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Existence of Homoclinic Solutions for Difference Equations on Integers Via Variational Method

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ABSTRACT: In this paper, we study the elastic membrane equation with dynamic boundary conditions, source term and a nonlinear weak damping localized on a part of the boundary and past history. Under some appropriate assumptions on the relaxation function the general decay for the energy have been established using the perturbed Lyapunov functionals and some properties of convex functions.

Key Words: Discrete boundary value problem, Nonlinear difference equation, Critical point, Lipschitz condition, Variational methods.

Contents

1. Introduction

The aim of this paper is to establish the existence of homoclinic solutions for the following discrete boundary value problem

$$\begin{cases} -\Delta(a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda f(k,u(k)) + \mu g(k,u(k)) + h(u(k)), & k \in \mathbb{Z}, \\ u(k) \to 0 & as \quad |k| \to \infty \end{cases} \tag{$P^{f,g,h}_{\lambda,\mu}$}$$

where $1 < p < +\infty$, $\lambda > 0$, $\mu \geq 0$, $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, $a, b : \mathbb{Z} \to (0, \infty)$, $f, g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions in the second variables, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function of order $p-1$ with Lipschitzian constant $L \geq 0$ such that $h(0) = 0$, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. A solution $u = u(k)$ of $(P^{f,g,h}_{\lambda,\mu})$ is homoclinic if $\lim_{|k|\to\infty} u(k) = 0$. Also we define the following conditions:

(A_1) $b(k) \geq \beta > 0$ for all $k \in \mathbb{Z}$, $b(k) \to +\infty$ as $|k| \to +\infty$.

There are many papers about existence of solutions to boundary value problems for finite difference equations with p-Laplacian operator which branching out in many fields such as biologic, economic, farm and other areas. There are various methods such fixed point, variational methods, critical point theory, Morse theory and the mountain-pass theorem. For background and recent results, we refer the reader to [1,2,3,4,5,6,7,8,12,13,19,20,22,23,24,26,27,28,29,30,32,33]. For example, Henderson and Thompson investigated existence multiple solutions for second order discrete boundary value problems in [19]. Wong and Xie proved three symmetric solutions of lidstone boundary value problem for partial equation in [33]. In [12] Cabada and Tersian studied the existence of homoclinic solutions for semilinear p-Laplacian difference equations with periodic coefficients based on the Brezis-Nirenberg's mountain pass theorem. In [13] Candito and D'Aguì studied discrete nonlinear Neumann problems to find three solutions. In addition, Iannizzotto and Tersian in [20] via critical point theory investigated the existence of multiple homoclinic solutions for the discrete p-Laplacian problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1)) + a(k)\phi_p(u(k)) = \lambda f(k,u(k)), & \forall k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as} & |k| \to \infty \end{cases} \tag{1.1}$$

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where $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$ is a positive and coercive weight function and $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function in the second variable. In [23] Kong studied the problem $(P^{f,g,h}_{\lambda,\mu})$, in the case $\mu = 0$ and $h \equiv 0$ and proved the problem has infinitely many homoclinic solutions. Also, a variant of the fountain theorem was utilized. Kong in [22] studied the following higher order difference equation defined on $\mathbb{Z}$ with p-Laplacian

$$\begin{cases} (-1)^n\Delta^n(a(k-n)\phi_p(\Delta^n u(k-n)) + b(k)\phi_p(u(k)) & \\ = \lambda f(k, u(k+1), u(k), u(k-1)), & \forall k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as} & |k| \to \infty \end{cases} \tag{1.2}$$

where $n > 1$ is an integer, $p > 1$ is a real number, $\lambda > 0$ is a parameter, $\phi_p = |t|^{p-2}t$ for $t \in \mathbb{R}$, Δ is the forward difference operator defined by $u(k) = u(k+1) - u(k)$ for $k \in \mathbb{Z}$, $\Delta^i u(k) = \Delta(\Delta^{i-1}u(k))$ for $i \in \mathbb{N}$, $a, b : \mathbb{Z} \to (0,\infty)$, and $f : \mathbb{Z}\times\mathbb{R}^3 \to \mathbb{R}$ is continuous in the second, third, and fourth variables. By using the critical point theory, sufficient conditions were obtained for the existence of infinitely many homoclinic solutions of the problem (1.2), based on the fountain theorem in combination with the variational technique. Stegliński in [28] determined a concrete interval of positive parameter λ, for prove the existence of homoclinic solutions for the problem (1.1), while in [30] dealt with the problem $(P^{f,g,h}_{\lambda,\mu})$, in the case $\mu = 0$ and $h \equiv 0$, and using both the general variational principle of Ricceri and the direct method introduced by Faraci and Kristály proved the existence of infinitely many homoclinic solutions for a the problem where the nonlinear term f has an appropriate oscillatory behavior at zero. In [4], sufficient conditions for the existence of at least one homoclinic solution for a nonlinear second-order difference equation with p-Laplacian were presented.

Inspired by the above results, in the present paper, we obtain the existence of at least three distinct nonnegative solutions for the problem $(P^{f,g,h}_{\lambda,\mu})$, in which two parameters are involved. Estimation of these two parameters λ and μ will be given. In particular, in Theorem 3.1 we establish the existence of at least three distinct nonnegative solutions for the problem $(P^{f,g,h}_{\lambda,\mu})$. Theorem 3.4 is a consequence of Theorem 3.1. The Examples 3.3 and 3.5 help us to illustrate our main results. In Example 3.3 the hypotheses of Theorem 3.1 are fulfilled and in Example 3.5 the condition of Theorem 3.4 are satisfied. As a special case of Theorem 3.4, we obtain Theorem 3.6 which under suitable conditions on f at zero and at infinity, ensures two positive solutions for the autonomous case of the problem. Finally, by the way of example, we point out Theorem 3.7, as simple consequence of Theorem 3.6.

2. Preliminaries

In this paper X denotes a finite dimensional real Banach space

$$X = \left\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] < \infty\} \right\}$$

with the norm

$$\|u\| = \Big(\sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] \Big)^{\frac{1}{p}},$$

and as is shown in [20] $(X, \|.\|)$ is a reflexive Banach space and the embedding $X \hookrightarrow \ell^p$ is compact and $I_\lambda : X \to \mathbb{R}$ is a functional satisfying the following structure hypothesis: $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$ where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$, and λ is a positive real parameter.

In this framework a finite dimensional variant of [8, Theorem 3.3] (see also Corollary 3.1 and Remark 3.9 of [8]) is the following:

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r_1,\ r_2,\ r_3$, with $r_2 > r_1$ and $r_2 > \inf_X \Phi$, and all $r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1,r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)},$$

$$\gamma(r_2,r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3},$$

$$\alpha(r_1,r_2,r_3) := \max\{\varphi(r_1),\varphi(r_2),\gamma(r_2,r_3)\}.$$

Theorem 2.1 ([8, Theorem 3.3])**.** *Assume that*

(B_1) Φ *is convex and* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(B_2) *for every* $u_1, u_2 \in X$ *such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 *with* $r_1 < r_2$, *such that*

(B_3) $\varphi(r_1) < \beta(r_1,r_2)$;

(B_4) $\varphi(r_2) < \beta(r_1,r_2)$;

(B_5) $\gamma(r_2,r_3) < \beta(r_1,r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}[$ *the functional* $\Phi - \lambda\Psi$ *admits three distinct critical points* u_1, u_2, u_3 *such that* $u_1 \in \Phi^{-1}(-\infty,r_1)$, $u_2 \in \Phi^{-1}[r_1,r_2)$ *and* $u_3 \in \Phi^{-1}(-\infty, r_2+r_3)$.

We refer the interested reader to the papers [11,14,15,16,17,18,21] in which Theorem 2.1 has been successfully employed to get the existence of at least three solutions for boundary value problems.

For Banach space X and the norm explained above the following inequality is obvious

$$\|u\|_\infty \leq \|u\|_p \leq \beta^{\frac{-1}{p}}\|u\| \quad \forall u \in X, \tag{2.1}$$

where (A_1) is satisfied. Let two functions $f, g : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be two continuous functions in the second variables and $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function of order $p-1$ with Lipschitzian constant $L \geq 0$, i.e,

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2|^{p-1}$$

for every $t_1, t_2 \in \mathbb{R}$, and $h(0) = 0$. Suppose the constant $L \geq 0$ satisfies $L\beta^{\frac{-1}{p}} < 1$. Corresponding to the function f, g and h, we introduce $F, G : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(k,t) := \int_0^t f(k,\xi)d\xi \quad \forall (k,t) \in \mathbb{Z}\times\mathbb{R},$$

$$G(k,t) := \int_0^t g(k,\xi)d\xi \quad \forall (k,t) \in \mathbb{Z}\times\mathbb{R}$$

and

$$H(t) := \int_0^t h(\xi)d\xi \quad \forall t \in \mathbb{R}.$$

3. Main results

Set

$$G^{\theta} := \sum_{k\in\mathbb{Z}} \max_{|\xi|\le\theta} G(k,\xi) \quad \forall\theta>0$$

and

$$G_{\eta} := \sum_{k\in\mathbb{Z}} \inf_{[0,\eta]} G(k,t) \quad \forall\eta>0.$$

If g is sign-changing, then $G^{\theta}\ge 0$ and $G_{\eta}\le 0$.

We formulate our main result as follows:

Fixing four positive constants θ_1, θ_1, θ_1, η and the the integer k_0, put

$$\begin{aligned}\delta_{\lambda,G} := \min\Big\{ &\frac{1}{p}\min\Big\{\frac{(1-L\beta^{-1})\beta\theta_1^p - \lambda p\sum_{k\in\mathbb{Z}}F(k,\theta_1)}{G^{\theta_1}},\\ &\frac{(1-L\beta^{-1})\beta\theta_2^p - \lambda p\sum_{k\in\mathbb{Z}}F(k,\theta_2)}{G^{\theta_2}},\\ &\frac{(1-L\beta^{-1})\beta(\theta_3^p-\theta_2^p) - \lambda p\sum_{k\in\mathbb{Z}}F(k,\theta_3)}{G^{\theta_3}}\Big\},\\ &\frac{\frac{1}{p}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p - \lambda(F(k_0,\eta)-\sum_{k\in\mathbb{Z}}F(k,\theta_1))}{G_{\eta}-G^{\theta_1}}\Big\}.\end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that there exist positive constants θ_1, θ_2, θ_3, η and the integer k_0 with*

$$\theta_1 < \beta^{\frac{-1}{p}}(a(k_0+1)+a(k_0)+b(k_0))^{\frac{1}{p}}\eta,$$

$$\frac{\beta^{\frac{-1}{p}}(1+L\beta^{-1})}{1-L\beta^{-1}}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)^{\frac{1}{p}}\eta < \theta_2$$

and $\theta_2<\theta_3$ such that

(A_2) $f(k,t)\ge 0 \quad \forall(k,t)\in\mathbb{Z}\times[0,\theta_3]$;

(A_3) $$\max\Big\{\frac{\sum_{k\in\mathbb{Z}}F(k,\theta_1)}{\theta_1^p},\frac{\sum_{k\in\mathbb{Z}}F(k,\theta_2)}{\theta_2^p},\frac{\sum_{k\in\mathbb{Z}}F(k,\theta_3)}{\theta_3^p}\Big\} < \frac{(1-L\beta^{-1})\beta}{1+L\beta^{-1}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}\Big(F(k_0,\eta)-\sum_{k\in\mathbb{Z}}F(k,\theta_1)\Big).$$

Then, for every

$$\begin{aligned}\lambda\in\Lambda := \Bigg(&\frac{\frac{1+L\beta^{-1}}{p}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}{F(k_0,\eta)-\sum_{k\in\mathbb{Z}}F(k,\theta_1)},\\ &\frac{(1-L\beta^{-1})\beta}{p}\times\min\Big\{\frac{\theta_1^p}{\sum_{k\in\mathbb{Z}}F(k,\theta_1)},\frac{\theta_2^p}{\sum_{k\in\mathbb{Z}}F(k,\theta_2)},\frac{\theta_3^p-\theta_2^p}{\sum_{k\in\mathbb{Z}}F(k,\theta_3)}\Big\}\Bigg)\end{aligned}$$

and for every nonnegative function $g:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$, there is $\delta_{\lambda,G}$ given by (3.1) such that for each $\mu\in[0,\delta_{\lambda,G})$, the problem $(P^{f,g,h}_{\lambda,\mu})$ has at least three nonnegative solutions u_1,u_2,u_3 such that $\max_{k\in\mathbb{Z}}|u_1(k)|<\theta_1$, $\max_{k\in\mathbb{Z}}|u_2(k)|<\theta_2$ and $\max_{k\in\mathbb{Z}}|u_3(k)|<\theta_3$.

Proof. Our goal is to apply Theorem 2.1 to the problem $(P^{f,g,h}_{\lambda,\mu})$. We consider the auxiliary problem $(P^{\hat{f},g,h}_{\lambda,\mu})$

$$\begin{cases} -\Delta(a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda\hat{f}(k,u(k)) & \\ +\mu g(k,u(k)) + h(u(k)), & k\in\mathbb{Z},\\ u(k)\to 0 & as \quad |k|\to\infty \end{cases}$$

where $\hat{f} : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function defined putting

$$\hat{f}(k,\xi) = \begin{cases} f(k,0) & if \quad \xi < 0, \\ f(k,\xi), & if \quad 0 \le \xi \le \theta_3, \\ f(k,\theta_3) & if \quad \xi > \theta_3. \end{cases} \tag{3.2}$$

From (A_1) any solution of the problem $(P^{\hat{f},g,h}_{\lambda,\mu})$ is nonnegative. In addition, if it satisfies also the condition $0 \le u(k) \le \theta_3$, and for every $k \in [1,T]$, clearly it turns to be also a nonnegative solution of $(P^{f,g,h}_{\lambda,\mu})$. Therefore, for our goal, it is enough to show that our conclusion holds for $(P^{\hat{f},g,h}_{\lambda,\mu})$. Let the functions Φ, Ψ for every $u \in X$, defined by

$$\Phi(u) = \frac{\|u\|^p}{p} - \sum_{k\in\mathbb{Z}} H(u(k)) \tag{3.3}$$

and

$$\Psi(u) = \sum_{k\in\mathbb{Z}} [F(k,u(k)) + \frac{\mu}{\lambda} G(k,u(k))]. \tag{3.4}$$

The function Ψ is differentiable at $u \in X$

$$\Psi'(u)(v) = \sum_{k\in\mathbb{Z}} [f(k,u(k)) + \frac{\mu}{\lambda} g(k,u(k))]v(k) \quad \forall v \in X.$$

From the definition of $(p-1)$-Lipschitz continues and $h(0) = 0$, so we deduce

$$\frac{1-L\beta^{-1}}{p}\|u\|^p \le \Phi(u) \le \frac{1+L\beta^{-1}}{p}\|u\|^p, \tag{3.5}$$

which the condition $L(1+L\beta^{-1}) < 1$ deduce that Φ is coercive. In addition Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \sum_{k\in\mathbb{Z}} \Big[a(k)\phi_p(\Delta u(k-1))\Delta v(k-1) + b(k)\phi_p(u(k))v(k) - h(u(k))v(k)\Big]$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semiconscious. Indeed, since $\sum_{k\in\mathbb{Z}} H(u(k))$ continuous on X, one has

$$\lim_{n\to\infty} \inf \Phi(u_n) = \lim_{n\to\infty} \inf \frac{\|u_n\|^p}{p} - \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} H(u_n(k)) \ge \frac{\|u\|^p}{p} - \sum_{k\in\mathbb{Z}} H(u(k)) = \Phi(u).$$

Therefore, the assumptions on Φ and Ψ, as requested in Theorem 2.1 are verified. We know the critical points of the function $\Phi - \lambda\Psi$ are the solutions for problem $(P^{f,g,h}_{\lambda,\mu})$. Define $w(k_0) = \eta$ and $w(k) = 0$ for every $k \in \mathbb{Z} \setminus \{k_0\}$. By using (3.3) and (3.5) we have

$$\frac{1-L\beta^{-1}}{p}\Big((a(k_0+1) + a(k_0) + b(k_0)\Big)\eta^p \le \Phi(w)$$

$$\le \frac{1+L\beta^{-1}}{p}\Big(a(k_0+1) + a(k_0) + b(k_0)\Big)\eta^p. \tag{3.6}$$

Setting $r_1 = \dfrac{(1-L\beta^{-1})\beta}{p}\theta_1^p$, $r_2 = \dfrac{(1-L\beta^{-1})\beta}{p}\theta_2^p$ and $r_3 = \dfrac{(1-L\beta^{-1})\beta}{p}(\theta_3^p - \theta_2^p)$,

so from (3.6) and from the conditions $\theta_1 < \beta^{\frac{-1}{p}}(a(k_0+1) + a(k_0) + b(k_0)))^{\frac{1}{p}}\eta$, $\dfrac{\beta^{\frac{-1}{p}}(1+L\beta^{-1})}{1-L\beta^{-1}}\Big((a(k_0+1) + a(k_0) + b(k_0)\Big)^{\frac{1}{p}}\eta < \theta_2$ and $\theta_2 < \theta_3$ we deduce

$$r_1 < \Phi(w) < r_2.$$

Now the estimation $\Phi(u) \leq r_1$ follows that

$$\begin{aligned}|u(k)|^p \leq \|u\|_\infty^p \leq \beta^{-1}\|u\|^p &\leq \frac{p}{(1-L\beta^{-1})\beta}\Phi(u)\\ &\leq \frac{p}{(1-L\beta^{-1})\beta} r_1 = \theta_1^p, \quad \forall k_0 \in \mathbb{Z}.\end{aligned}$$

From the definition of r_1, it follows that

$$\Phi^{-1}(-\infty, r_1) = \{u \in X; \Phi(u) < r_1\} \subseteq \{u \in X; |u| \leq \theta_1\},$$

Hence, by using the assumption (A_2), one has

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)} \sum_{k\in\mathbb{Z}} F(k,u(k)) \leq \sum_{k\in\mathbb{Z}} \max_{|t|\leq\theta_1} F(k,t) \leq \sum_{k\in\mathbb{Z}} F(k,\theta_1).$$

Similarity, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)} \sum_{k\in\mathbb{Z}} F(k,u(k)) \leq \sum_{k\in\mathbb{Z}} \max_{|t|\leq\theta_2} F(k,t) \leq \sum_{k\in\mathbb{Z}} F(k,\theta_2)$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \sum_{k\in\mathbb{Z}} F(k,u(k)) \leq \sum_{k\in\mathbb{Z}} \max_{|t|\leq\theta_3} F(k,t) \leq \sum_{k\in\mathbb{Z}} F(k,\theta_3)$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r_1) &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u) - \Psi(u)}{r_1 - \Phi(u)}\\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)} \sum_{k\in\mathbb{Z}}[F(k,u(k)) + \frac{\mu}{\lambda}G(k,u(k))]}{r_1}\\ &\leq \frac{\sum_{k\in\mathbb{Z}} \max_{|t|\leq\theta_1}[F(k,u(k)) + \frac{\mu}{\lambda}G(k,u(k))]}{r_1}.\end{aligned}$$

By replacing $r_1 = \dfrac{(1-L\beta^{-1})\beta}{p}\theta_1^p$, we conclude

$$\varphi(r_1) \leq \frac{p}{(1-L\beta^{-1})\beta}\left(\frac{\sum_{k\in\mathbb{Z}} F(k,\theta_1)}{\theta_1^p} + \frac{\mu}{\lambda}\frac{G^{\theta_1}}{\theta_1^p}\right)$$

and

$$\begin{aligned}\varphi(r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(\infty,r_2)} \Psi(u)}{r_2}\\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \sum_{k\in\mathbb{Z}}[F(k,u(k)) + \frac{\mu}{\lambda}G(k,u(k))]}{r_2}\\ &\leq \frac{\sum_{k\in\mathbb{Z}} \max_{|t|\leq\theta_2}[F(k,u(k)) + \frac{\mu}{\lambda}G(k,u(k))]}{r_2}.\end{aligned}$$

Then by replacing $r_2 = \dfrac{(1-L\beta^{-1})\beta}{p}\theta_2^p$, we deduce

$$\varphi(r_2) \leq \frac{p}{(1-L\beta^{-1})\beta}\left(\frac{\sum_{k\in\mathbb{Z}} F(k,\theta_2)}{\theta_2^p} + \frac{\mu}{\lambda}\frac{G^{\theta_2}}{\theta_2^p}\right)$$

and

$$\begin{aligned}\gamma(r_2,r_3) &\le \frac{\sup_{u\in\Phi^{-1}(\infty,r_2+r_3)}\Psi(u)}{r_3}\\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\sum_{k\in\mathbb{Z}}[F(k,u(k))+\frac{\mu}{\lambda}G(k,u(k))]}{r_3}\\ &\le \frac{p}{(1-L\beta^{-1})\beta}\frac{\sum_{k\in\mathbb{Z}}\max_{|t|\le\theta_3}[F(k,t)+\frac{\mu}{\lambda}G(k,t)]}{\theta_3^p-\theta_2^p}\\ &\le \frac{p}{(1-L\beta^{-1})\beta}\left(\frac{\sum_{k\in\mathbb{Z}}F(k,\theta_3)}{\theta_3^p-\theta_2^p}+\frac{\mu}{\lambda}\frac{G^{\theta_3}}{\theta_3^p-\theta_2^p}\right).\end{aligned}$$

Now for $u\in\Phi^{-1}(-\infty,r_1)$ we have

$$\begin{aligned}\beta(r_1,r_2) &\ge \frac{F(k_0,\eta)-\sum_{k\in\mathbb{Z}}F(k,\theta_1)+\frac{\mu}{\lambda}(G_\eta-G^{\theta_1})}{\Phi(w)-\Phi(u)}\\ &\ge \frac{F(k_0,\eta)-\sum_{k\in\mathbb{Z}}F(k,\theta_1)+\dfrac{\mu}{\lambda}(G_\eta-G^{\theta_1})}{\dfrac{1+L\beta^{-1}}{p}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}.\end{aligned}$$

Due to (A_3) we get

$$\alpha(r_1,r_1,r_1)<\beta(r_1,r_2).$$

Therefore, the assumptions (B_1) and (B_2) of Theorem 2.1 are verified. We verify $\Phi-\lambda\Psi$ admits three distinct critical points. Let two local minima u_2 and u_3 for $\Phi-\lambda\Psi$ are solutions for the problem $(P^{f,g,h}_{\lambda,\mu})$ and we want to prove that they are nonnegative. Let u_1 be a nontrivial solution of the problem $(P^{f,g,h}_{\lambda,\mu})$. Arguing by a contradiction, assume that the set $A=\{k\in\mathbb{Z}: u_1(k)<0\}$ is nonempty and of positive measure. Put $\bar v(k)=\min\{0,u_1\}$ for all $k\in\mathbb{Z}$ and we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\Big[a(k)\phi_p(\Delta u_1(k-1))\Delta\bar v(k-1)\\ &+b(k)\phi_p(u_1(k))\bar v(k)-h(u_1(k))\bar v(k)\Big]\\ &-\lambda\sum_{k\in\mathbb{Z}}[f(k,u_1(k))+\mu g(k,u_1(k))]\bar v(k)=0\quad\forall\bar v(k)\in X.\end{aligned}$$

Thus, from our sign assumptions on the data one has

$$\begin{aligned}0&\le(1-L\beta^{-1})\Big(\sum_{k\in A}[a(k)|\Delta u(k-1)|^p+b(k)|u(k)|^p]\Big)\\ &\le\sum_{k\in A}\Big[a(k)\phi_p(\Delta u_1(k-1))\Delta u_1(k-1)\\ &+b(k)\phi_p(u_1(k))u_1(k)-h(u_1(k))u_1(k)\Big]\le 0.\end{aligned}$$

Hence, since $1>L\beta^{-1}$, $u_1=0$ and this is absurd. Then, we deduce $u_2\ge0$ and $u_3\ge0$. Thus, it follows that $su_2+(1-s)u_3\ge0$ for all $s\in[0,1]$, and that $(\lambda f+\mu g)(k,su_2+(1-s)u_3)\ge0$, and consequently, $\Psi(su_2+(1-s)u_3)\ge0$, for all $s\in[0,1]$. Hence from Theorem 2.1 for every

$$\begin{aligned}\lambda\in\Bigg]&\frac{\dfrac{1+L\beta^{-1}}{p}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}{F(k_0,\eta)-\sum_{k\in\mathbb{Z}}F(k,\theta_1)},\\ &\frac{(1-L\beta^{-1})\beta}{p}\times\min\{\frac{\theta_1^p}{\sum_{k\in\mathbb{Z}}F(k,\theta_1)},\frac{\theta_2^p}{\sum_{k\in\mathbb{Z}}F(k,\theta_2)},\frac{\theta_3^p-\theta_2^p}{\sum_{k\in\mathbb{Z}}F(k,\theta_3)}\}\Bigg[\end{aligned}$$

and $\mu \in [0, \delta_{\lambda,G})$, the functional $\Phi - \lambda\Phi$ has three critical points u_1, u_2 and u_3 in X in order to $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_3 + r_2$, that is $\max_{k\in\mathbb{Z}} |u_1(k)| < \theta_1$, $\max_{k\in\mathbb{Z}} |u_2(k)| < \theta_2$, and $\max_{k\in\mathbb{Z}} |u_3(k)| < \theta_3$.

Remark 3.2. *If either $f(k,t) \neq 0$ for some $k \in \mathbb{Z}$ or $g(k,t) \neq 0$ for some $k \in \mathbb{Z}$, or both hold true the solutions of Theorem 3.1 are not trivial.*

Now, we present the following example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.3. *Consider the following problem*

$$\begin{cases} -\Delta(a(k)\Delta u(k-1) + b(k)u(k)) = \lambda f(k, u(k)) & \\ +\mu g(k, u(k)) + h(u(k)), & k \in \mathbb{Z}, \\ u(k) \to 0 & \text{as} \quad |k| \to \infty, \end{cases} \tag{3.7}$$

where $a(k) = \dfrac{1}{|k|}$ for every $k \in \mathbb{Z} \setminus \{0\}$ and

$$b(k) = \begin{cases} 10^{-4k} & k \in \mathbb{Z}^+, \\ 10^{4k} & k \in \mathbb{Z}^- \end{cases}$$

and

$$f(k,t) = \begin{cases} \frac{2}{3} \times 5(\frac{1}{2})^{|k|+1} t^4 & t < 1, \\ \frac{2}{3} \times (\frac{1}{2})^{|k|+1} \dfrac{5}{t} & t \geq 1, \end{cases}$$

also, we define $h(t) = \dfrac{1}{10^3}(1 - \cos t)$ for $t \in \mathbb{R}$. So, by the definition of F and H its obvious

$$F(k,t) = \begin{cases} \frac{2}{3}(\frac{1}{2})^{|k|+1} t^5 & t < 1, \\ \frac{2}{3}(\frac{1}{2})^{|k|+1}(1 + 5\ln(t)) & t \geq 1 \end{cases}$$

and $H(t) = \dfrac{1}{10^3}(t - \sin t)$ for every $t \in \mathbb{R}$. We suppose $\theta_1 = 10^{-4}$, $\theta_2 = 10^2$, $\theta_3 = 10^4$ and $\eta = 1$, in addition by selecting $\beta = 10^8$, $L = 10^{-4}$ and fixed $k_0 = 1$, since the hypotheses of Theorem 3.1 are fulfilled since $\theta_1 < 10^{-4}(\dfrac{3}{2} + 10^{-4})^{\frac{1}{2}}$, $\dfrac{10^{-4}(1 + 10^{-12})}{1 - 10^{-12}}(\dfrac{3}{2} + 10^{-12})^{\frac{1}{2}} < \theta_2$ and $\theta_2 < \theta_3$, and $f(k,t) \geq 0$ for each $(k,t) \in \mathbb{Z} \setminus \{0\} \times [0, 10^4]$, and taking into account that

$$\begin{aligned} &\max\left\{\frac{\sum_{k\in\mathbb{Z}} F(k, 10^{-4})}{10^{-8}}, \frac{\sum_{k\in\mathbb{Z}} F(k, 10^2)}{10^4}, \frac{\sum_{k\in\mathbb{Z}} F(k, 10^4)}{10^8}\right\} \\ &= \max\left\{\frac{10^{-20}}{10^{-8}}, \frac{1 + 10\ln(10)}{10^4}, \frac{1 + 20\ln(10)}{10^8}\right\} = \frac{1 + 10\ln(10)}{10^4} \end{aligned}$$

and

$$\begin{aligned} &\frac{(1 - 10^{-12})10^8}{1 + 10^{-12}(\frac{3}{2} + 10^{-4})}\Big(F(1,1) - \sum_{k\in\mathbb{Z}} F(k, 10^{-4})\Big) \\ &\approx \frac{(1 - 10^{-12})10^8}{2} \frac{\frac{1}{6} - 10^{-20}}{\frac{1 + 10^{-12}}{2}(\frac{3}{2} + 10^{-4})}, \end{aligned}$$

so the condition (A_3) of Theorem 3.1 is verified because

$$\frac{1 + 10\ln(10)}{10^4} < \frac{(1 - 10^{-12})10^8}{2} \frac{\frac{1}{6} - 10^{-20}}{\frac{1 + 10^{-12}}{2}(\frac{3}{2} + 10^{-4})}.$$

Then, for every

$$\lambda \in \Lambda := \Big(\frac{\frac{1+10^{-12}(1.5+10^{-12})}{3}}{\frac{1}{6}-10^{-20}}, \frac{1-10^{-12}(10^8)}{3} \times \frac{10^4}{24}\Big)$$

we set nonnegative function $g(t) = |\sin t|$, *and constant* $\lambda = 1$. *Since*

$$\begin{aligned}\min\Big\{&\frac{(1-10^{-12})10^{-8} - 3\times 10^{-20}}{0.99}, \frac{(1-10^{-12})10^{16} - 3(1+10\ln 10)}{0.17},\\ &\frac{(1-10^{-12})(10^{20}-10^{16}) - 3(1+10\ln 10)}{0.17}\Big\} = \frac{(1-10^{-12})10^{-8} - 3\times 10^{-20}}{0.99}.\end{aligned}$$

and

$$\begin{aligned}\min\Big\{&\frac{(1-10^{-12})10^{-8} - 3\times 10^{-20}}{0.99}, \frac{(1+10^{-12})(\frac{3}{2}+10^{-4}) - (1-10^{-20})}{1-0.99}\Big\}\\ &= \frac{(1-10^{-12})10^{-8} - 3\times 10^{-20}}{0.99}.\end{aligned}$$

So, there is $\delta_{\lambda,G} = \frac{(1-10^{-12})10^{-8}-3\times10^{-20}}{0.99}$ *such that for each* $\mu \in [0, \delta_{\lambda,G})$, *the problem* (3.7) *has at least three nonnegative solutions* u_1, u_2, u_3 *such that* $\max_{k\in\mathbb{Z}} |u_1(k)| < 10^{-4}$, $\max_{k\in\mathbb{Z}} |u_2(k)| < 10^2$ *and* $\max_{k\in\mathbb{Z}} |u_3(k)| < 10^4$.

For positive constants θ_1, θ_2 and η, put

$$\begin{aligned}\delta'_{\lambda,G} := \min\Big\{&\frac{1}{p}\min\Big\{\Big(\frac{(1-L\beta^{-1})\beta\theta_1^p - p\lambda\sum_{k\in\mathbb{Z}} F(k,\theta_1)}{G^{\theta_1}},\\ &\frac{(1-L\beta^{-1})\beta\theta_4^p - 2p\lambda\sum_{k\in\mathbb{Z}} F(k, \frac{1}{\sqrt[p]{2}}\theta_4)}{2G^{\frac{1}{\sqrt[p]{2}}\theta_4}},\\ &\frac{(1-L\beta^{-1})\beta\theta_4^p - 2p\lambda\sum_{k\in\mathbb{Z}} F(k,\theta_4)}{2G^{\theta_4}}\Big)\Big\},\\ &\frac{\frac{(1+L\beta^{-1})}{p}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}{G_\eta - G^{\theta_1}}\\ &- \frac{\lambda(F(k_0,\eta) - \sum_{k\in\mathbb{Z}} F(k,\theta_1))}{G_\eta - G^{\theta_1}}\Big\}.\end{aligned} \tag{3.8}$$

□

We, now deduce the following consequence of Theorem 3.1.

Theorem 3.4. *Assume that there exist three positive constants* θ_1, θ_4, η *and the integer* k_0 *which we put*

$$\theta_1 < \min\{\eta, \beta^{\frac{-1}{p}}(a(k_0+1)+a(k_0)+b(k_0))^{\frac{1}{p}}\eta\}$$

and

$$\beta^{\frac{-1}{p}}\frac{\sqrt[p]{2}(1+L\beta^{-1})}{2(1-L\beta^{-1})}\Big((a(k_0+1)+a(k_0)+b(k_0))^{\frac{1}{p}}\Big)\eta < \theta_4$$

such that

(A_4) $f(k,t) \geq 0$ *for each* $(k_0, t) \in \mathbb{Z}\times[0,\theta_4]$,

(A_5) $\max\Big\{\frac{\sum_{k\in\mathbb{Z}} F(k,\theta_1)}{\theta_1^p}, \frac{2\sum_{k\in\mathbb{Z}} F(k,\theta_4)}{\theta_4^p}\Big\}$
$$< \frac{1-L\beta^{-1}}{1-L\beta^{-1}+\beta^{-1}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)} \times \frac{F(k_0,\eta)}{\eta^p}.$$

Then, for every

$$\lambda \in \Big(\frac{1-L\beta^{-1}+\beta^{-1}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}{p\beta^{-1}F(k_0,\eta)},$$
$$\frac{(1-L\beta^{-1})\beta}{p} \times \min\{\frac{\theta_1^p}{\sum_{k\in\mathbb{Z}} F(k,\theta_1)}, \frac{\theta_4^p}{2\sum_{k\in\mathbb{Z}} F(k,\theta_4)}\}\Big)$$

and for every nonnegative function $g : \mathbb{Z}\times\mathbb{R}\to\mathbb{R}$, there is $\delta'_{\lambda,G} > 0$ given by (3.8) such that for each $\mu \in [0, \delta'_{\lambda,G})$, the problem $(P^{f,g,h}_{\lambda,\mu})$ has at least three nonnegative solutions u_1, u_2, u_3 such that $\max_{k\in\mathbb{Z}} |u_1(k)| < \theta_1$, $\max_{k\in\mathbb{Z}} |u_2(k)| < \frac{1}{\sqrt[p]{2}}\theta_4$, $\max_{k\in\mathbb{Z}} |u_3(k)| < \theta_4$.

Proof. We put $\theta_2 = \frac{1}{\sqrt[p]{2}}\theta_4$ and $\theta_3 = \theta_4$. So from (A_5) we get

$$\begin{aligned}\frac{\sum_{k\in\mathbb{Z}} F(k,\theta_2)}{\theta_2^p} &= \frac{2\sum_{k\in\mathbb{Z}} F(k,\frac{1}{\sqrt[p]{2}}\theta_4)}{\theta_4^p} \le \frac{2\sum_{k\in\mathbb{Z}} F(k,\theta_4)}{\theta_4^p} \\ &< \frac{1-L\beta^{-1}}{1-L\beta^{-1}+\beta^{-1}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)} \times \frac{F(k_0,\eta)}{\eta^p}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\frac{\sum_{k\in\mathbb{Z}} F(k,\theta_3)}{\theta_3^p-\theta_2^p} &= \frac{2\sum_{k\in\mathbb{Z}} F(k,\theta_4)}{\theta_4^p} \\ &< \frac{1-L\beta^{-1}}{1-L\beta^{-1}+\beta^{-1}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)} \times \frac{F(k_0,\eta)}{\eta^p}.\end{aligned} \tag{3.10}$$

From (A_5) and taking into account $\theta_1 < \eta$ we have

$$\begin{aligned}&\frac{(1-L\beta^{-1})\beta}{1+L\beta^{-1}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}(F(k_0,\eta) - \sum_{k\in\mathbb{Z}} F(k,\theta_1)) \\ &> \frac{(1-L\beta^{-1})\beta}{1+L\beta^{-1}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}(F(k_0,\eta)) \\ &- \frac{(1-L\beta^{-1})\beta}{1+L\beta^{-1}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}(\sum_{k\in\mathbb{Z}} F(k,\theta_1)) \\ &> \frac{1-L\beta^{-1}}{1-L\beta^{-1}+\beta^{-1}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)} \times \frac{F(k_0,\eta)}{\eta^p}.\end{aligned} \tag{3.11}$$

Hence, from (3.9), (3.10), and (A_5), the assumption (A_3) of Theorem 3.1 is satisfied, and since the critical points of function $\phi - \lambda\psi$ are the solutions of the problem $(P^{f,g,h}_{\lambda,\mu})$ we have the conclusion. □

Now, we present the following example in which the conditions of Theorem 3.1 are satisfied.

Example 3.5. *Consider the problem*

$$\begin{cases} -\Delta(a(k)\Delta u(k-1)+b(k)u(k)) = \lambda f(k,u(k)) & \\ +\mu g(k,u(k))+h(u(k)), & k\in\mathbb{Z}, \\ u(k)\to 0 & as \quad |k|\to\infty, \end{cases} \tag{3.12}$$

where $a(k)=10^{-k}$ *for every* $k\in\mathbb{Z}$ *and* $b(k)=|\cos k\pi|$ *and*

$$f(k,t)=\begin{cases} \frac{2}{3}\times 3(\frac{1}{2})^{|k|+1}t^2 & t<1, \\ \frac{2}{3}\times(\frac{1}{2})^{|k|+1}\dfrac{3}{t} & t\geq 1, \end{cases}$$

also $h(t)=\dfrac{1}{1+t^2}-1$ *for* $t\in\mathbb{R}$, *so by the definitions of* F *and* H *it is obvious and*

$$F(k,t)=\begin{cases} \frac{2}{3}(\frac{1}{2})^{|k|+1}t^3 & t<1, \\ \frac{2}{3}(\frac{1}{2})^{|k|+1}(1+3\ln(t)) & t\geq 1 \end{cases}$$

and $H(t)=\arctan t-t$ *for every* $t\in\mathbb{R}$. *We suppose that* $\theta_1=10^{-3}$, $\theta_4=10^3$ *and* $\eta=1$, *in addition by choosing* $\beta=10^4$, $L=10^{-3}$ *and fixed* $k_0=1$, *since the assumptions of Theorem 3.1 are satisfied since* $\theta_1<10^{-2}(10^{-1}+10^{-2}+1)^{\frac{1}{2}}$ *and* $10^{-2}\dfrac{(\sqrt[2]{2})(1+10^{-7})}{2(1-10^{-7})}(10^{-1}+10^{-2}+1)^{\frac{1}{2}}<\theta_4$, *and* $f(k,t)\geq 0$ *for each* $(k,t)\in\mathbb{Z}\times[0,10^3]$, *and*

Since

$$\max\left\{\frac{\sum_{k\in\mathbb{Z}}F(k,10^{-3})}{10^{-6}},\frac{2\sum_{k\in\mathbb{Z}}F(k,10^3)}{10^6}\right\}$$
$$=\max\left\{\frac{10^{-9}}{10^{-6}},\frac{2(1+9\ln 10)}{10^6}\right\}=\frac{10^{-9}}{10^{-6}}$$

and

$$\frac{1-10^{-7}}{1-10^{-7}+10^{-4}(1+10^{-7})(10^{-1}+10^{-2}+1)}\times\frac{F(1,1)}{1}$$
$$=\frac{1-10^{-7}}{1-10^{-7}+10^{-4}(1+10^{-7})(10^{-1}+10^{-2}+1)}\times\frac{1}{6}$$

so the condition (A_5) *of Theorem 3.4 is verified because*

$$\frac{10^{-9}}{10^{-6}}<\frac{1-10^{-7}}{1-10^{-7}+10^{-4}(1+10^{-7})(10^{-1}+10^{-2}+1)}\times\frac{1}{6}.$$

Then, for every

$$\lambda\in\Lambda\simeq\Big(\frac{1-10^{-7}+10^{-4}(1+10^{-7})(10^{-1}+10^{-2}+1)}{2\times 10^{-4}\times\frac{1}{6}},\frac{(1-10^{-7})10^4}{2}\times\frac{10^{-6}}{10^{-9}}\Big)$$

and for every nonnegative function $g=|\sin t|$ *and constant* $\lambda=1$ *since*

$$\min\left\{\frac{(1-10^{-7})10^{-2}-2\times 10^{-9}}{1},\frac{(1-10^{-7})10^{10}-4(\frac{10^{-9}}{10^{-6}})}{2},\right.$$
$$\left.\frac{(1-10^{-7})10^{10}-4(\frac{10^{-9}}{10^{-6}})}{2\times 0.17}\right\}=\frac{(1-10^{-7})10^{-2}-2\times 10^{-9}}{1}$$

and

$$\min\Big\{\frac{(1-10^{-7})10^{-2}-2\times 10^{-9}}{1},\frac{\frac{(1-10^{-7})}{3}(10^{-1}+10^{-2}+1)+10^{-9}}{0.99-1}\Big\}$$
$$=\frac{(1-10^{-7})10^{-2}-2\times 10^{-9}}{1}.$$

So, there is $\delta'_{\lambda,G}=\frac{(1-10^{-7})10^{-2}-2\times10^{-9}}{1}$ *such that for each* $\mu\in[0,\delta'_{\lambda,G})$*, the problem* (3.12) *has at least three nonnegative solutions* u_1,u_2,u_3 *such that* $\max_{k\in\mathbb{Z}}|u_1(k)|<10^{-3}$, $\max_{k\in\mathbb{Z}}|u_2(k)|<\dfrac{10^3}{\sqrt[2]{2}}$ *and* $\max_{k\in\mathbb{Z}}|u_3(k)|<10^3$.

Now, we deduce the following straightforward consequence of Theorem 3.4.

Theorem 3.6. *Let* f *be a nonnegative continuous and nonzero function such that*

$$\lim_{t\to 0^+}\frac{f(t)}{|t|^{p-1}}=\lim_{t\to+\infty}\frac{f(t)}{|t|^{p-1}}=0 \tag{3.13}$$

for every $\lambda>\lambda*$ *where*

$$\lambda^*=\inf\Big\{\frac{1-L\beta^{-1}+\beta^{-1}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}{p\beta^{-1}F(\eta)}$$
$$:\eta>0,F(\eta)>0\Big\}$$

there exists

$$\tilde{\mu}:=\min\Big\{\frac{1}{p}\min\Big\{\frac{(1-L\beta^{-1})\beta\theta_1^p-p\lambda F(\theta_1)}{G^{\theta_1}},$$
$$\frac{(1-L\beta^{-1})\beta\theta_4^p-2p\lambda F(\frac{1}{\sqrt[p]{2}}\theta_4)}{2G^{\frac{1}{\sqrt[p]{2}}\theta_4}},\frac{(1-L\beta^{-1})\beta\theta_4^p-2p\lambda F(k,\theta_4)}{2G^{\theta_4}}\Big\},$$
$$\frac{(\frac{1+L\beta^{-1}}{p})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p-\lambda(F(\eta)-F(\theta_1))}{G_\eta-G^{\theta_1}}\Big\},$$

where θ_1, θ_4 *and* η *are positive constants, so the problem*

$$\left\{\begin{array}{ll}-\Delta(a(k)\phi_p(\Delta u(k-1)))+b(k)\phi_p(u(k))=\lambda f(u(k)) & \\ +\mu g(u(k))+h(u(k)) & k\in\mathbb{Z},\\ u(k)\to 0 & as\quad |k|\to\infty,\end{array}\right. \tag{3.14}$$

where $g:\mathbb{R}\to\mathbb{R}$ *is nonnegative continuous and nonzero function, for each* $\mu\in[0,\tilde{\mu})$ *has at least two distinct positive solutions.*

Proof. Fix $\lambda>\lambda^*$ which

$$\lambda>\frac{1-L\beta^{-1}+\beta^{-1}(1+L\beta^{-1})\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)\eta^p}{p\beta^{-1}F(\eta)},$$

where $F(t)=\int_0^t f(\xi)d\xi$ for all $t\in\mathbb{R}$ and $\eta>0$, $F(\eta)>0$.

From (3.13) there is $\theta_1>0$ such that $\theta_1<\min\{\eta,\beta^{\frac{-1}{p}}\Big(a(k_0+1)+a(k_0)+b(k_0)\Big)^{\frac{1}{p}}\eta$ and

$$\beta^{\frac{-1}{p}}\frac{\sqrt[p]{2}(1+L\beta^{-1})}{2(1-L\beta^{-1})}\Big(a(k_0+1)+a(k_0)+b(k_0))^{\frac{1}{p}}\Big)\eta<\theta_4$$

such that $\dfrac{F(\theta_1)}{\theta_1} < \dfrac{1 - L\beta^{-1}}{2p\beta^{-1}\lambda}$ and $\dfrac{F(\theta_4)}{\theta_4} < \dfrac{1 - L\beta^{-1}}{2p\beta^{-1}\lambda}$. □

Finally, we point out the following simple consequence of Theorem 3.4 when $\mu = 0$ and $h \equiv 0$.

Theorem 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $tf(t) > 0$ for all $t \neq 0$ and*

$$\lim_{t\to 0} \frac{f(t)}{t^{p-1}} = \lim_{t\to +\infty} \frac{f(t)}{t^{p-1}} = 0. \tag{3.15}$$

Then, for every $\lambda > \lambda^{**}$ where

$$\begin{aligned}\lambda^{**} = &\frac{1 - L\beta^{-1} + \beta^{-1}(1 + L\beta^{-1})\Big(a(k_0+1) + a(k_0) + b(k_0)\Big)\eta^p}{p\beta^{-1}F(\eta)} \\ &\times \max\left\{\inf_{\eta>0} \frac{\eta^p}{F(\eta)}; \inf_{\eta<0} \frac{-\eta^p}{F(\eta)}\right\}\end{aligned}$$

the problem (3.14) in the case $\mu = 0$ and $h \equiv 0$ possesses at least four distinct nontrivial solutions.

Proof. Setting

$$f_1(t) = \begin{cases} 0 & t < 0, \\ f(t) & t \geq 0 \end{cases}$$

and

$$f_2(t) = \begin{cases} 0 & t < 0, \\ -f(-t) & t \geq 0 \end{cases}$$

and employing Theorem 3.6 to f_1 and f_2 we have the result. □

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