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Elastic Membrane Equation with Dynamic Boundary Conditions and Infinite Memory *

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ABSTRACT: In this paper, we study the elastic membrane equation with dynamic boundary conditions, source term and a nonlinear weak damping localized on a part of the boundary and past history. Under some appropriate assumptions on the relaxation function the general decay for the energy have been established using the perturbed Lyapunov functionals and some properties of convex functions.

Key Words: Elastic membrane equation, Energy decay, Balakrishnan-Taylor damping, Dynamic boundary conditions, Infinite memory.

Contents

1. Introduction

The objective of this work is to study the following problem

$$\begin{cases} u_{tt} - M(t)\Delta u + \int_0^\infty g(s)\Delta u\,(t-s)\,ds = 0, & in\ \Omega \times (0,+\infty)\,, \\ u = 0 & on\ \Gamma_0 \times [0,+\infty)\,. \\ u_{tt}(t) = -\frac{\partial u(t)}{\partial \nu} - \frac{\partial u_t(t)}{\partial \nu} + \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(t-s)ds - h(u_t) - f(u), & on\ \Gamma_1 \times [0,+\infty)\,, \\ u(x,-t) = u_0(x), u_t(x,0) = u_1(x), & in\ \bar{\Omega}. \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\Gamma_0,\ \Gamma_1$ have positive measure $\lambda_{n-1}(\Gamma_i)\,,\ i = 0,1,\ \nu$ denotes the unit outer normal vector pointing towardthe exterior of Ω and $M(t) = \xi_0 + \xi_1 \left\|\nabla u\,(t)\right\|_2^2 + \sigma\left(\nabla u\,(t)\,, \nabla u_t\,(t)\right)$, where u is the plate transverse displacement, x is the spatial coordinate in the direction of the fluid flow, and t is the time. The viscoelastic structural damping terms are denote by σ, ξ_1 is the nonlinear stiffness of the membrane, ξ_0 is an in-plane tensile load. All quantities are physically non-dimensionalized ξ_0, ξ_1, σ and α are fixed positive. Equation (1.1) is related to the flutter panel equation with memory term this equation arises in a wind tunnel experiment for a panel at supersonic speeds. For a derivation of this model see, for instance, Dowell [14] Holmes [24, 25], Bass [5] . For more results concerning Balakrishnan-Taylor equation, one can refer to Zaraï and Tatar [2, 3], For viscoelastic wave equation with Dirichlet boundary condition, the problems are truly overworked. Many existence and stability results have been established, Cavalcanti and Oquendo [10], Fabrizio and Polidoro [15], Messaoudi [30, 34]. For linear Cauchy viscoelastic problem, one can refer to Kafini and Mustafa [26]. With respect to viscoelastic wave equation with boundary stabilization, Cavalcanti $[8-11]$ considered the following system

$$\begin{cases} u_{tt}(t) - \Delta u(t) + \int_0^\infty g(s)\Delta u(t-s)ds = 0, & in\ \Omega \times \mathbb{R}^+, \\ u = 0, & on\ \Gamma_0 \times \mathbb{R}^+, \\ \frac{\partial u(t)}{\partial \nu} - \int_0^\infty g(t-s)\frac{\partial u}{\partial \nu}(s)ds + h(u_t) = 0 & on\ \Gamma_1 \times \mathbb{R}^+, \\ u(0) = u_0(x), u_t(0) = u_1(x), x \in \Omega & x \in \Omega. \end{cases}$$

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Under the following assumptions on functions h,

$$\begin{cases} C_1 |s|^p \leq |h(s)| \leq C_2 |s|^{\frac{1}{p}}, & if\ \ |s| \leq 1 \\ C_3 |s| \leq |h(s)| \leq C_4 |s|, & if\ \ |s| > 1 \end{cases}$$

the authors first proved the global existence of solutions, and obtained the energy decays exponentially if $p = 1$ and decays polynomially if $p > 1$.The results were generalized by Cavalcanti et al.[9]. They obtained the same results without imposing a growth condition on h and under a weaker assumption on g Messaoudi and Mustafa [26] extended these results and established an explicit and general decay rate result by exploiting some properties of convex functions. Recently, using the same method as in [26], Messaoudi et al.[33] considered the above wave system with infinite memory $\int_0^\infty g(s)\Delta u(t-s)ds$ and obtained a general decay result using multiplier method. Gerbi and Said-Houari [21] studied a viscoelastic wave equation with dynamic boundary conditions of the form

$$\begin{cases} u_{tt}(t) - \Delta u(t) - \alpha \Delta u_t + \int_0^\infty g(t-s)\Delta u(s)ds = |u|^{p-2} u, & in\ \Omega \times \mathbb{R}^+, \\ u = 0, & on\ \Gamma_0 \times \mathbb{R}^+, \\ u_{tt}(t) = -\left[-\frac{\partial u}{\partial \nu} + \alpha \frac{\partial u_t}{\partial \nu} - \int_0^\infty g(t-s)\frac{\partial u(s)}{\partial \nu} ds + h(u_t) \right], & on\ \Gamma_1 \times \mathbb{R}^+, \\ u(0) = u_0(x), u_t(0) = u_1(x), & x \in \Omega, \end{cases}$$

Using the Faedo-Galerkin method and fixed point theorem, they proved the existence and uniqueness of a local in time solution, and proved the solution exists globally in time under some restrictions on the initial data. They also proved if $\alpha > 0$, the solution is unbounded and grows as an exponential function, if $\alpha = 0$, then the solution ceases to exist and blows up in finite time. Ferhat and Hakem [11] considered the same viscoelastic wave equation as in [14] but with the following dynamic boundary conditions

$$u_{tt}(t) = -\begin{bmatrix} -\frac{\partial u}{\partial \nu} - \int_0^\infty g(t-s)\frac{\partial u(s)}{\partial \nu} ds + \sigma \frac{\partial u_t}{\partial \nu} \\ +\mu_1 h(u_t) + \mu_2 h(u_t(t-\tau)) \end{bmatrix}, \quad on\ \Gamma_1 \times \mathbb{R}^+$$

They established a general decay result by introducing suitable energy and Lyapunov functionals and some properties of convex functions. Ferhat and Hakem [12] investigated the following system

$$\begin{cases} u_{tt}(t) - \Delta u(t) - \alpha \Delta u_t + \delta(t) \int_0^\infty g(t-s)\Delta u(s)ds = |u|^{p-2} u, & in\ \Omega \times \mathbb{R}^+, \\ u = 0, & on\ \Gamma_0 \times \mathbb{R}^+, \\ u_{tt}(t) = -\begin{bmatrix} \frac{\partial u}{\partial \nu} - \sigma(t) \int_0^\infty g(t-s)\frac{\partial u(s)}{\partial \nu} ds + \alpha \frac{\partial u_t}{\partial \nu} \\ +\mu_1 |u|^{m-1} u_t + \mu_2 |u_t(t-\tau)|^{m-1} u_t(t-\tau) \end{bmatrix} & on\ \Gamma_1 \times \mathbb{R}^+, \\ u(0) = u_0(x), u_t(0) = u_1(x), & x \in \Omega, \end{cases}$$

They proved the global existence and energy decay of solutions for this system. Ferhat and Hakem [18] considered a weak viscoelastic wave equation with dynamic boundary conditions and Kelvin Voigt damping and delay term acting on the boundary in a bounded domain, and proved the asymptotic behavior by making use an appropriate Lyapunov functional. Recently, Benaissa and Ferhat [6] considered a viscoelastic wave equation with dynamic boundary conditions and infinite memory

$$\begin{cases} u_{tt}(t) - \Delta u(t) + \int_0^\infty g(s)\Delta u(t-s)ds = 0, & in\ \Omega \times \mathbb{R}^+, \\ u = 0, & on\ \Gamma_0 \times \mathbb{R}^+, \\ u_{tt}(t) = -\left[\frac{\partial u(t)}{\partial \nu} - \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(t-s)ds \right], & on\ \Gamma_1 \times \mathbb{R}^+, \\ u(0) = u_0(x), u_t(0) = u_1(x), & x \in \Omega, \end{cases}$$

and established an exponential decay result of energy by exploiting the frequency domain method which consists in combining a contradiction argument and a special analysis for the resolvent of the operator under the assumption $-\zeta_0 g(t) \leq g'(t) \leq \zeta_0 g(t)$. For more results concerned with wave equation with boundary stabilization, one can refer to Doronin and Larkin [13], Muñoz Rivera and Andrade [35], Gerbi and Said-Houari [20 − 22], Liu and Yu [29], Since there are few works on wave equation with dynamic boundary conditions, source term and a nonlinear weak damping localized on a part of the boundary and past history, motivated by above scenario, we study in the present work the stability of solutions to problem (1.1) − (1.4). The main objective of the present work is to establish an explicit and general decay result using multiplier method and some properties of convex functions. Our result is obtained without imposing any restrictive growth assumption on the damping term. We end this section by establishing the usual history setting of problem (1.1) − (1.4). Following the same arguments of Dafermos [12], we introduce a new variable

$$\eta(x,t,s) = u(t) - u(t-s),$$

which gives us

$$\eta_t + \eta_s = u_t.$$

Assuming $\xi_0 - \int_0^\infty g(s)ds = l$ then we can get a new system, which is equivalent to problem (1.1)

$$\begin{cases} u_{tt}(t) - \left(l + \xi_1 \|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\right)\Delta u(t) - \int_0^\infty g(s)\Delta\eta(s)ds = 0 & in\ \Omega\times\mathbb{R}^+, \\ u = 0, & on\ \Gamma_0\times\mathbb{R}^+, \\ \quad u_{tt}(t) = -\left(l + \xi_1 \|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\right)\frac{\partial u(t)}{\partial \nu} & \\ \quad -\int_0^\infty g(s)\frac{\partial \eta}{\partial \nu}(s)ds - h(u_t) - f(u), & on\ \Gamma_1\times\mathbb{R}^+, \\ u(x,-t) = u_0(x), u_t(x,0) = u_1(x), & x\in\Omega. \end{cases}$$

The rest of this paper is as follows. In *Sect.*2, we give some assumptions and our main results. In *Sect.*3, we establish the general decay result of the energy. In this paper we will use a lot of concepts and techniques contained in Feng [16].

2. Assumptions and Main Results

In this section, we present some materials and assumptions used in this paper.

$L^q(\Omega), (1 \le q \le \infty)$, and $H^1(\Omega)$ denote Lebesgue integral and Sobolev spaces $\|.\|_q$ and $\|.\|_{q,\Gamma_1}$ are the norm in the space $L^q(\Omega)$ and $L^q(\Gamma_1)$, respectively. For simplicity, we write $\|.\|$ and $\|.\|_{\Gamma_1}$ instead of $\|.\|_2$ and $\|.\|_{2,\Gamma_1}$, respectively C is used to denote a generic positive constant. Denote

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) \colon u\backslash_{\Gamma_0} = 0\right\}$$

then we have the embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Gamma_1)$. We will usually use the following Green's formula

$$\int_\Omega \nabla u(x)\nabla w(x)dx = -\int_\Omega \Delta u(x)w(x)dx + \int_{\Gamma_1}\frac{\partial u}{\partial \nu}(x)w(x)d\Gamma, \forall w \in H^1_{\Gamma_0}(\Omega).$$

To deal with the new variable η, we introduce a weighted L^2 space

$$M = L^2_g\left(\mathbb{R}^+; H^1_{\Gamma_0}(\Omega)\right) = \left\{\zeta \colon \mathbb{R}^+ \longrightarrow H^1_{\Gamma_0}(\Omega) \colon \int_0^\infty g(s)\|\nabla\zeta(s)\|^2 ds < \infty\right\},$$

which is Hilbert space endowed with inner product and norm

$$\langle\zeta,\vartheta\rangle_M = \int_0^\infty g(s)\left(\int_\Omega \nabla\zeta(s)\nabla\vartheta(s)dx\right)ds,$$

and

$$\|\zeta\|^2_M = \int_0^\infty g(s)\|\nabla\zeta(s)\|^2 ds.$$

The phase space $\hat{H}$ is defined by

$$\hat{H} = H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times M.$$

In the sequel, we shall give some assumptions. For the relaxation function g, we assume:

(A1) $g(t) \colon \mathbb{R}^+ \hookrightarrow \mathbb{R}^+$ is a nonincreasing C^1 function satisfying

$$g(0) > 0\ and\ \xi_0 - \int_0^\infty g(s)ds = l > 0. \tag{2.1}$$

In addition, there exists an increasing strictly convex function $G\colon \mathbb{R}^+ \hookrightarrow \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)$ satisfying

$$G(0) = G'(0) = 0 \textit{ and } \lim_{t \hookrightarrow +\infty} G'(t) = +\infty \tag{2.2}$$

such that

$$\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty \tag{2.3}$$

$(A2)$ $h\colon \mathbb{R} \hookrightarrow \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing function $h_0 \in C^1(\mathbb{R}^+)$ with $h_0(0) = 0$ and positive constants c_1, c_2 and ε such that

$$\begin{cases} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|); & if \ \ |s| \le \varepsilon \\ c_1|s| \le |h(s)| \le c_2|s| & if \ \ |s| > \varepsilon \end{cases} \tag{2.4}$$

Moreover we suppose that the function $H(s) = \sqrt{s} h_0(\sqrt{s})$ is a strictly convex C^2 function on $(0, r^2]$ for some $r > 0$ when h_0 is nonlinear.

$(A3)$ We assume $f\colon \mathbb{R} \hookrightarrow \mathbb{R}$ that for some $c_0 > 0$,

$$|f(u) - f(v)| \le c_0(1 + |u|^p + |v|^p)|u - v|, \ \forall u, \ v \in \mathbb{R} \tag{2.5}$$

where

$$0 < p < \frac{2}{n-2} \ if \ n \ge 3 \ and \ p > 0 \ if \ n = 1, 2.$$

In addition, we assume that

$$f(u)u \ge F(u) \ge 0; \forall u \in \mathbb{R} \tag{2.6}$$

where $F(z) = \int_0^z f(s)ds$. Assumptions $(2.5) - (2.6)$ include nonlinear terms of the form

$$f(u) \approx |u|^p u + |u|^\alpha u, \ 0 < \alpha < p.$$

$(A4)$ There exists a positive constant m_0 such that

$$\|\nabla u_0(., s)\| \le m_0. \tag{2.7}$$

The same arguments as in [5], [15] and [32], we can prove the global existence of solutions to problem $(1.5) - (1.8)$ given in the following theorem.

Theorem 2.1. *Suppose assumptions* $(A1) - (A4)$ *hold. If the initial data* $(u_0(.,0), u_1, \eta_0) \in \hat{H}$ *then problem* $(1.5) - (1.8)$ *has a unique weak solution such that for any* $T > 0$,

$$u(t) \in L^\infty\left([0,T]; H^1_{\Gamma_0}(\Omega)\right), u_t(t) \in L^\infty\left([0,T]; L^2(\Omega)\right) \ and \ \eta \in L^\infty([0,T]; M).$$

The energy functional of problem $(1.5) - (1.8)$ *is defined by*

$$E(t) = \frac{1}{2}\left\{\|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1} + \int_{\Gamma_1} F(u)d\Gamma + \|\eta\|^2_M + l\|\nabla u\|^2 + \frac{b}{2}\|\nabla u\|^4\right\} \tag{2.8}$$

Then we can get the stability result of energy to problem $(1.5) - (1.8)$ *given in following theorem.*

Theorem 2.2. *Suppose* $(A1) - (A4)$ *hold, let* $(u_0(.,0), u_1, \eta_0) \in \hat{H}$, *then there exist positive constants* $k_2, k_3, k_4, \varepsilon_1, \varepsilon_0$ *such that the energy* $E(t)$ *defined by* (2.8) *satisfies*

$$E(t) \le k_4 W_1^{-1}(k_2 t + k_3), \forall t \in \mathbb{R}^+, \tag{2.9}$$

where

$$W_1(\tau) = \int_\tau^1 \frac{1}{W_2(s)} ds \text{ and } W_2(t) = tG'(\varepsilon_1 t) H'(\varepsilon_0 t)$$

3. General Decay

In this section, we shall study the general decay of energy to problem (1.5) – (1.8) to prove Theorem 2. For this purpose, we need the following technical lemmas.

Lemma 3.1. *Under the assumptions of Theorem2, the energy functional $E(t)$ is non-increasing and satisfies that for any $t \geq 0$,*

$$E'(t) \leq -\sigma \left(\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 \right)^2 - \int_{\Gamma_1} h(u_t) u_t d\Gamma + \frac{1}{2} \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 ds \tag{3.1}$$

Proof. Multiplying (1.5) by u_t, and using integration by parts, boundary conditions (1.6) – (1.7) and Green's formula, we can obtain that

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(\|u_t(t)\|^2 + \|u_t(t)\|_{\Gamma_1}^2 + l\|\nabla u\|^2 + \frac{b}{2}\|\nabla u\|^4 + \int_{\Gamma_1} F(u) d\Gamma \right) \\ \qquad +\sigma \left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 \right)^2 + \int_{\Gamma_1} h(u_t) u_t d\Gamma \\ \qquad + \int_\Omega \nabla u_t \int_0^\infty g(s) \nabla \eta(s) ds dx = 0 \end{array} \tag{3.2}$$

Note that

$$\int_\Omega \nabla u_t \int_0^\infty g(s) \nabla \eta(s) ds dx = \frac{1}{2}\frac{d}{dt}\|\eta(t)\|_M^2 - \frac{1}{2}\int_0^\infty g'(s)\|\nabla \eta(s)\|^2 ds \tag{3.3}$$

Inserting (3.3) into (3.2), we can get the desired estimate (3.1). Using $(A2)$, we know that $h(u_t)u_t \geq 0$. Then $E'(t) \leq 0$. The proof is complete. □

Lemma 3.2. *Under the assumptions of Theorem2, the functional $\phi(t)$ defined by*

$$\phi(t) = \int_\Omega u_t(t) u(t) dx + \int_{\Gamma_1} u_t(t) u(t) d\Gamma + \frac{\sigma}{4}\|\nabla u\|^4 \tag{3.4}$$

satisfies that for any $t \geq 0$,

$$\begin{aligned} \phi'(t) \quad \leq \quad & \|u_t(t)\|^2 + \|u_t(t)\|_{\Gamma_1}^2 - (l - \delta_1(1+c))\|\nabla u\|^2 - \xi_1 \|\nabla u\|^4 \\ & + C_1 \int_0^\infty g(s)\|\nabla \eta(s)\|^2 ds + C_2 \int_{\Gamma_1} h^2(u_t) d\Gamma \end{aligned} \tag{3.5}$$

Proof. Differentiating $\phi(t)$ with respect to t, we obtain

$$\phi'(t) = \|u_t(t)\|^2 + \|u_t(t)\|_{\Gamma_1}^2 + \int_\Omega u_{tt}(t) u(t) dx + \int_{\Gamma_1} u_{tt}(t) u(t) d\Gamma + \sigma \|\nabla u\|^2 (\nabla u(t), \nabla u_t(t)) \tag{3.6}$$

We infer from (1.7) and Green's formula that

$$\begin{aligned} & \int_\Omega u_{tt}(t) u(t) dx + \int_{\Gamma_1} u_{tt}(t) u(t) d\Gamma \\ = \quad & \int_\Omega \left[\left(l + \xi_1 \|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t) \right) \Delta u(t) + \int_0^\infty g(s) \Delta \eta(s) ds \right] u dx \\ & + \int_{\Gamma_1} \left[- \left(l + \xi_1 \|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t) \right) \frac{\partial u}{\partial \nu} - \int_0^\infty g(s) \frac{\partial \eta}{\partial \nu}(s) ds - h(u_t) - f(u) \right] u d\Gamma \\ = \quad & -\left(l + \xi_1 \|\nabla u\|^2 \right) \|\nabla u\|^2 - (\sigma(\nabla u, \nabla u_t))\|\nabla u\|^2 - \int_\Omega \nabla u \int_0^\infty g(s) \nabla \eta(s) ds dx \\ & - \int_{\Gamma_1} u h(u_t) d\Gamma - \int_{\Gamma_1} u f(u) d\Gamma \end{aligned}$$

which, combined with (3.6), gives us

$$\begin{aligned}\phi'(t) &= \|u_t(t)\|^2 + \|u_t(t)\|_{\Gamma_1}^2 - \left(l + \xi_1 \|\nabla u\|^2\right)\|\nabla u\|^2 \\ &\quad - \int_\Omega \nabla u \int_0^\infty g(s)\nabla\eta(s)\,ds dx - \int_{\Gamma_1} uh(u_t)d\Gamma - \int_{\Gamma_1} uf(u)d\Gamma\end{aligned} \tag{3.7}$$

Using Young's inequality, Holder's inequality and Poincaré's inequality, we know that for any $\delta_1 > 0$,

$$\begin{aligned}&-\int_\Omega \nabla u \int_0^\infty g(s)\nabla\eta(s)\,ds dx \\ &\leq \delta_1 \|\nabla u\|^2 + \frac{1}{4\delta_1}\int_\Omega \left[\int_0^\infty g(s)\nabla\eta(s)\,ds\right]^2 dx \\ &\leq \delta_1 \|\nabla u\|^2 + \frac{1}{4\delta_1}\int_\Omega \left[\int_0^\infty g(s)ds\right]\left[\int_0^\infty g(s)\left|\nabla\eta(s)\right|^2 ds\right] dx\end{aligned}$$

$$\leq \delta_1 \|\nabla u\|^2 + \frac{(\xi_0 - l)}{4\delta_1}\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds \tag{3.8}$$

and

$$-\int_{\Gamma_1} uh(u_t)d\Gamma \leq C_1\delta_1 \|\nabla u\|^2 + \frac{1}{4\delta_1}\int_{\Gamma_1} h^2(u_t)d\Gamma \tag{3.9}$$

Inserting (3.8) – (3.9) into (3.7), we get for any $\delta_1 > 0$

$$\begin{aligned}\phi'(t) &\leq \|u_t(t)\|^2 + \|u_t(t)\|_{\Gamma_1}^2 - (l - \delta_1(1 + C_1))\|\nabla u\|^2 - \xi_1 \|\nabla u\|^4 \\ &\quad + \frac{(\xi_0 - l)}{4\delta_1}\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds \int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds \\ &\quad + \frac{1}{4\delta_1}\int_{\Gamma_1} h^2(u_t)d\Gamma - \int_{\Gamma_1} uf(u)d\Gamma\end{aligned} \tag{3.10}$$

Now, we take $\delta_1 > 0$ so small that

$$l - \delta_1(1 + C_1) > \frac{l}{2}.$$

Thus, (3.5) follows from (3.10) and (2.6) with

$$C = \max\left\{\frac{1}{4\delta_1}, \frac{\xi_0 - l}{4\delta_1}\right\}.$$

The proof is complete. □

Lemma 3.3. *Define the functional* $\psi(t)$ *as*

$$\psi(t) = -\int_\Omega u_t(t)\int_0^\infty g(s)\eta(s)\,ds dx - \int_{\Gamma_1} u_t(t)\int_0^\infty g(s)\eta(s)\,ds d\Gamma.$$

Under the assumptions of Theorem 2, then the functional $\psi(t)$ *satisfies for any* $\delta_2 > 0$,

$$\begin{aligned}\psi'(t) &\leq -\frac{3}{4}(\xi_0 - l)\|u_t\|^2 - \frac{3}{4}(\xi_0 - l)\|u_t\|_{\Gamma_1}^2 + \delta_2\|\nabla u\|^2 + \sigma^2 E(0)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\right)^2 \\ &\quad + K_1\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds + K_2\int_0^\infty g'(s)\|\nabla\eta(s)\|^2 ds + \frac{1}{2}\int_{\Gamma_1} h^2(u_t)d\Gamma.\end{aligned} \tag{3.11}$$

Proof. Differentiating $\psi(t)$ with respect to t, we can obtain that

$$\begin{aligned}\psi'(t) &= \underbrace{-\int_\Omega u_{tt}(t)\int_0^\infty g(s)\eta(s)\,dsdx - \int_{\Gamma_1} u_{tt}(t)\int_0^\infty g(s)\eta(s)\,dsd\Gamma}_{:=I_1} \\ &\quad \underbrace{-\int_\Omega u_t(t)\int_0^\infty g(s)\eta_t(s)\,dsdx}_{:=I_2} \underbrace{-\int_{\Gamma_1} u_t(t)\int_0^\infty g(s)\eta_t(s)\,dsd\Gamma}_{:=I_3}. \end{aligned} \tag{3.12}$$

Clearly

$$\begin{aligned} & -\int_\Omega u_{tt}(t)\int_0^\infty g(s)\eta(s)\,dsdx \\ = & \int_\Omega \Big(-\left(l+\xi_1\|\nabla u\|^2+\sigma(\nabla u,\nabla u_t)\right)\Delta u(t) \\ & -\int_0^\infty g(s)\Delta\eta(s)\,ds\Big)\left(\int_0^\infty g(s)\eta(s)\,ds\right)dx \\ = & -\int_{\Gamma_1}\left(l+\xi_1\|\nabla u\|^2+\sigma(\nabla u,\nabla u_t)\right)\frac{\partial u}{\partial\nu}\left(\int_0^\infty g(s)\eta(s)\,ds\right)d\Gamma \\ & +\int_\Omega\left(l+\xi_1\|\nabla u\|^2+\sigma(\nabla u,\nabla u_t)\right)\nabla u(t)\int_0^\infty g(s)\nabla\eta(s)\,dsdx \\ & +\int_\Omega\left(\int_0^\infty g(s)\nabla\eta(s)\,ds\right)^2dx \\ & -\int_{\Gamma_1}\left(\int_0^\infty g(s)\frac{\partial\eta}{\partial\nu}(s)\,ds\right)\left(\int_0^\infty g(s)\eta(s)\,ds\right)d\Gamma \end{aligned}$$

which, using (1.7), yields

$$\begin{aligned} I_1 &= \int_\Omega\left(l+\xi_1\|\nabla u\|^2+\sigma(\nabla u,\nabla u_t)\right)\nabla u(t)\int_0^\infty g(s)\nabla\eta(s)\,dsdx \\ &\quad +\int_\Omega\left(\int_0^\infty g(s)\nabla\eta(s)\,ds\right)^2dx+\int_{\Gamma_1}f(u)\int_0^\infty g(s)\eta(s)\,dsd\Gamma \\ &\quad +\int_{\Gamma_1}h(u_t)\int_0^\infty g(s)\eta(s)\,dsd\Gamma. \end{aligned} \tag{3.13}$$

Since $E(t)$ is nonincreasing, then we can infer from (2.9) that

$$\frac{l}{2}\|\nabla u\|^2 \le E(t) \le E(0)$$

which gives us

$$\|\nabla u\|^2 \le \frac{2}{l}E(0). \tag{3.14}$$

Performing Hölder's and Young's inequalities, (2.5) and (2.8), we infer that for any $\delta_3>0$

$$\begin{aligned} & \int_\Omega\left(l+\xi_1\|\nabla u\|^2+\sigma(\nabla u,\nabla u_t)\right)\nabla u(t)\int_0^\infty g(s)\nabla\eta(s)\,dsdx \\ \le & \left(l+\frac{2\xi_1}{l}E(0)\right)\int_\Omega\nabla u(t)\int_0^\infty g(s)\left|\nabla\eta(s)\right|dsdx \\ \le & \ \delta_3\|\nabla u\|^2+\frac{1}{4\delta_3}\left(l+\frac{2\xi_1}{l}E(0)\right)^2\int_\Omega\left[\int_0^\infty g(s)\left|\nabla\eta(s)\right|ds\right]^2dx \\ \le & \ \delta_3\|\nabla u\|^2+\frac{\xi_0-l}{4\delta_3}\left(l+\frac{2\xi_1}{l}E(0)\right)^2\left[\int_0^\infty g(s)\|\nabla\eta(s)\|^2\,ds\right] \end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\sigma\left(\nabla u,\nabla u_t\right)\int_\Omega \nabla u(t)\int_0^\infty g(s)\nabla\eta\left(s\right)dsdx\\
\leq\ &\sigma^2\left(\frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2\right)^2 E(0)+\frac{\xi_0-l}{2l}\int_0^\infty g(s)\left\|\nabla\eta\left(s\right)\right\|^2 ds
\end{aligned}\tag{3.16}$$

$$\int_\Omega\left(\int_0^\infty g(s)\nabla\eta\left(s\right)ds\right)^2 dx\leq\left(\xi_0-l\right)\int_0^\infty g(s)\left\|\nabla\eta\left(s\right)\right\|^2 ds \tag{3.17}$$

$$\int_{\Gamma_1}h(u_t)\int_0^\infty g(s)\eta\left(s\right)dsd\Gamma\leq\frac{1}{2}\int_{\Gamma_1}h^2(u_t)d\Gamma+\frac{\xi_0-l}{2}C_1\int_0^\infty g(s)\left\|\nabla\eta\left(s\right)\right\|^2 ds \tag{3.18}$$

$$\begin{aligned}
&\int_{\Gamma_1}f(u)\int_0^\infty g(s)\eta\left(s\right)dsd\Gamma\\
\leq\ &\delta_3 C_1 E^p\left(0\right)\left\|\nabla u\right\|^2+\frac{\xi_0-l}{4\delta_3}C_1\int_0^\infty g(s)\left\|\nabla\eta\left(s\right)\right\|^2 ds.
\end{aligned}\tag{3.19}$$

Combining (3.15) – (3.19) with (3.13), we have for any, $\delta_3>0$

$$\begin{aligned}
I_1\ \leq\ &\left(\frac{\xi_0-l}{4\delta_3}\left(l+\frac{2b}{l}E(0)\right)^2+\frac{\xi_0-l}{2l}\right.\\
&\left.+\left(\xi_0-l\right)+\frac{\xi_0-l}{2}C_1+\frac{\xi_0-l}{4\delta_3}C_1\right)\int_0^\infty g(s)\left\|\nabla\eta\left(s\right)\right\|^2 ds\\
&+\sigma^2\left(\frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2\right)^2 E(0)+\delta_3\left\|\nabla u\right\|^2+\frac{1}{2}\int_{\Gamma_1}h^2(u_t)d\Gamma.
\end{aligned}\tag{3.20}$$

Noting that

$$\begin{aligned}
\int_0^\infty g(s)\eta_t\left(s\right)ds &= -\int_0^\infty g(s)\eta_s\left(s\right)ds+\int_0^\infty u_t\left(t\right)g(s)ds\\
&= \int_0^\infty g'(s)\eta\left(s\right)ds+\left(1-l\right)u_t
\end{aligned}$$

we can derive that

$$\begin{aligned}
I_2 &= -\left(\xi_0-l\right)\left\|u_t\right\|^2-\int_\Omega u_t\left(t\right)\int_0^\infty g'(s)\eta\left(s\right)dsdx\\
&\leq -\frac{3\left(\xi_0-l\right)}{4}\left\|u_t\right\|^2+\frac{1}{\xi_0-l}\int_\Omega\left(\int_0^\infty -g'(s)ds\right)\left(\int_0^\infty -g'(s)\eta\left(s\right)ds\right)dx\\
&\leq -\frac{3\left(\xi_0-l\right)}{4}\left\|u_t\right\|^2-\frac{g(0)C_*^2}{\xi_0-l}\int_0^\infty g'(s)\left\|\nabla\eta\left(s\right)\right\|^2 ds.
\end{aligned}\tag{3.21}$$

The same arguments give us

$$I_3\leq-\frac{3\left(\xi_0-l\right)}{4}\left\|u_t\right\|_{\Gamma_1}^2-\frac{g(0)C_*^2}{\xi_0-l}\int_0^\infty g'(s)\left\|\nabla\eta\left(s\right)\right\|^2 ds. \tag{3.22}$$

Inserting (3.20) – (3.22) into (3.12), we can get (3.11) with

$$\begin{aligned}
K_1 &= \frac{\xi_0-l}{4\delta_3}\left(l+\frac{2\xi_1}{l}E(0)\right)^2+\frac{\xi_0-l}{2l}\\
&\quad+\left(1-l\right)+\frac{\xi_0-l}{2}C_1+\frac{\xi_0-l}{4\delta_3}C_1\\
K_2 &= \frac{2g(0)C_*^2}{\xi_0-l}.
\end{aligned}$$

The proof is done. □

Define the functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) := E(t) + \varepsilon_1 \phi(t) + \varepsilon_2 \psi(t),$$

where ε_1 and ε_2 are positive constants will be chosen later. It is easy to verify that for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough,

$$\frac{1}{2}E(t) \le \mathcal{L}(t) \le \frac{3}{2}E(t). \tag{3.23}$$

Lemma 3.4. *There exists a positive constant m such that for any $t \ge 0$,*

$$\mathcal{L}'(t) \le -mE(t) + C\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds + C\int_{\Gamma_1} h^2(u_t)d\Gamma. \tag{3.24}$$

Proof. It follows from (3.1), (3.5) and (3.11) that for any $t \ge 0$,

$$\begin{aligned}\mathcal{L}'(t) \quad \le \quad & -\left[\frac{3}{4}(\xi_0 - l)\varepsilon_2 - \varepsilon_1\right]\|u_t\|^2 - \left[\frac{3}{4}(\xi_0 - l)\varepsilon_2 - \varepsilon_1\right]\|u_t\|^2_{\Gamma_1} \\ & -[l\varepsilon_1 + \delta_1(1+C)\varepsilon_1 - \delta_2\varepsilon_2]\|\nabla u\|^2 \\ & -\xi_1\varepsilon_1\|\nabla u\|^4 - \sigma(1-\sigma E(0)\varepsilon_2)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\right)^2 \\ & +\left(\frac{1}{2} - K_2\varepsilon_2\right)\int_0^\infty g'(s)\|\nabla\eta(s)\|^2 ds \\ & +\left(\left(\frac{\xi_0 - l}{4\delta_1}\right)\varepsilon_1 + K_2\varepsilon_2\right)\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds \\ & +\left(\frac{1}{4\delta_1}\varepsilon_1 + \frac{\varepsilon_2}{2}\right)\int_{\Gamma_1} h^2(u_t)d\Gamma.\end{aligned} \tag{3.25}$$

At this point we choose $\delta_2 > 0$ satisfying ϵ

$$\delta_2 < \frac{3l}{8}(\xi_0 - l)$$

which gives us

$$\frac{2}{l}\delta_2\varepsilon_2 < \frac{3}{4}(\xi_0 - l)\varepsilon_2.$$

For any fixed ,$\delta_2 > 0$ we take $\varepsilon_2 > 0$ small enough so that (3.23) remains valid and further. For fixed δ_2 and ε_2, we pick $\varepsilon_1 > 0$ so small that (3.23) remains valid and further

$$\frac{2}{l}\delta_2\varepsilon_2 < \varepsilon_1 < \min\left\{\frac{1}{2C}, \frac{3}{4}(\xi_0 - l)\varepsilon_2\right\},$$

which gives us

$$\frac{3}{4}(\xi_0 - l)\varepsilon_2 > 0, (\varepsilon_1 + \delta_1(1+C)\varepsilon_1 - \delta_2\varepsilon_2) > 0, \left(\frac{1}{2} - K_2\varepsilon_2\right) > 0.$$

Therefore, there exists a positive constant m such that for any $t \ge 0$,

$$\mathcal{L}'(t) \le -mE(t) + C\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds + C\int_{\Gamma_1} h^2(u_t)d\Gamma.$$

which completes the proof. □

The same arguments as in [17], we can get the following lemma.

Lemma 3.5. *Under the assumptions of* $Theorem 2$, *there exists a positive* $\gamma > 0$ *such that for any* $\varepsilon_0 > 0$

$$G'(\varepsilon_0 E(t)) \int_0^\infty g(s) \|\nabla\eta(s)\|^2 ds \leq -\gamma_1 E'(t) + \gamma_1 E(t) G'(\varepsilon_0 E(t)). \tag{3.26}$$

Proof. of $Theorem 2$. We distinguish the following two cases to prove $Theorem 2$.

Case 1. *The function* h_0 *is linear.It follows from* $(A2)$ *that*

$$c_1 |s| \leq |h(s)| \leq c_2 |s|, \forall s \in \mathbb{R},$$

which implies

$$h^2(s) \leq c_2 s h(s), \forall s \in \mathbb{R}^+. \tag{3.27}$$

Combining (3.1) *and* (3.27) *with* (3.24), *we arrive at for any* $t > 0$,

$$\mathcal{L}'(t) \leq -mE(t) + C \int_0^\infty g(s) \|\nabla\eta(s)\|^2 ds - CE'(t). \tag{3.28}$$

Let $\epsilon(t) = \mathcal{L}(t) + CE(t)$. *Using* (3.23), *we know that* $\epsilon(t) \sim E(t)$. *Then* (3.28) *gives us*

$$\epsilon'(t) \leq -mE(t) + C \int_0^\infty g(s) \|\nabla\eta(s)\|^2 ds. \tag{3.29}$$

Multiplying (3.29) *by* $G'(\varepsilon_0 E(t))$ *and using* (3.26), *we obtain*

$$\begin{aligned} G'(\varepsilon_0 E(t)) \epsilon'(t) &\leq -mG'(\varepsilon_0 E(t)) E(t) - C\gamma_1 E'(t) \\ &\quad + C\gamma_1 \varepsilon_0 E(t) G'(\varepsilon_0 E(t)) \\ &= -(m - C\gamma_1 \varepsilon_0) E(t) G'(\varepsilon_0 E(t)) - C\gamma_1 E'(t). \end{aligned}$$

Now we take $\varepsilon_0 > 0$ *so small that* $m - C\gamma_1 \varepsilon_0 > 0$, *and denote* $\epsilon_1'(t) = G'(\varepsilon_0 E(t)) \epsilon'(t) + C\gamma_1 E'(t)$ *we can get there exists some* $K_1 > 0$ *such that*

$$\epsilon_1(t) \sim E(t) \text{ and } \epsilon_1'(t) \leq -K_1 G'(\varepsilon_1 \epsilon_1(t)) \epsilon_1(t) \tag{3.30}$$

which yields $(W_1(\epsilon_1))' \geq K_1$, *where*

$$W_1(\tau) = \int_\tau^1 \frac{1}{CsG'(\epsilon_1 s)} ds$$

for $0 < \tau \leq 1$. *Integrating the last inequality in* (3.30) *over* $[0, t]$, *we have for any* $t > 0$,

$$\epsilon_1(t) \leq W_1^{-1}(K_1 t + K_2). \tag{3.31}$$

Then (2.9) *follows from* (3.31), (2.8) *and* $\epsilon_1 \sim E$.

Case 2. *The function* h_0 *is nonlinear on* $[0, \varepsilon]$.

Following the arguments as in [20], *we first suppose that* $\max\{r, h_0(r)\} < \varepsilon$ *otherwise we choose* r *smaller.*

Let $\varepsilon_1 = \min\{r, h_0(r)\}$ *It follows from* $(A2)$ *that for* $\varepsilon_1 \leq |s| \leq \varepsilon$

$$|h(s)| \leq \frac{h_0^{-1}(|s|)}{|s|} |s| \leq \frac{h_0^{-1}(|\varepsilon|)}{|\varepsilon_1|} |s|,$$

and

$$|h(s)| \geq \frac{h_0(|s|)}{|s|} |s| \leq \frac{h_0(|\varepsilon_1|)}{|\varepsilon_1|} |s|.$$

Tthen we have

$$\begin{cases} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|), for \ |s| < \varepsilon_1, \\ c_1|s| \le |h(s)| \le c_2|s|, for \ |s| \ge \varepsilon_1, \end{cases} \tag{3.32}$$

which gives us for all $|s| \le \varepsilon_1$,

$$H\left(h^2(s)\right) = |h(s)|\, h_0(|h(s)|) \le sh(s).$$

Then

$$h^2(s) \le H^{-1}(sh(s)), \forall |s| \le \varepsilon_1. \tag{3.33}$$

As in [19], *we denote*

$$\Gamma_{11} = \{x \in \Gamma_1 : |u_t(t)| > \varepsilon_1\}, \Gamma_{12} = \{x \in \Gamma_1 : |u_t(t)| \le \varepsilon_1\}.$$

It follows from (3.32) *that on* Γ_{12} ,

$$u_t h(u_t) \le \varepsilon_1 h_0^{-1}(\varepsilon_1) \le h_0(r)\, r = H^2(r). \tag{3.34}$$

We define $J(t)$ *by*

$$J(t) = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t h(u_t)\, d\Gamma$$

Since H^{-1} *is concave, we infer from Jensen's inequality that*

$$H^{-1}(J(t)) \ge C \int_{\Gamma_{12}} H^{-1}(u_t h(u_t))\, d\Gamma. \tag{3.35}$$

Using (3.33) *and* (3.35), *we conclude that*

$$\begin{aligned} \int_{\Gamma_1} h^2(u_t)\, d\Gamma &\le \int_{\Gamma_{12}} H^{-1}(u_t h(u_t))\, d\Gamma + \int_{\Gamma_{11}} h^2(u_t)\, d\Gamma \\ &\le CH^{-1}(J(t)) - CE'(t), \end{aligned}$$

which, together with (3.24), *yields for any* $t > 0$,

$$\mathbb{k}'(t) \le -mE(t) + C\int_0^\infty g(s) \|\nabla\eta(s)\|^2 ds + CH^{-1}(J(t))$$

where $\mathbb{k}(t) = \mathcal{L}(t) + CE(t)$ *and* $\mathbb{k}(t) \sim E(t)$ *For* $\varepsilon_0 \le r^2$ *and* $C_0 > 0$ *and the fact* $E' < 0, H' > 0$ *and* $H" > 0$, *we obtain that the functional* $\mathbb{k}_1(t)$ *defined by*

$$\mathbb{k}_1(t) = H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathbb{k}(t) + C_0 E(t),$$

is equivalent $E(t)$ *to and*

$$\begin{aligned} \mathbb{k}_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H"\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathbb{k}(t) + H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathbb{k}'(t) + C_0 E'(t) \\ &\le -mE(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + CH^{-1}(J(t))\, H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + C_0 E'(t) \\ &\quad + CH'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds. \end{aligned} \tag{3.36}$$

Now, we denote the conjugate function of the convex function H *by* H^* *see, for example, Arnold* [1], *and Lasieckaand Tataru* [28], *i.e.,*

$$H^*(s) = \sup_{t \in \mathbb{R}^+} (st - H(t)).$$

Then

$$H^{*}(s)=s\left(H'\right)^{-1}(s)-H\left[\left(H'\right)^{-1}(s)\right],$$

is the Legendre transform of H, which satisfies

$$AB\leq H^{*}(A)+H(B).$$

For $A=H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$ *and* $B=H^{-1}(J(t))$, *and noting the fact* $H^{*}(s)\leq s\left(H'\right)^{-1}(s)$ *and using* (3.36), *we shall see that*

$$\begin{aligned}
\Bbbk_1'(t) &\leq -mE(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+C\varepsilon_0\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-CE'(t)\\
&\quad +C_0E'(t)+CH'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\int_0^{\infty}g(s)\left\|\nabla\eta(s)\right\|^2ds\\
&\leq -\left(mE(0)-C\varepsilon_0\right)\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-\left(C-C_0\right)E'(t)\\
&\quad +CH'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\int_0^{\infty}g(s)\left\|\nabla\eta(s)\right\|^2ds. \qquad (3.37)
\end{aligned}$$

In (3.37), *we choose* $\varepsilon_0>0$ *so small that* $mE(0)-C\varepsilon_0>0$ *and* C_0 *so large that* $C-C_0<0$ *to get for any* $t>0$,

$$\Bbbk_1'(t)\leq -K\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+CH'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\int_0^{\infty}g(s)\left\|\nabla\eta(s)\right\|^2ds. \qquad (3.38)$$

Multiplying (3.38) *by* $G'\left(\varepsilon_0E(t)\right)$ *and using* (3.26), *we obtain*

$$\begin{aligned}
G'\left(\varepsilon_0E(t)\right)\Bbbk_1'(t) &\leq -K\frac{E(t)}{E(0)}G'\left(\varepsilon_0E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\
&\quad -\gamma_2E'(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\
&\quad +\gamma_2\varepsilon_0E(t)G'\left(\varepsilon_0E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\
&\leq -K\frac{E(t)}{E(0)}G'\left(\epsilon_0E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-CE'(t)\\
&\quad +\gamma_2\epsilon_0E(t)G'\left(\epsilon_0E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right). \qquad (3.39)
\end{aligned}$$

Define the functional $\Bbbk_2(t)$ *by*

$$\Bbbk_2(t)=G'\left(\epsilon_0E(t)\right)\Bbbk_1(t)+CE(t).$$

It is easy to verify that $\Bbbk_2(t)\sim E(t)$. *i.e., there exist two positive constants* β_1 *and* β_2 *such that*

$$\beta_1\Bbbk_2(t)\leq E(t)\leq\beta_2\Bbbk_2(t). \qquad (3.40)$$

Noting the fact $E'(t)\leq 0$ *and* $G" >0$ *we infer from* (3.39) *that*

$$\Bbbk_2'(t)\leq -\left(K-\gamma_2\epsilon_1\right)\frac{E(t)}{E(0)}G'\left(\epsilon_1\frac{E(t)}{E(0)}\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right), \qquad (3.41)$$

with $\epsilon_1=\epsilon_0E(0)$. *For a suitable choice of* ϵ_0, *we get from* (3.41) *that for some constant* $K_1>0$,

$$\Bbbk_2'(t)\leq -K_1\frac{E(t)}{E(0)}G'\left(\epsilon_1\frac{E(t)}{E(0)}\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)=-K_1W_2\left(\frac{E(t)}{E(0)}\right), \qquad (3.42)$$

where $W_2(t) = tH'(\varepsilon_0 t)\, G'(\epsilon_1 t)$ *Denote* $R(t) = \frac{\beta_1 \mathbb{k}_2(t)}{E(0)}$ *It follows from* (3.40) *that*

$$R(t) \sim E(t). \tag{3.43}$$

By (3.42), *we get for some* $\mathbb{k}_2(t) > 0$,

$$R'(t) \leq -K_2 W_2(R(t)), \tag{3.44}$$

which implies $(W_1(R(t)))' \geq K_2$, *where*

$$W_1(t) = \int_t^1 W_2(s)ds, \ \textit{for } t \in (0,1].$$

Integrating (3.44) *over* $[0,t]$ *we have for any* $t > 0$,

$$R(t) \leq W_1^{-1}(K_2 t + K_3). \tag{3.45}$$

Then (2.9) *follows from* (3.43) *and* (3.45). *The proof is done.*

□

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