, \beta_1, \bar{\alpha}_1$ *and* $\bar{\beta}_1$ *are independent from* N.

Proof. We have, for $1 \leq n \leq N$

$$U_n \quad = \quad h \sum_{i=0}^{n-1} w_i (t_n - t_i) K(t_i, U_i, V_i) + f(t_n).$$

Then

$$|U_n - f(t_n)| \quad \leq \quad h \sum_{i=0}^{n-1} w_i (t_n - t_i) |K(t_i, U_i, V_i)|$$
$$\leq \quad T^2 W M.$$

Therefore

$$\alpha - T^2 W M \leq f(t_n) - T^2 W M \leq \quad U_n \quad \leq T^2 W M + f(t_n) \leq T^2 W M + \beta$$
$$\alpha_1 \leq \quad U_n \quad \leq \beta_1.$$

On the other hand, we have for $1 \leq n \leq N$

$$V_n \quad = \quad h \sum_{i=0}^{n} w_i K(t_i, U_i, V_i) + f'(t_n),$$

then

$$|V_n - f'(t_n)| \quad \leq \quad h \sum_{i=0}^{n} w_i |K(t_i, U_i, V_i)|$$
$$\leq \quad T W M.$$

We obtain

$$\bar{\alpha} - T W M \leq f'(t_n) - T W M \leq \quad V_n \quad \leq T W M + f'(t_n) \leq T W M + \bar{\beta}$$
$$\bar{\alpha}_1 \leq \quad V_n \quad \leq \bar{\beta}_1.$$

It is clear that

$$\alpha_1 \leq U_0 \quad = \quad f(t_0) \leq \beta_1,$$
$$\bar{\alpha}_1 \leq V_0 \quad = \quad f'(t_0) \leq \bar{\beta}_1.$$

$\square$

3.1. System study

The next theorem shows the existence and uniqueness of the solution of the system (3.1)-(3.4), under the properties (H)

Theorem 3.2. *For h sufficiently small, the system* (3.1)-(3.4) *has a unique solution.*

Proof. It is clear that we can obtain the solution of (3.3) by the recurrence formula, it remains to prove the existence of the solution of the equation (3.4).
For all $n \geq 1$, we define

$$\psi_n : \mathbb{R} \rightarrow \mathbb{R}$$
$$X \rightarrow \psi_n(X).$$

Such that

$$\psi_n(X) = f'(t_n) + h \sum_{i=0}^{n-1} w_i K(t_i, U_i, V_i) + h w_n K(t_n, U_n, X).$$

For all $X, Y \in \mathbb{R}$, we have

$$|\psi_n(X) - \psi_n(Y)| = |h w_n K(t_n, U_n, X) - h w_n K(t_n, U_n, Y)|,$$
$$\leq h W \bar{L}_{\bar{\alpha}_1, \bar{\beta}_1} |X - Y|.$$

Consequently, ψ_n is a contraction for h small enough. Using Banach's fixed point, we get the result. □

3.2. Error analysis

In this part, we will show that the numerical method constructed previously converges to the exact solution of the equation. For $N \in \mathbb{N}^*$, we define for $0 \leq n \leq N$,

$$\varepsilon_n := |U_n - u(t_n)| + |V_n - u'(t_n)|.$$

The method converges if

$$\lim_{h \to 0} \left(\max_{0 \leq n \leq N} \varepsilon_n \right) = 0.$$

We define local consistency error by

$$\delta(h, t_n) = \left| \int_0^{t_n} (t_n - s) K(s, u(s), u'(s)) ds - h \sum_{i=0}^{n-1} (t_n - t_i) w_i K(t_i, u(t_i), u'(t_i)) \right| +$$
$$+ \left| \int_0^{t_n} K(s, u(s), u'(s)) ds - h \sum_{i=0}^{n} w_i K(t_i, u(t_i), u'(t_i)) \right|.$$

The approximation method (3.1)-(3.4) is said consistent with (1.16), if

$$\lim_{h \to 0} \left(\max_{0 \leq n \leq N} \delta(h, t_n) \right) = 0.$$

Theorem 3.3. *If the approximation method* (3.1)-(3.4) *is consistent with* (1.16)*, then*

$$\lim_{h \to 0} \left(\max_{0 \leq n \leq N} \varepsilon_n \right) = 0.$$

Proof. For $n \geq 1$.

$$
\begin{aligned}
\varepsilon_n &= |U_n - u(t_n)| + |V_n - u'(t_n)| \\
&= \left| h \sum_{i=0}^{n-1} (t_n - t_i) w_i K(t_i, U_i, V_i) - \int_0^{t_n} (t_n - s) K(s, u(s), u'(s)) ds \right| + \\
&\quad + \left| h \sum_{i=0}^{n} w_i K(t_i, U_i, V_i) - \int_0^{t_n} K(s, u(s), u'(s)) ds \right|, \\
&\leq \left| h \sum_{i=0}^{n-1} (t_n - t_i) w_i K(t_i, u(t_i), u'(t_i)) - \int_0^{t_n} (t_n - s) K(s, u(s), u'(s)) ds \right| + \\
&\quad + \left| h \sum_{i=0}^{n} w_i K(t_i, u(t_i), u'(t_i)) - \int_0^{t_n} K(s, u(s), u'(s)) ds \right| + \\
&\quad + \left| h \sum_{i=0}^{n-1} (t_n - t_i) w_i K(t_i, U_i, V_i) - h \sum_{i=0}^{n-1} (t_n - t_i) w_i K(t_i, u(t_i), u'(t_i)) \right| + \\
&\quad + \left| h \sum_{i=0}^{n} w_i K(t_i, U_i, V_i) - h \sum_{i=0}^{n} w_i K(t_i, u(t_i), u'(t_i)) \right|, \\
&\leq \delta(h, t_n) + hW \sum_{i=0}^{n-1} (t_n - t_i) \left[L_{\alpha,\beta} |U_i - u(t_i)| + \bar{L}_{\bar{\alpha},\bar{\beta}} |V_i - u'(t_i)| \right] + \\
&\quad + hW \sum_{i=0}^{n-1} \left[L_{\alpha,\beta} |U_i - u(t_i)| + \bar{L}_{\bar{\alpha},\bar{\beta}} |V_i - u'(t_i)| \right] + hW \left[L_{\alpha,\beta} |U_n - u(t_n)| + \bar{L}_{\bar{\alpha},\bar{\beta}} |V_n - u'(t_n)| \right].
\end{aligned}
$$

Then

$$
\begin{aligned}
\varepsilon_n &\leq \delta(h, t_n) + hW L_{\alpha,\beta} |U_n - u(t_n)| + hW \bar{L}_{\bar{\alpha},\bar{\beta}} |V_n - u'(t_n)| \\
&\quad + hW \sum_{i=0}^{n-1} (t_n - t_i + 1) L_{\alpha,\beta} |U_i - u(t_i)| + hW \sum_{i=0}^{n-1} (t_n - t_i + 1) \bar{L}_{\bar{\alpha},\bar{\beta}} |V_i - u'(t_i)|.
\end{aligned}
$$

For h small enough

$$
\alpha = \min((1 - hW L_{\alpha,\beta}), (1 - hW \bar{L}_{\bar{\alpha},\bar{\beta}})) > 0,
$$

and

$$
\varepsilon_n \leq \frac{1}{\alpha} \delta(h, t_n) + \frac{1}{\alpha} hW \sum_{i=0}^{n-1} (t_n - t_i + 1)(L_{\alpha,\beta} |U_i - u(t_i)| + \bar{L}_{\bar{\alpha},\bar{\beta}} |V_i - u'(t_i)|),
$$

then

$$
\varepsilon_n \leq \frac{hW \max((t_n - t_i + 1) L_{\alpha,\beta}, (t_n - t_i + 1) \bar{L}_{\bar{\alpha},\bar{\beta}})}{\alpha} \sum_{i=0}^{n-1} \varepsilon_i + \frac{1}{\alpha} \delta(h, t_n).
$$

Applying Theorem 7.1 from [6], we get

$$
\begin{aligned}
\varepsilon_n \leq \frac{1}{\alpha} \Big(1 + & \frac{hW(t_n - t_i + 1) \max(L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}})}{\alpha} \Big)^{n-1} \Big(\max_{1 \leq i \leq n} \delta(h, t_i) + hW(t_n - t_i + 1) \\
& \times \max(L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}}) \varepsilon_0 \Big),
\end{aligned}
$$

but

$$\left(1 + \frac{hW(t_n - t_i + 1)\max(L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}})}{\alpha}\right)^{n-1} \leq \left(1 + \frac{TW(t_n - t_i + 1)\max(L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}})}{N\alpha}\right)^{N},$$

and

$$\lim_{N \to +\infty} \left(1 + \frac{TW(t_n - t_i + 1)\max(L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}})}{N\alpha}\right)^N < +\infty.$$

Then, $\exists \theta > 0$ such that

$$\forall N \in \mathbb{N}; \qquad \max \frac{1}{\alpha}\left(1 + \frac{hW(t_n - t_i + 1)\max(L_{\alpha,\beta}, \bar{L}_{\bar{\alpha},\bar{\beta}})}{\alpha}\right)^{n-1} \leq \theta.$$

And the desired result is obtained. □

4. Experimental results

Our study allows us to solve a rather complicated problem where the numerical results show their efficiency and accuracy. To show this efficiency and precision of the proposed method in this paper, and to illustrate the approximation performance of our modeling of the mechanical model as well as the solving of the integro-differential nonlinear equation of Volterra (1.16), we complete the study with results obtained from an experimental simulation of Earthquake machine, using the Nyström approximation method proved in the previous section.

To perform the calculus, we need to clarify some parameters that intervene in our equation (1.16), indeed the physical parameters ϱ, m, M, R and F are

$$\varrho = 10N.m^{-1}, \qquad m = 10g, \qquad M = 50000g \qquad R = 2N, \qquad F = 3N$$

in an interval $t \in [0, 50]$ with a discretization $N = 1000$.

We choose $\{w_i\}_{0 \leq i \leq N}$ of the Trapezoidal rule i.e. for $1 \leq i \leq N - 1$, $w_i = 1$, and $w_0 = w_N = \frac{1}{2}$.

For $1 \leq n \leq N$, U_n is calculated directly from the previous terms, but V_n is approached using Banach fixed point sequence with the stopping condition of the form $|X_{old} - X_{new}| \leq 10^{-7}$.

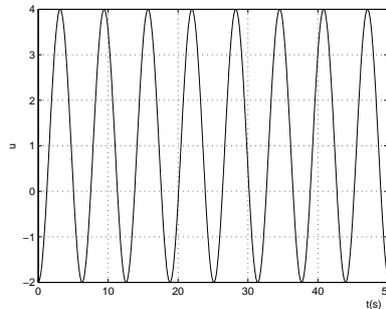The results of this numerical calculus for these values are illustrated in the figures 2, 3 and 4
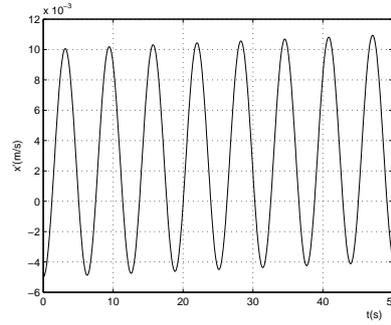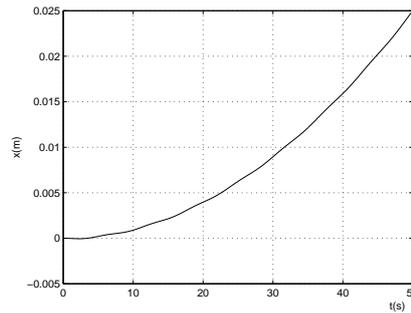

Figure 2: The seismic function: u

Figure 3: The velocity of mass M: x'

Figure 4: The movement of M: x

The results presented in these graphs are compatible with the mechanism of the earthquake machine, and they show the efficiency of our proceeding and the good vision for this phenomenon modeling.

From different values of parameters that we have put, it can be seen that the velocity $x'(t)$ can take a negative values in some moments. This latter is due to the theoretical approach of replacing the function δ with another smoother function δ_ξ defined in (1.13), in addition to the error spread of the numerical approximation using the Nyström method.

The reader at this point can ask the following question: What happens when ξ approaches zeros? The answer is 'nothing !', because the curves of u, x and x' keep the same shape if we replace δ_ξ by δ in our program, as it is entailed in the figures 2, 3 and 4 . Since the studied equation remains well constructed and admits a unique solution, although theoretically we can not treat the case $\xi = 0$ using Picard's successive method. In addition, the numerical method is also convergent in this case. We explore another analytical and numerical track to study our equation when $\xi = 0$.

During its movement, the mass M jumps into different distances over time, as shown in the figures 4. This jump reflects the null values of speed recorded in our results which proves our good vision of approximation of the physical phenomenon.

The following table shows that each time the discretization N is increased , the minimum of the velocity converges to 0.

Table 1: Comparison of the different velocity values according to N.

N	250	500	750	1000	250	1500	1750	2000
$\min(x')(m/s)$	-0.0198	-0.0099	-0.0066	-0.0050	-0.0040	-0.0033	-0.0029	-0.0025

5. Conclusion

The main purpose of the current paper has been to construct a modeling of an earthquake machine mechanism, and to study the existence and uniqueness of the solution of an integro-differential nonlinear Volterra equation that results from this modeling . Developed numerical tests show the effectiveness of this modeling of the problem as well as Nyström method used to approximate the solution of this equation.

Currently, we are studying the case where the resistance is not constant. This situation is more realistic and will give better results, but gives rise to new mathematical challenges.

References

1. X. Le Pichon, J. Francheteau and J. Bonnin , *Plate Tectonics* (Elsevier Scientific Publishing Company Amsterdam, London, New York, 1973)

2. G. D. Williams, Tectonics and seismic sequence stratigraphy: an introduction, *Tectonics and Seismic Sequence Stratigraphy. Geological Society Special Publication* No. 71 (1993) 1–13.

3. H. Guebbai, M.Z. Aissaoui, I. Debbar and B. Khalla, *Analytical and numerical study for an integro-differential nonlinear Volterra equation*, Applied Mathematics and Computation **229**(2014) 367–373.

4. S. Segni, M. Ghiat and H. Guebbai, *New approximation method for Volterra nonlinear integro-diferential equation*, Asian-European Journal of Mathematics, Vol. 12, No. 1, 1950016 Doi:10.1142/S1793557119500165 (2019)

5. M. Ghiat and H. Guebbai, *Annalytical and numerical study for an integro-differential nonlinear Volterra equation with weakly singular kernel*, Com. Appl. Math, Doi: https://doi.org/10.1007/s40314-018-0597-3 (2018)

6. P. Linz, *Analytical and numerical methods for Volterra equations* (SIAM Studies in Applied Mathematics, Philadelphia, 1985).

7. A. Atkinson and W. Han , *Theoretical Numerical Analysis: A Functional Analysis Framework* (Springer-Verlag, New York, 2001).

8. Tokui Sato, *Sur L'équation Intégrale non Linéaire de Volterra*, Compositio Mathematica. tome 11 (1953) 271–290.

9. H. Brunner, *The numerical treatment of Volterra integro-differential equations with unbounded delay*, J. Comput. Appl. Math. **28** (1989) 5–23.

10. R. K. Nagle, E. B. Saff and A. D. Snider , *Fundamentals of Differential Equations* (Pearson international edition, 2008)

Selma Salah,
Laboratoire de Mathématiques appliquées et de modélisation,
Université 8 mai 1945 Guelma,
Algeria.
E-mail address: `salah.selma@univ-guelma.dz, selmasalah@hotmail.fr`

and

Hamza Guebbai ,
Laboratoire de Mathématiques appliquées et de modélisation,
Université 8 mai 1945 Guelma,
Algeria.
E-mail address: `guebbai.hamza@univ-guelma.dz`

and

Samir Lemita,
Laboratoire de Mathématiques appliquées et de modélisation,
Université 8 mai 1945 Guelma,
Algeria.
E-mail address: `lemita.samir@univ-guelma.dz`

and

Mohamed Zine Aissaoui,
Laboratoire de Mathématiques appliquées et de modélisation,
Université 8 mai 1945 Guelma,
Algeria.
E-mail address: `aissaoui.zine@univ-guelma.dz`