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Nonlinear parabolic systems in Musielack-Orlicz space

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ABSTRACT: In this paper, we discuss the solvability of the nonlinear parabolic systems associated to the nonlinear parabolic equation: for $i = 1, 2$

$$(\mathcal{S})\begin{cases} \dfrac{\partial u_i}{\partial t} - div(a(x,t,u_i,\nabla u_i)) + g_i(x,t,u_i,\nabla u_i)) = f_i(x,u_1,u_2) & \text{in} \quad Q_T, \\ u_i(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_i(x,(t=0)) = u_{i,0}(x) & \text{in} \quad \Omega, \end{cases}$$

with the source f is merely integrable. The operator $A(u) = div\Big(a(x,t,u_i,\nabla u_i)\Big)$ is a generalized Leray-Lions operator defined on the inhomogeneous Musielak-Orlicz spaces (the vector field $a(x,t,u_i,\nabla u_i)$ have a growth prescribed by a generalized N-function). The non linearity g_i is a Carathéodory function satisfy the some condition.

Key Words: Parabolic problems, Sobolev-Muiselak-Orlicz inhomogeneous spaces, Renormalized Solutions.

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1. Introduction

From a physical point of view, the study of non-linear partial differential equations governed by general non-linear operators, with non-polynomial growths (growths described by N-functions or ϕ-functions) are considerable at the application level. For example, non-standard growth operators ($\phi(x,t) = |t|^{p(x)}$), in this context include models from fluid mechanics (non-Newtonian fluid), image processing (see Rajagopal, Rusi$\widetilde{c}$ka [19]) for more details. For more general operators with growth described by modular functions $\phi(x,t)$, Polish and Czechoslovak mathematicians have developed a functional framework for this type of operators, the framework of modular spaces that is the extension of Orlicz spaces appeared in the literature in the 1930s, for more details we refer to (Musielak [18], Kovacic and Rakosnich [16]).

From a mathematical point of view, the weak formulation of the PDE's is very difficult in general due to the fact that the terms of the system are not well defined, so the uniqueness of the solution is not generally accessible (see Serrin's counter-example [22]) so a difficulty to find the physically observable solution, to overcome this problem we will use the notion of a renormalized solution first introduced by R.-J. Diperna and P.-L. Lions [9]. The approach of solving the system (S) and going through approximation theorems using the notion of general modular convergence see([15]).

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Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let $T > 0$, $Q_T = \Omega \times (0,T))$. Consider the nonlinear parabolic system

$$(\mathcal{S}) \begin{cases} \dfrac{\partial u_i}{\partial t} + A(u) + g_i(x,t,u_i,\nabla u_i) = f_i(x,u_1,u_2) & \text{in} \quad Q_T, \\ u_i(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_i(x,(t=0)) = u_{i,0}(x) & \text{in} \quad \Omega, \end{cases}$$

$A(u) = -div(a(x,t,u_i,\nabla u_i))$ is a Leray-lions operators defined on the Inhomogeneous Musielak-Orlicz-Sobolev spaces $W_0^{1,x}L_\varphi(Q_T)$, where $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function and $g_i : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory functions, with sign conditions, the source terms f_i is merely integrable.

The resolution of the system (S) within the framework of the Classical Sobolev spaces is well known, we cite as an example the works on the attractors of A. El hachimi and E. Elouardi [10,11]. For type (S) systems with degenerate operator we refer to [4,5]. And the resolution of the system (S) in the case of non-polynomial growths [20].

It our purpose to solve the system (S) in the case of operators with such general growths including non-standard and non-polynomial growths, we show the existence of at least one renormalized solution of the system (S). This is the case when we are dealing with non-linear parabolic system, as in the problem

$$(\mathcal{S}) \begin{cases} \dfrac{\partial u_i}{\partial t} - div(\dfrac{\phi(x,|\nabla u_i|)}{|\nabla u_i|}\nabla u_i) + b(|u_i|)\phi(x,,\nabla u_i) = f_i(x,u_1,u_2) & \text{in} \quad Q_T, \\ u_i(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_i(x,(t=0)) = u_{i,0}(x) & \text{in} \quad \Omega. \end{cases}$$

The plan of this paper: In the section 2 present the mathematical tools, which will be used in the following sections. In section 3, we give some useful Lemmas. In section 4, we give basics assumptions and the definition of a renormalized solution of (S). Finally, we establish Theorem 5.1 the existence of such a solution of the system (S) in section 5.

2. Musielak-Orlicz spaces - Notations and properties

2.1. Musielak-Orlicz function

Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying conditions :

Φ_1: $\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $\varphi(x,0) = 0$,$\varphi(x,0) > 0$ for $t > 0$, $\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty$).

Φ_2: $\varphi(.,t)$ is a measurable function for all $t \geq 0$.

A function φ which satisfies the conditions Φ_1 and Φ_2 is called a Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp.globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0 \text{ (resp. for all } t \geq 0 \text{).}$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$, for every positive constant c, we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0 \text{ (resp.} \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0\text{)}.$$

Remark 2.1. *[7]. If $\gamma \prec\prec \varphi$ near infinity,then $\forall\epsilon > 0$ there exist $k(\epsilon) > 0$ such that for almost all $x \in \Omega$, we have*

$$\gamma(x,t) \leq k(\epsilon)\varphi(x,\epsilon t) \quad \forall t \geq 0$$

2.2. Musielak-Orlicz space

For a Musielak-Orlicz function φ and a measurable function $u:\Omega\to\mathbb{R}$, we define the functionnal

$$\varrho_{\varphi,\Omega}(u)=\int_\Omega \varphi(x,|u(x)|)dx.$$

The set $K_\varphi(\Omega)=\{u:\Omega\to\mathbb{R} \text{ measurable}: \quad \varrho_{\varphi,\Omega}(u)<\infty\}$ is called the Musielak-Orlicz class. The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$; that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega)=\{u:\Omega\to\mathbb{R} \quad \text{measurable}: \quad \varrho_{\varphi,\Omega}(\frac{u}{\lambda})<\infty, \quad \text{for some} \quad \lambda>0\}.$$

For any Musielak-Orlicz function φ, we put $\psi(x,s)=\sup_{t\geq 0}(st-\varphi(x,s))$.

ψ is called the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to s. We say that a sequence of function $u_n\in L_\varphi(\Omega)$ is modular convergent to $u\in L_\varphi(\Omega)$ if there exists a constant $\lambda>0$ such that $\lim_{n\to\infty}\varrho_{\varphi,\Omega}(\frac{u_n-u}{\lambda})=0$.

This implies convergence for $\sigma(\Pi L_\varphi,\Pi L_\psi)$ [see [6]].

In the space $L_\varphi(\Omega)$, we define the following two norms

$$\|u\|_\varphi=\inf\Big\{\lambda>0:\int_\Omega\varphi(x,\frac{|u(x)|}{\lambda})dx\leq 1\Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega}=\sup_{\|v\|_\psi\leq 1}\int_\Omega|u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [6]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$. The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is by denoted $E_\varphi(\Omega)$. It is a separable space and $(E_\varphi(\Omega))^*=L_\varphi(\Omega)$. We have $E_\varphi(\Omega)=K_\varphi(\Omega)$, if and only if φ satisfies the Δ_2-condition for large values of t or for all values of t, according to whether Ω has finite measure or not.

We define

$$W^1L_\varphi(\Omega)=\Big\{u\in L_\varphi(\Omega):D^\alpha u\in L_\varphi(\Omega),\quad \forall\alpha\leq 1\Big\},$$

$$W^1E_\varphi(\Omega)=\Big\{u\in E_\varphi(\Omega):D^\alpha u\in E_\varphi(\Omega),\quad \forall\alpha\leq 1\Big\},$$

where $\alpha=(\alpha_1,...,\alpha_N)$, $|\alpha|=|\alpha_1|+...+|\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\overline{\varrho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq 1}\varrho_{\varphi,\Omega}(D^\alpha u),$$

and

$$\|u\|^1_{\varphi,\Omega}=\inf\{\lambda>0:\overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda})\leq 1\} \text{ for } u\in W^1L_\varphi(\Omega).$$

These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$, respectively. Then pair $(W^1L_\varphi(\Omega),\|u\|^1_{\varphi,\Omega})$ is a Banach space if φ satisfies the following condition (see [18]).

$$\text{There exists a constant} \quad c>0 \quad \text{such that} \quad \inf_{x\in\Omega}\varphi(x,1)>c.$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\prod_{\alpha\leq 1}L_\varphi(\Omega)=\prod L_\varphi$ We denote by $\mathcal{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(\mathbb{R})$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi,\Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$. For two complementary Musielak-Orlicz functions φ and ψ, we have [?].

- The Young inequality

$$st \leq \varphi(x,s) + \psi(x,t) \text{ for all } s,t \geq 0\ ,\ x \in \Omega.$$

- The Hölder inequality

$$\Big| \int_\Omega u(x)v(x)dx \Big| \leq \|u\|_{\varphi,\Omega} \||v\||_{\psi,\Omega} \text{ for all } u \in L_\varphi(\Omega), v \in L_\psi(\Omega).$$

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W_0^1L_\varphi(\Omega)$) if, for some $\lambda > 0$.

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

The following spaces of distributions will also be used

$$W^{-1}L_\psi(\Omega) = \Big\{ f \in \mathcal{D}^{'}(\Omega) : f = \sum_{\alpha\leq 1} (-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in L_\psi(\Omega) \Big\},$$

and

$$W^{-1}E_\psi(\Omega) = \Big\{ f \in \mathcal{D}^{'}(\Omega) : f = \sum_{\alpha\leq 1} (-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in E_\psi(\Omega) \Big\}.$$

2.3. Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $Q_T = \Omega \times]0,T[$ with some given $T > 0$. let φ be a Musielak-Orlicz function.For each $\alpha \in N^N$, denote by D_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in R^N$. The inhomogeneous Musielak-Orlicz-Sovolev spaces of order 1 are defined as follows

$$W^{1,x}L_\varphi(Q_T) = \Big\{ u \in L_\varphi(Q_T) : \forall |\alpha| \leq 1, \quad D_x^\alpha u \in L_\varphi(Q) \Big\},$$

$$W^{1,x}E_\varphi(Q_T) = \Big\{ u \in E_\varphi(Q_T) : \forall |\alpha| \leq 1, \quad D_x^\alpha u \in E_\varphi(Q_T) \Big\}.$$

The last is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq m} \|D_x^\alpha u\|_{\varphi,Q}.$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q_T)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u \in W^{1,x}L_\varphi(Q_T)$ then the function $: t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1L_\varphi(\Omega)$. If, further, $u \in W^{1,x}E_\varphi(Q_T)$ then this function is a $W^1E_\varphi(\Omega)$-valued and is strongly measurable.

Furthermore the following imbedding holds

$$W^{1,x}E_\varphi(Q_T) \subset L^1(0,T; W^1E_\varphi(\Omega)).$$

The space $W^{1,x}L_\varphi(Q_T)$ is not in general separable, if $u \in W^{1,x}L_\varphi(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function

$$t \mapsto u(t) = \|u(t)\|_{\varphi,\Omega},$$

is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_\varphi(Q_T)$ of $\mathcal{D}(\Omega)$. We can easily show that when Ω is a Lipschitz domain then each element u of the closure of $\mathcal{D}(\Omega)$ with respect of the weak* topology $\sigma(\Pi L_{\varphi}, \Pi E_{\psi})$ is limit, in $W^{1,x}L_{\varphi}(Q_T)$, of some subsequence $(u_i) \in \mathcal{D}(\Omega)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_{Q_T} \varphi\Big(x, \frac{D_x^{\alpha} u_i - D_x^{\alpha} u}{\lambda}\Big) dxdt \to 0 \quad \text{as} \quad i \to \infty,$$

this implies that (u_i) converge to u in $W^{1,x}L_{\varphi}(Q_T)$ for the weak topology $\sigma(\Pi L_{\varphi}, \Pi L_{\psi})$. Consequently

$$\mathcal{D}(Q_T)^{\sigma(\Pi L_{\varphi}, \Pi E_{\psi})} = \mathcal{D}(Q_T)^{\sigma(\Pi L_{\varphi}, \Pi L_{\psi})},$$

this space will be denoted by $W_0^{1,x}L_{\varphi}(Q_T)$. Furthermore $W_0^{1,x}E_{\varphi}(Q_T) = W_0^{1,x}L_{\varphi}(Q_T) \cap \Pi E_{\varphi}$. F being the dual space of $W_0^{1,x}E_{\varphi}(Q_T)$. It is also, except for an isomorphism, the quotient of ΠL_{ψ} by the polar set $W_0^{1,x}E_{\varphi}(Q_T)^{\perp}$, and will be denoted by $F = W^{1,x}L_{\psi}(Q_T)$ and it is shown that this space will be equipped with the usual quotient norm where the inf is taken on all possible decomposition and is denoted by $F_0 = W^{-1,x}E_{\psi}(Q_T)$.

Lemma 2.2. *[17]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions :*

- *There exists a constant $c > 0$ such that*

$$\inf_{x \in \Omega} \varphi(x, 1) > c. \tag{2.1}$$

- *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$, we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \text{ for all } t \geq 1. \tag{2.2}$$

- $$\int_K \varphi(y, \lambda) dx < \infty, \text{ for all compact set } K \subset \Omega, \tag{2.3}$$

- $$\text{There exists a constant } C > 0 \text{ such that } \psi(y,t) \leq C \text{ a.e. in } \Omega. \tag{2.4}$$

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_{\varphi}(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_{\varphi}(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1 L_{\varphi}(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in in $W^{-1}L_{\psi}$ on an element u of $W_0^1 L_{\varphi}(\Omega)$ is well defined. It will be denoted by $< S, u >$.

2.4. Truncation Operator

T_k, $k > 0$, denotes the truncation function at level k defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2.3. *[7]. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitz, with $F(0) = 0$. Let φ be an Musielak-Orlicz function and let $u \in W_0^1 L_{\varphi}(\Omega)$(resp.$u \in W^1 E_{\varphi}(\Omega)$). Then $F(u) \in W^1 L_{\varphi}(\Omega)$ (resp. $u \in W_0^1 E_{\varphi}(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(x) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega; \, u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega; \, u(x) \in D\}. \end{cases}$$

Lemma 2.4. *Assume that Ω satisfies the segment property and let $u \in W_0^1 L_{\varphi}(\Omega)$. Then, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_{\varphi}(\Omega).$$

Furthermore, if $u \in W_0^1 L_{\varphi}(\Omega) \cap L^{\infty}(\Omega)$ then $\|u_n\|_{\infty} \leq (N+1)\|u\|_{\infty}$.

Let Ω be an open subset of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function satisfying

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty \quad \text{a.e.} \quad x \in \Omega, \tag{2.5}$$

and the conditions of Lemma 2.2. We may assume without loss of generality that

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty \quad \text{a.e.} \quad x \in \Omega. \tag{2.6}$$

Define a function $\varphi^* : \Omega \times [0, \infty)$ by $\varphi^*(x, s) = \displaystyle\int_0^s \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt$ $x \in \Omega$ and $s \in [0, \infty)$. φ^* its called the Sobolev conjugate function of φ (see [1] for the case of Orlicz function).

Theorem 2.5. *([18])Let Ω be a bounded Lipschitz domain and let φ be a Musielak-Orlicz function satisfying (2.5), (2.6) and the conditions of Lemma 2.2. Then*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_{\varphi^*}(\Omega)$$

where φ^* is the Sobolev conjugate function of φ. Moreover, if ϕ is any Musielak-Orlicz function increasing essentially more slowly than φ^* near infinity, then the imbedding

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_\phi(\Omega) \text{ is compact.}$$

Corollary 2.6. *Under the same assumptions of Lemma 2.2, we have*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow\hookrightarrow L_\varphi(\Omega),$$

Lemma 2.7. *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

Proof. We adopte the same techniques as in [15]. □

3. Technical lemma

Lemma 3.1. *[17] Under the assumptions of lemma 2.2, and by assuming that $\varphi(x,t)$ decreases with respect to one of coordinate of x, there exists a constant $c_1 > 0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x, |u|) dx \leq \int_\Omega \varphi(x, c_1 |\nabla u|)\, dx. \tag{3.1}$$

Theorem 3.2. *Let Ω be a bounded Lipchitz domain and let φ be a Musielak-Orlicz function satisfying the same conditions of Lemma 2.2. Then there exists a constant $\lambda > 0$ such that*

$$\|u\|_\varphi \leq \lambda \|\nabla u\|_\varphi, \quad \forall \in W_0^1 L_\varphi(\Omega).$$

Corollary 3.3. *[12] Let φ be a Musielak-fonction and let (u_n) be a sequence of $W^{1,x} L_\varphi(Q_T)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,x} L_\varphi(Q_T)$ for $\sigma(\Pi L_\varphi, \Pi L_\Psi)$ and $\frac{\partial u_n}{\partial t} = h_n + k_n$ in $D'(Q_T)$ with (u_n) bounded in $W^{-1,x} L_\Psi(Q_T)$ and (k_n) bounded in the space $M(Q_T)$ of measures in Q_T, then $u_n \longrightarrow u$ strongly in $L^1_{loc}(Q_T)$. If further $u_n \in W_0^{1,x} L_\varphi(Q_T)$ then $u_n \longrightarrow u$ strongly in $L^1(Q_T)$.*

4. Essential assumptions

Let Ω be an bonded open subset of $\mathbb{R}^N$ $(N \geq 2)$ satisfying the segment property, and let φ and γ be two Musielak-Orlicz functions such that φ and its complementary ψ satisfies conditions of Lemma 2.2 and $\gamma \prec\prec \varphi$. $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ defined by

$$A(u) = -\text{div}(a(x, t, u, \nabla u)),$$

where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$.

$$|a(x,s,\xi)| \leq \beta(c(x) + \psi_x^{-1}(\gamma(x,\nu_1|s|)) + \psi_x^{-1}(\varphi(x,\nu_2|\xi|))),\ \beta > 0,\ c(x) \in E_\psi(\Omega), \tag{4.1}$$

$$(a(x,s,\xi) - a(x,s,\xi^*)(\xi - \xi^*) > 0, \tag{4.2}$$

$$a(x,s,\xi)\xi \geq \alpha\varphi(x,|\xi|). \tag{4.3}$$

The non linear terms $g_i : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|g_i(x,t,s,\xi)| \leq b(|s|)(c_2(x,t) + \varphi(x,|\xi|)) \text{ and } g_i(x,t,s,\xi)s \geq 0, \tag{4.4}$$

where $c_2(x,t) \in L^1(Q_T)$ and $b : \mathbb{R}^+ \to \mathbb{R}$ is a continuous and nondecreasing.

The function $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function belongs to $L^1(Q_T)$ with

$$f_1(x,0,s) = f_2(x,s,0) = 0 \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}, \tag{4.5}$$

and for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$,

$$\operatorname{sign}(s_i)f_i(x,s_1,s_2) \geq 0. \tag{4.6}$$

$$u_{i,0} \quad \text{is an element of} \quad L^1(Q_T). \tag{4.7}$$

In this paper, for any measurable subset E of Q_T, we denote by $meas(E)$ the Lebesgue measure of E. For any measurable function v defined on Q_T and for any real number s, $\chi_{\{v<s\}}$ (respectively, $\chi_{\{v=s\}}, \chi_{\{v>s\}}$) denote the characteristic function of the set

$$\{(x,t) \in Q_T\ ;\ v(x,t) < s\} \text{ (respectively, } \{(x,t) \in Q_T; v(x,t) = s\}, \{(x,t) \in Q_T; v(x,t) > s\}).$$

Definition 4.1. *A couple of functions (u_1,u_2) defined on Q_T is called a renormalized solution of (1) if for $i = 1,2$ the function u_i satisfies*

$$T_K(u_i) \in W_0^{1,x}L_\varphi(Q_T), \tag{4.8}$$

$$\int_{\{\ m\leq|u_i|\leq m+1\}} a(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt \to 0 \quad \text{as } m \to +\infty, \tag{4.9}$$

for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have

$$\begin{aligned} &\frac{\partial S(u_i)}{\partial t} - div(S'(u_i)a(x,t,u_i,\nabla u_i)) + S''(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i \\ &+g_i(x,u_i,\nabla u_i)S'(u_i) = f_i(x,u_1,u_2)S'(u_i), \end{aligned} \tag{4.10}$$

$$S(u_i)(t=0) = S(u_{i,0}) \quad \text{in } \Omega. \tag{4.11}$$

Remark 4.2.

Due to (4.8), each term in (4.10) has a meaning in $W^{-1,x}L_\psi(Q_T) + L^1(Q_T)$.

Indeed, if K such that $suppS \subset [-K,K]$, the following identifications are made in (4.10)

- $S'(u_i)a(x,t,u_i,\nabla u_i)$ can be identified with

$$S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i)) \text{ a.e. in } Q_T.$$

Since indeed $|T_K(u_i)| \leq K$ a.e. in Q_T.

As a consequence of (4.1) , (4.8) and $S'(u_i) \in L^\infty(Q_T)$, it follows that

$$S'(u_i)a(x,T_K(u_i),\nabla T_K(u_i)) \in (L_\psi(Q_T))^N.$$

- $S'(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i$ can be identified with

$$S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i) \text{ a.e. in } Q_T.$$

  with (4.8) it has

$$S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i) \in L^1(Q_T).$$

- $S'(u_i)g_i(x,t,T_K(u_i),\nabla T_k u_i) \in (L^\infty(Q_T))^N$ and $S'(u_i)f_i(x,u_1,u_2)$ identifies with

$$S'(u_i)f_1(x,T_K(u_1),u_2) \text{ a.e. in } Q_T \text{ (or } S'(u_i)f_2(x,u_1,T_K(u_2)) \text{ a.e. in } Q_T).$$

  Indeed, since $|T_K(u_i)| \le K$ a.e. in Q_T.

- The datum $f_i(x,u_1,u_2)$ belongs to $L^1(\Omega \times (0,T))$, and using (4.8) and of

$$S'(u_i) \in L^\infty(Q_T),$$

  one has

$$S'(u_1)f_1(x,T_K(u_1),u_2) \in L^1(Q_T) \text{ and } S'(u_2)f_2(x,u_1,T_K(u_2)) \in L^1(Q_T).$$

Remark 4.3.

Due to (4.8), each term in (4.10) has a meaning in $W^{-1,x}L_\psi(Q_T) + L^1(Q_T)$.

Indeed, if K such that $suppS \subset [-K,K]$, the following identifications are made in (4.10)

- $S'(u_i)a(x,t,u_i,\nabla u_i)$ can be identified with

$$S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i)) \text{ a.e. in } Q_T.$$

  Since indeed $|T_K(u_i)| \le K$ a.e. in Q_T.
  As a consequence of (4.1) , (4.8) and $S'(u_i) \in L^\infty(Q_T)$, it follows that

$$S'(u_i)a(x,T_K(u_i),\nabla T_K(u_i)) \in (L_\psi(Q_T))^N.$$

- $S'(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i$ can be identified with

$$S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i) \text{ a.e. in } Q_T.$$

  with (4.8) it has

$$S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i) \in L^1(Q_T).$$

- $S'(u_i)g_i(x,t,T_K(u_i),\nabla T_k u_i) \in (L^\infty(Q_T))^N$ and $S'(u_i)f_i(x,u_1,u_2)$ identifies with

$$S'(u_i)f_1(x,T_K(u_1),u_2) \text{ a.e. in } Q_T \text{ (or } S'(u_i)f_2(x,u_1,T_K(u_2)) \text{ a.e. in } Q_T).$$

  Indeed, since $|T_K(u_i)| \le K$ a.e. in Q_T.

- The datum $f_i(x,u_1,u_2)$ belongs to $L^1(\Omega \times (0,T))$, and using (4.8) and of

$$S'(u_i) \in L^\infty(Q_T),$$

  one has

$$S'(u_1)f_1(x,T_K(u_1),u_2) \in L^1(Q_T) \text{ and } S'(u_2)f_2(x,u_1,T_K(u_2)) \in L^1(Q_T).$$

5. Existence result

Lets ϕ and ν two Musielak-functions satisfies conditions of Lemma2.1. We shall prove the following existence theorem,

Theorem 5.1. *Assume that (4.1)-(4.7) hold true. There at least a renormalized solution (u_1, u_2) of Problem (1).*

Proof. We divide the prof of Theorem 5.1 in 5 steps.

Step 1: Approximate problem. Let us introduce the following regularization of the data : for $n > 0$ and $i = 1, 2$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \text{ a.e. in } Q_T, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \tag{5.1}$$

$$g_{i,n}(x,t,s,\xi) = g(x,t,T_n(s),\xi) \text{ a.e. in } Q_T,\ \forall s \in \mathbb{R}, \tag{5.2}$$

$$f_{1,n}(x,s_1,s_2) = f_1(x,T_n(s_1),s_2) \text{ a.e. in } \Omega, \forall s_1, s_2 \in \mathbb{R}, \tag{5.3}$$

$$f_{2,n}(x,s_1,s_2) = f_2(x,s_1,T_n(s_2)) \text{ a.e. in } \Omega, \forall s_1, s_2 \in \mathbb{R}, \tag{5.4}$$

$$\begin{gathered} u^0_{i,n} \in C_0^\infty(\Omega), \\ u^0_{i,n}(x) \to u_{i,0} \text{ in } L^1(\Omega) \text{ as n tends to } +\infty, \\ \text{and} \quad ||u^0_{i,n}||_{L^1} \le ||u_{i,0}||_{L^1},\ f_{i,n} \to f_i \text{ in } L^1(\Omega) \text{ as n tends to} +\infty \\ \text{and} \quad ||f_{i,n}||_{L^1} \le ||f_i||_{L^1}. \end{gathered} \tag{5.5}$$

Let us now consider the regularized problem

$$\frac{\partial u_{i,n}}{\partial t} - \operatorname{div}(a_n(x,u_{i,n},\nabla u_{i,n})) + g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})) = f_{i,n}(x,u_{1,n},u_{2,n}) \text{ in } Q_T, \tag{5.6}$$

$$u_{i,n} = 0 \text{ on } (0,T) \times \partial\Omega, \tag{5.7}$$

$$u_{i,n}(t=0) = u^0_{i,n} \text{ in } \Omega. \tag{5.8}$$

Since $g_{i,n}$ for $i = 1, 2$ is bonded for any fixed n, As a consequence, proving the existence of a weak solution $u_{i,n} \in W_0^{1,x} L_\varphi(Q_T)$ of (5.6)-(5.8) is an easy task (see e.g. [21]).

Step 2 : A priori estimates

Proposition 5.2. *assume that (4.1) are satisfied, and let $u_{i,n}$ be a solution of the approximate problem (1). Then for all k, n, we have*

$$||T_k(u_{i,n})||_{W_0^{1,x} L_\varphi(Q_T)} \le Ck. \tag{5.9}$$

$$\lim_{k\to+\infty} meas\{(x,t) \in Q_T : |u_{i,n} \ge k| = 0\}, \quad \textit{uniformly with respect to} \, n. \tag{5.10}$$

$$\int_{Q_t} g_{i,n}(x,t,u_{i,n}) \nabla T_k(u_{i,n}) dx\, dt \le C_g, \quad where\, C_g \textit{ is a constant not depending on} \, n. \tag{5.11}$$

Proof. Let $\tau \in (0,T)$ and using $T_k(u_{i,n})\chi_{(0,\tau)}$ as a test function in problem (5.6), we get

$$\begin{aligned} &\int_{Q_t} \frac{\partial u_{i,n}}{\partial t} T_k(u_{i,n}) \chi_{(0,\tau)} dx\, dt + \int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_k(u_{i,n}) dx\, dt \\ &\quad + \int_{Q_t} g_{i,n}(x,t,u_{i,n},\nabla T_k(u_{i,n})) T_k(u_{i,n}) dx\, dt = \int_{Q_t} f_{i,n} T_k(u_{i,n})\, dx\, dt \ge 0, \end{aligned} \tag{5.12}$$

implies that,

$$
\begin{aligned}
\int_{\Omega} \hat{T}_k(u_{i,n}(\tau))dx + \int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})dx\,dt = \int_{Q_t} f_{i,n}T_k(u_{i,n})\,dx\,dt \\
- \int_{Q_t} g_{i,n}(x,t,u_{i,n},\nabla T_k(u_{i,n}))T_k(u_{i,n})dx\,dt + \int_{\Omega} \hat{T}_k u_{i,0})\,dx,
\end{aligned}
\tag{5.13}
$$

where

$$
\hat{T}_k(s) = \int_0^s T_k(t)dt = \begin{cases} \frac{s^2}{2} & \text{if} \quad |s| \le k, \\ k|s| - \frac{s^2}{2} & \text{if} \quad |s| > k. \end{cases}
\tag{5.14}
$$

Due to the definition of $\hat{T}_k$ and (5.5), we have

$$
0 \le \int_{\Omega} \hat{T}_k(u_{i,0n}(t))dx \le k\int_{\Omega} |u_{i,0}|dx \le ||u_{i0}||_{L^1(\Omega)}.
\tag{5.15}
$$

Consider now for $\theta, \epsilon > 0$ a function $\rho_\theta^\epsilon \in C^1(\mathbb{R})$ such that

$$
\varrho_\theta^\epsilon(s) = \begin{cases} 0 & \text{if} \quad |s| \le \theta, \\ sign(s) & \text{if} \quad |s| > \theta + \epsilon, \end{cases}
\tag{5.16}
$$

and

$$
(\varrho_\theta^\epsilon)'(s) \ge 0, \quad \forall s \in \mathbb{R},
$$

then, by using $\varrho_\theta^\epsilon(u_{i,n})$ as a test function in (5.5) and following [1], we can see that

$$
\int_{\{|u_{i,n}|>\theta\}} |g_{i,n}(x,t,u_{i,n},\nabla T_k(u_{i,n}))|dx\,dt \le \int_{\{|u_{i,n}|>\theta\}} |f_{i,n}|dx\,dt + \int_{\{|u_{i,n}|>\theta\}} |u_{i,0n}|dx\,dt,
\tag{5.17}
$$

and, so by letting $\theta \to 0$ and using Fatou's a lemma, we deduce that $g_{i,n}(x,t,u_{i,n},\nabla T_k(u_{i,n}))$ is a bounded sequence in $L^1(\Omega)$, then we obtain (5.11). By using (5.5) and (5.15) (5.11) permit to deduce from (5.13) that

$$
\begin{aligned}
&\int_{\Omega} \hat{T}_k(u_{i,n}(\tau))dx + \int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})dx\,dt \\
&\quad = \int_{Q_t} f_{i,n}T_k(u_{i,n})\,dx\,dt - \int_{Q_t} g_{i,n}(x,t,u_{i,n},\nabla T_k(u_{i,n}))T_k(u_{i,n})dx\,dt + \int_{\Omega} \hat{T}_k u_{i,0})dx \\
&\quad \le k||f_n||_{L^1(Q_t)} + kC_g + k||u_0||_L^1(\Omega) \\
&\quad \le kC_0.
\end{aligned}
\tag{5.18}
$$

□

Where here and below C_0 denote positive constants not depending on n and k. By using (5.26) and the fact that $\hat{T}_k(u_{i,n}) \ge 0$, permit to deduce that

$$
\int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})dx\,dt \le kC_0,
\tag{5.19}
$$

Which implies by virtu of (4.3) that

$$
\int_{Q_t} \varphi_{i,n}(x,\nabla T_k(u_{i,n})dx\,dt \le kC_1.
$$

We deduce from that above inequality (5.15) that

$$
\int_{\Omega} \hat{T}_k(u_{i,n}(t))dx \le kC_0,
$$

for almost any $t \in (0,T)$. And then, by (5.26), we conclude that $T_k(u_{i,n})$ is bounded in $W^{1,x}L_\varphi(Q_T)$ independently of n and for any $k \geq 0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_{i,n}) \rightharpoonup \psi_{i,k}, \tag{5.20}$$

weakly in $W_0^{1,x}L_\varphi(Q_T)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$ strongly in $E_\varphi(Q_T)$ and a.e in Q_T.

Since Lemma 3.1 and (5.26), we get also,

$$\begin{aligned} \varphi(x,k)\ meas\Big\{\{|u_{i,n}|>k\}\times[0,T]\Big\} &\leq \int_0^T \int_{\{|u_{i,n}|>k\}} \varphi(x, T_k(u_{i,n}))dxdt \\ &\leq \int_{Q_T} \varphi(x, T_k(u_{i,n}))dxdt \\ &\leq diam Q_T \int_{Q_T} \varphi(x, \nabla T_k(u_{i,n}))dxdt. \end{aligned}$$

Then

$$meas\Big\{\{|u_{i,n}|>k\}\times[0,T]\Big\} \leq \frac{diam(Q_T).C_i.k}{\varphi(x,k)}.$$

Which implies that:

$$\lim_{k\to+\infty} \text{meas}\Big\{\{|u_{i,n}|>k\}\times[0,T]\Big\} = 0, \text{ uniformly with respect to } n.$$

Now we shall prove the following proposition.

Proposition 5.3. *Let $u_{i,n}$ be a solution of the approximate problem, then :*

$$u_{i,n} \to u_i \quad a.e\ in \quad Q_T. \tag{5.21}$$

$$a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})) \rightharpoonup X_{i,k} \quad in \quad (L_\psi(Q_T))^N for \quad \sigma(\Pi L_\psi, \Pi E_\varphi) \tag{5.22}$$

for some $X_{i,k} \in (L_\psi(Q_T))^N$

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \int_{m\leq|u_{i,n}|\leq m+1} a_i(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} dxdt = 0 \tag{5.23}$$

Proof. The first we give the **proof of** (5.21) **and** (5.22). Consider now a function non decreasing $\zeta_k \in C^2(\mathbb{R})$ such that $\zeta_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $\zeta_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $\zeta_k'(u_{i,n})$, we get

$$\frac{\partial(\zeta_k(u_{i,n}))}{\partial t} - div\Big(a_n(x,t,u_{i,n},\nabla u_{i,n})\zeta_k'(u_{i,n})\Big) + a_n(x,t,u_{i,n},\nabla u_{i,n})\zeta_k''(u_{i,n})\nabla u_{i,n} \tag{5.24}$$

$$+\zeta_k'(u_n)g_k(u_n) = f_{i,n}\zeta_k'(u_n \quad \text{in } D'(Q_T)$$

Using (5.24),we can deduce that $\zeta_k(u_{i,n})$ is bounded in $W_0^{1,x}L_\varphi(Q_T)$ and $\frac{\partial \zeta_k u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_\psi(Q_T)$ independently of n.

Hence Corollary 3.3 implies that $\zeta_k(u_{i,n})$ is compact in $L^1(Q_T)$. Due to the choice of ζ_k, we conclude that for each k, the sequence $T_k(u_{i,n})$ converges almost everywhere in Q_T, which implies that the sequence $u_{i,n}$ converge almost everywhere to some measurable function u_i in Q_T. Then by the same argument in [3], we have

$$u_{i,n} \to u_i \text{ a.e. } Q_T, \tag{5.25}$$

where u_i is a measurable function defined on Q_T.

$$\int_{Q_t} \varphi(x, \nabla T_k(u_{i,n}))\, dx\, dt \leq C_i.k \tag{5.26}$$

Then, by (5.26), we conclude that $T_k(u_{i,n})$ is bounded in $W^{1,x}L_\varphi(Q_T)$ independently of n and for any $k \geq 0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_{i,n}) \rightharpoonup \psi_{i,k}, \tag{5.27}$$

weakly in $W_0^{1,x}L_\varphi(Q_T)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$, strongly in $E_\varphi(Q_T)$ and a.e in Q_T.

Since Lemma 3.1 and (5.26), we get also,

Now we shall to prove the boundness of $a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n}))$ in $(L_\psi(Q_T))^N$

Let $\phi \in (E_\varphi(Q_T))^N$ with $||\phi|| = 1$. In view of the monotonicity of a, one easily has,

$$\int_{Q_T} [a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n}) - a_n(x,t,T_k(u_{i,n}), \phi)][\nabla T_k(u_{i,n}) - \phi] dx\, dt \geq 0$$

which gives

$$\int_{Q_T} [a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})).\phi] dx\, dt \leq \int_{Q_T} [a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n}).\nabla T_k(u_{i,n}))] dx\, dt$$
$$+ \int_{Q_T} [a_n(x,t,T_k(u_{i,n}), \phi)][\nabla T_k(u_{i,n}) - \phi] dx\, dt$$

using (4.1) and (5.26), we easily see that

$$\int_{Q_T} [a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})).\phi] dx\, dt \leq C_3.$$

And so, we conclude that $a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})$ is bounded sequence in $L_\psi(Q_T))^N$, and we obtain (5.22).

The second we give the **proof of** (5.23). Considering the following function

$$v = T_1(u_{i,n} - T_m(u_{i,n}))$$

as test function in (5.5) we obtain,

$$\begin{aligned} &< \frac{\partial u_{i,n}}{\partial t}, T_1(u_{i,n} - T_m(u_{i,n})) > + \int_{m \leq u_{i,n} \leq m+1} [a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})).\nabla T_k(u_{i,n}) dx\, dt \\ &\quad = \int_{Q_T} f_{i,n} T_1(u_{i,n} - T_m(u_{i,n})) dx\, dt - \int_{Q_T} g_{in}(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n}).T_1(u_{i,n} - T_m(u_{i,n})) dx\, dt \end{aligned} \tag{5.28}$$

we get

$$\begin{aligned} &\int_\Omega U_{i,n}^m(u_{in}(T) dx + \int_{m \leq u_{i,n} \leq m+1} [a_n(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})).\nabla T_k(u_{i,n}) dx\, dt \\ &\qquad \leq \int_{Q_T} |f_{i,n} T_1(u_{i,n} - T_m(u_{i,n}))| dx\, dt + \int_\Omega U_{i,n}^m(u_{i,0n}) dx \\ &\qquad\quad + \int_{Q_T} |g_{i,n}(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n}) T_1(u_{i,n} - T_m(u_{i,n}))| dx\, dt \\ &\qquad \leq \int_{Q_T} |f_{i,n} + g_{i,n}(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})||T_1(u_{i,n} - T_m(u_{i,n}))| dx\, dt + \int_\Omega U_{i,n}^m(u_{i,0n}) dx, \end{aligned} \tag{5.29}$$

where $U_{in}^m(u_{in})(r) = \displaystyle\int_0^r \frac{\partial u_{i,n}}{\partial t} T_1(s - T_m(s)) ds$ and we use $U_{i,n}^m(u_{i,0n}(T)) \geq 0$ and (5.5) we obtain that,

$$\begin{aligned} &\lim_{n \to +\infty} \int_{m \leq u_{i,n} \leq m+1} a_n(x,t,u_{i,n}, \nabla T_k(u_{i,n})) \nabla T_k(u_{i,n})\, dx\, dt \\ &\qquad \leq \int_{|u_{in}| > m} (|f_i| + C_{g_i}) dx\, dt + \int_{|u_{in}| > m} |u_{i0}| dx. \end{aligned} \tag{5.30}$$

Finally by (5.5), and (5.30), we obtain (5.23). $\square$

Step 3. Let $\upsilon_{i,j} \in \mathcal{D}(Q_T)$ be a sequence such that $\upsilon_{i,j} \to u_i$ in $W_0^{1,x}L_\varphi(Q_T)$ for the modular convergence.
This specific time regularization of $T_k(\upsilon_{i,j})$ (for fixed $k \geq 0$) is defined as follows.
Let $(\alpha^\mu_{i,0})_\mu$ be a sequence of functions defined on Ω such that

$$\alpha^\mu_{i,0} \in L^\infty(\Omega) \cap W_0^1 L_\varphi(\Omega) \quad \text{for all } \mu > 0 \tag{5.31}$$

$$\|\alpha^\mu_{i,0}\|_{L^\infty(\Omega)} \leq k \text{ for all } \mu > 0.$$

and

$$\alpha^\mu_{i,0} \text{ converges to } \quad T_k(u_{i,0}) \text{ a.e. in } \Omega \quad \text{ and } \frac{1}{\mu}\|\alpha^\mu_{i,0}\|_{\varphi,\Omega} \text{ converges to } \quad 0 \quad \mu \to +\infty$$

For $k \geq 0$ and $\mu > 0$, let us consider the unique solution $(T_k(\upsilon_{i,j}))_\mu \in L^\infty(Q) \cap W_0^{1,x}L_\varphi(Q_T)$ of the monotone problem

$$\frac{\partial (T_k(\upsilon_{i,j}))_\mu}{\partial t} + \mu((T_k(\upsilon_{i,j}))_\mu - T_k(\upsilon_{i,j})) = 0 \text{ in } D'(\Omega), \tag{5.32}$$

$$(T_k(\upsilon_{i,j}))_\mu(t=0) = \alpha^\mu_{i,0} \text{ in } \Omega. \tag{5.33}$$

Remark that due to

$$\frac{\partial (T_k(\upsilon_{i,j}))_\mu}{\partial t} \in W_0^{1,x}L_\varphi(Q_T) \tag{5.34}$$

We just recall that,

$$(T_k(\upsilon_{i,j}))_\mu \to T_k(u_i) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly} * \quad \text{in } L^\infty(Q_T) \quad \text{and} \tag{5.35}$$

$$(T_k(\upsilon_{i,j}))_\mu \to (T_k(u_i))_\mu \quad \text{in} \quad W_0^{1,x}L_\varphi(Q_T) \quad \text{for the modular convergence as} \quad j \to +\infty. \tag{5.36}$$

$$(T_k(u_i))_\mu \to T_k(u_i) \quad \text{in} \quad W_0^{1,x}L_\varphi(Q_T) \quad \text{for the modular convergence as} \quad \mu \to +\infty. \tag{5.37}$$

$$||(T_k(\upsilon_{i,j}))_\mu||_{L^\infty(Q_T)} \leq max(||(T_k(u_i))||_{L^\infty(Q_T)}, ||\alpha^\mu_0||_{L^\infty(\Omega)}) \leq k, \ \forall \ \mu > 0 \ ,\forall \ k > 0 \tag{5.38}$$

Now, we introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that, for any $m \geq 1$

$$S_m(r) = r \text{ for } |r| \leq m, \quad \text{supp}(S'_m) \subset [-(m+1), (m+1)], \quad \|S''_m\|_{L^\infty(\mathbb{R})} \leq 1. \tag{5.39}$$

Through setting, for fixed $K \geq 0$,

$$W^n_{i,j,\mu} = T_K(u_{i,n}) - T_K(\upsilon_{i,j})_\mu \quad \text{and} \quad W^n_{i,\mu} = T_K(u_{i,n}) - T_K(u_i)_\mu \tag{5.40}$$

we obtain upon integration,

$$\begin{aligned} &\int_{Q_T} \langle \frac{\partial S_m(u_i,n)}{\partial t}, W^n_{i,j,\mu} \rangle \, dx\, dt + \int_{Q_T} S'_m(u_{i,n}) a_n(x, u^n_i, \nabla u_{i,n}) \nabla W^n_{i,j,\mu} \, dx\, dt \\ &+ \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} a_n(x, u_{i,n}, \nabla u_{i,n}) \nabla u_{i,n} \, dx\, dt + \int_{Q_T} g_{i,n}(x, u_{1,n}, u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu} \, dx\, dt \\ &\qquad = \int_{Q_T} f_{i,n}(x, u_{1,n}, u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu} \, dx\, dt. \end{aligned} \tag{5.41}$$

Next we pass to the limit as n tends to $+\infty$, j tends to $+\infty$, μ tends to $+\infty$ and then m tends to $+\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $K \geq 0$

$$\liminf_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\langle\frac{\partial S_m(u_{i,n})}{\partial t}\ ,W^n_{i,j,\mu}\rangle\,dx\,dt\geq 0\quad\text{for any } m\geq K, \tag{5.42}$$

$$\lim_{m\to+\infty}\overline{\lim}_{\mu\to+\infty}\lim_{j\to+\infty}\overline{\lim}_{n\to+\infty}\Big|\int_{Q_T}S''_m(u_{i,n})W^n_{i,j,\mu}a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\Big|=0, \tag{5.43}$$

$$\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{Q_T}g_{i,n}(x,u_{1,n},u_{2,n})S'_m(u_{i,n})W^n_{i,j,\mu}\,dx\,dt=0\text{ for any } m\geq 1. \tag{5.44}$$

$$\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{Q_T}f_{i,n}(x,u_{1,n},u_{2,n})S'_m(u_{i,n})W^n_{i,j,\mu}\,dx\,dt=0 \tag{5.45}$$

for any $m\geq 1$.

$$\limsup_{n\to+\infty}\int_{Q_T}a(x,t,u_{i,n},\nabla T_K(u_{i,n}))\nabla T_K(u_{i,n})\,dx\,dt\leq\int_{Q_T}X_{i,K}\nabla T_K(u_i)\,dx\,dt \tag{5.46}$$

.

$$\int_{Q_T}[a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))-a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))][\nabla T_k(u_{i,n})-\nabla T_k(u_i)]dx\,dt\to 0. \tag{5.47}$$

Proof of (5.42):

Lemma 5.4.

$$\int_{Q_T}\langle\frac{\partial u_{i,n}}{\partial t},S'_m(u_{i,n})W^n_{i,j,\mu}\rangle dx\,dt\geq\epsilon(n,j,\mu,m) \tag{5.48}$$

Proof. We can follow the same proof in [21]. □

Proof of (5.43):

If we take $n>m+1$, we get for any $m\geq 1$ fixed

$$\begin{aligned}&\Big|\int_{Q_T}S''_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}W^n_{i,j,\mu}\,dx\,dt\Big|\\&\leq\|S''_m\|_{L^\infty(\mathbb{R})}\|W^n_{i,j,\mu}\|_{L^\infty(Q_T)}\int_{\{m\leq|u_{i,n}|\leq m+1\}}a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt,\end{aligned}$$

for any $m\geq 1$, and any $\mu>0$. In view (5.38) and (5.39), we can obtain

$$\begin{aligned}&\limsup_{n\to+\infty}\Big|\int_{Q_T}S''_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}W^n_{i,j,\mu}\,dx\,dt\Big|\\&\leq 2K\limsup_{n\to+\infty}\int_{\{m\leq|u_{i,n}|\leq m+1\}}a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt,\end{aligned} \tag{5.49}$$

for any $m\geq 1$. Using (5.23) we pass to the limit as $m\to+\infty$ in (5.49) and we obtain (5.43).
Proof of (5.44):
Since $g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})\to g(x,t,u_{i,n},\nabla u)$ a.e.in Q_T, thanks to (4.4) and (5.17) and Vitali's theorem,it suffices to prove that $g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})$ are uniformly equi-integrable in Q . Let $E\subset Q_T$ be a measurable subset of Q_T ,then for any $m>O$, one has

$$\begin{aligned}\int_E|g_{i,n}(x,t,u_i,\nabla u_i|\,dx\,dt=&\int_{E\cap u_{i,n}\leq m}|g_{i,n}(x,t,u_i,\nabla u_i)|\,dx\,dt\\&+\int_{E\cap u_{i,n}>m}|g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})|\,dx\,dt,\end{aligned}$$

on the other hand,

$$\int_{E\cap u_{i,n}>m} |g_{i,n}(x,t,u_i),\nabla u_i)|\,dx\,dt, \leq \frac{1}{m}\int_{Q_T} |g_{i,n}(x,t,u_{i,n},\nabla u_{i,n}|\,dx\,dt \leq \frac{C}{m}$$

where C is the constant in (5.19), therefore, there exists $m = m(\epsilon)$ large enough such that

$$\begin{aligned}\int_{E\cap u_{i,n}\leq m} |g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})|\,dx\,dt &\leq \int_E |g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})|\,dx\,dt\\ &\leq b(m)\int_E (c_2(x,t)+\varphi(x,\nabla T_m(u_n))\,dx\,dt\\ &\leq b(m)\int_E (c_2(x,t)+\frac{1}{\alpha}d(x,t)\,dx\,dt\\ &\leq \frac{b(m)}{\alpha}\int_E a_n(x,t,u_{i,n},\nabla u_{i,n}).\nabla u_{i,n}\,dx\,dt,\end{aligned}$$

where we have used (5.17) and (4.4), therefore, it is easy to see that there exists $\mu > 0$ such that

$$|E| < \nu \Rightarrow \int_{E\cap u_{i,n}\leq m} |g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})|\,dx\,dt, \leq \frac{\epsilon}{2}\forall n.$$

which shows that $g_{i,n}(x,t,u_{i,n},\nabla u_{i,n}$ are uniformly equi-integrable in Q are required.
Moreover, we get

$$g_{i,n}(x,t,u_{i,n}),\nabla u_{i,n} \to g_i(x,t,u_i,\nabla u_i)) \quad \text{strongly in} \quad L_1(Q) \tag{5.50}$$

For fixed $n \geq 1$ and $n > m+1$, we have

$$g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})S'_m(u_{1,n}) = g_{i,n}(x,t,T_m u_{i,n},\nabla T_m u_{i,n})S'_m(u_{i,n})$$

In view (5.39),(5.40),(5.50) the theorem allow us to get, for

$$\lim_{n\to+\infty}\int_{Q_T} g_{i,n}(x,t,u_{i,n}),\nabla u_{i,n})S'_m(u_{i,n})W^n_{i,j,\mu}\,dx\,dt = \int_{Q_T} g_i(x,t,u_i,\nabla u_i))S'_m(u_i)W_{i,j,\mu}\,dx\,dt$$

Using (5.36), we follow a similar way we get as $j \to +\infty$

$$\lim_{j\to+\infty}\int_{Q_T} g_i(x,t,u_i,\nabla u_i))S'_m(u_i)W_{i,j,\mu}\,dx\,dt = \int_{Q_T} g_i(x,t,u_i,\nabla u_i))S'_m(u_i)(T_K(u_i)-T_K(u_i)_\mu)\,dx\,dt$$

we fixed $m > 1$, and using (5.37), we have

$$\lim_{\mu\to+\infty}\int_{Q_T} g_i(x,t,u_i,\nabla u_i))S'_m(u_i)(T_K(u_i)-T_K(u_i)_\mu)\,dx\,dt = 0$$

Then we conclude the proof of (5.44).
Proof of (5.45):
For fixed $n \geq 1$ and $n > m+1$, we have

$$f_{1,n}(x,u_{1,n},u_{2,n})S'_m(u_{1,n}) = f_1(x,T_{m+1}(u_{1,n}),T_n(u_{2,n}))S'_m(u_{1,n}),$$

$$f_{2,n}(x,u_{1,n},u_{2,n})S'_m(u_{2,n}) = f_2(x,T_n(u_{1,n}),T_{m+1}(u_{2,n}))S'_m(u_{2,n}),$$

In view (5.3),(5.4),(5.40) and Lebegue's the theorem allow us to get, for

$$\lim_{n\to+\infty}\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})S'_m(u_{i,n})W^n_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)W_{i,j,\mu}\,dx\,dt$$

Using (5.36), we follow a similar way we get as $j \to +\infty$

$$\lim_{j\to+\infty} \int_{Q_T} f_i(x,u_1,u_2) S'_m(u_i) W_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2) S'_m(u_i)(T_K(u_i) - T_K(u_i)_\mu)\,dx\,dt$$

we fixed $m > 1$, and using (5.37), we have

$$\lim_{\mu\to+\infty} \int_{Q_T} f_i(x,u_1,u_2) S'_m(u_i)(T_K(u_i) - T_K(u_i)_\mu)\,dx\,dt = 0$$

Then we conclude the proof of (5.45).
Proof of (5.46):
If we pass to the lim-sup when n ,j and μ tends to $+\infty$ and then to the limit as m tends to $+\infty$ in (5.41). We obtain using (5.42)-(5.43), for any $K \geq 0$,

$$\lim_{m\to+\infty} \limsup_{\mu\to+\infty} \limsup_{j\to+\infty} \limsup_{n\to+\infty} \int_{Q_T} S'_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \big(\nabla T_K(u_{i,n}) - \nabla T_K(\upsilon_{i,j})_\mu\big)\,dx\,dt \leq 0.$$

Since

$$S'_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_K(u_{i,n}) = a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_K(u_{i,n})$$

for $n > K$ and $K \leq m$. Then, for $K \leq m$,

$$\begin{aligned} &\limsup_{n\to+\infty} \int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_K(u_{i,n})\,dx\,dt \\ &\leq \lim_{m\to+\infty} \limsup_{\mu\to+\infty} \limsup_{j\to+\infty} \limsup_{n\to+\infty} \int_{Q_T} S'_m(u_{i,n}) a_n(x,u_{i,n},\nabla u_{i,n}) \nabla T_K(\upsilon_{i,j})_\mu\,dx\,dt \end{aligned} \tag{5.51}$$

Thanks to (5.39), we have in the right hand side of (5.51), for $n > m+1$,

$$S'_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) = S'_m(u_{i,n}) a\Big(x,t,T_{m+1}(u_{i,n}), \nabla T_{m+1}(u_{i,n})\Big) \text{ a.e. in } Q_T.$$

Using (5.22), and fixing $m \geq 1$, we get

$$S'_m(u_{i,n}) a_n(u_{i,n},\nabla u_{i,n}) \rightharpoonup S'_m(u_i) X_{i,m+1} \text{ weakly in } (L_\psi(Q_T))^N.$$

when $n \to +\infty$.
We can pass to limit as $j \to +\infty$ and $\mu \to +\infty$, and using (5.36)-(5.37)

$$\begin{aligned} &\limsup_{\mu\to+\infty} \limsup_{j\to+\infty} \limsup_{n\to+\infty} \int_{Q_T} S'_m(u_{i,n})) a_n(x,t,u_{i,n}), \nabla u_{i,n})) \nabla T_K(\upsilon_{i,j})_\mu\,dx\,dt \\ &= \int_{Q_T} S'_m(u_i) X_{i,m+1} \nabla T_K(u_i)\,dx\,dt \\ &= \int_{Q_T} X_{i,m+1} \nabla T_K(u_i)\,dx\,dt \end{aligned} \tag{5.52}$$

where $K \leq m$, since $S'_m(r) = 1$ for $|r| \leq m$.
On the other hand, for $K \leq m$, we have

$$a\big(x,t,T_{m+1}(u_{i,n}), \nabla T_{m+1}(u_{i,n})\big) \chi_{\{|u_{i,n}|<K\}} = a\big(x,t,T_K(u_{i,n}), \nabla T_K(u_{i,n})\big) \chi_{\{|u_{i,n}|<K\}},$$

a.e. in Q_T. Passing to the limit as $n \to +\infty$, we obtain

$$X_{i,m+1} \chi_{\{|u_i|<K\}} = X_{i,K} \chi_{\{|u_i|<K\}} \quad \text{a.e. in } Q_T - \{|u_i| = K\} \text{ for } K \leq n. \tag{5.53}$$

Then

$$X_{m+1} \nabla T_K(u_i) = X_K \nabla T_K(u_i) \quad \text{a.e. in } Q_T. \tag{5.54}$$

Then we obtain (5.46).
Proof of (5.47):
Let $K \geq 0$ be fixed. Using (4.3) we have

$$\int_{Q_T} \big[a(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u_i))\big]\big[\nabla T_K(u_{i,n}) - \nabla T_K(u_i)\big]\,dx\,dt \geq 0, \tag{5.55}$$

In view (1) and (5.27), we get

$$a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \quad \text{a.e. in } Q_T,$$

as $n \to +\infty$, and by Lebesgue's theorem, we obtain

$$a\big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\big) \to a\big(x,t,T_K(u_i),\nabla T_K(u_i)\big) \quad \text{strongly in } (L_\psi(Q_T))^N. \tag{5.56}$$

Using (5.46), (5.27), (5.22) and (5.56), we can pass to the lim-sup as $n \to +\infty$ in (5.55) to obtain (5.47).
To finish this step, we prove this lemma:

Lemma 5.5. *For $i = 1, 2$ and fixed $K \geq 0$, we have*

$$X_{i,K} = a\big(x,t,T_K(u_i),\nabla T_K(u_i)\big) \quad \textit{a.e. in } Q. \tag{5.57}$$

Also, as $n \to +\infty$,

$$a\big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\big)\nabla T_K(u_{i,n}) \rightharpoonup a\big(x,t,T_K(u_i),DT_K(u_i)\big)\nabla T_K(u_i), \tag{5.58}$$

weakly in $L^1(Q_T)$.

Proof. **Proof of** (5.57):
It's easy to see that

$$a_n(x,t,T_K(u_{i,n}),\xi) = a(x,t,T_K(u_{i,n}),\xi) = a_K(x,t,T_K(u_{i,n}),\xi) \quad \text{a.e. in } Q_T$$

for any $K \geq 0$, any $n > K$ and any $\xi \in \mathbb{R}^N$.
In view of (5.22), (5.47) and (5.56) we obtain

$$\begin{aligned} &\lim_{n\to+\infty} \int_{Q_T} a_K\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})\,dx\,dt \\ &= \int_{Q_T} X_{i,K}\nabla T_K(u_i)\,dx\,dt. \end{aligned} \tag{5.59}$$

Since (1), (4.2) and (5.27), imply that the function $a_K(x,s,\xi)$ is continuous and bounded with respect to s. Then we conclude that (5.57).
Proof of (5.58):
Using (4.3) and (5.47), for any $K \geq 0$ and any $T' < T$, we have

$$\big[a(x,t,T_K(u_{i,n},\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u))\big]\big[\nabla T_K(u_{i,n}) - \nabla T_K(u_i)\big] \to 0 \tag{5.60}$$

strongly in $L^1(Q_{T'})$ as $n \to +\infty$.
On the other hand with (5.27), (5.22), (5.56) and (5.57),we get

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_i) \rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_{i,n}) \rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_i)\to a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i),$$

strongly in $L^1(Q)$, as $n\to+\infty$.
It's results from (5.60), for any $K\geq 0$ and any $T'<T$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i) \tag{5.61}$$

weakly in $L^1(Q_{T'})$ as $n\to+\infty$.then for $T'=T$, we have (5.58). $\square$

Finally we should prove that u_i satisfies (4.9).
Step 4: Passing to the limit.
We first show that u satisfies (4.9)

$$\begin{aligned}
&\int_{m\leq|u_{i,n}|\leq m+1\}} a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\\
&=\int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\Big[\nabla T_{m+1}(u_{i,n})-\nabla T_m(u_{i,n})\Big]\,dx\,dt\\
&=\int_{Q_T} a_n\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big)\nabla T_{m+1}(u_{i,n})\,dx\,dt\\
&\quad-\int_{Q_T} a_n\Big(x,t,T_m(u_{i,n}),\nabla T_m(u_{i,n})\Big)\nabla T_m(u_{i,n})\,dx\,dt
\end{aligned}$$

for $n>m+1$. According to (5.58), one can pass to the limit as $n\to+\infty$; for fixed $m\geq 0$ to obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{m\leq|u_{i,n}|\leq m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\\
&=\int_Q a\Big(x,t,T_{m+1}(u_i),\nabla T_{m+1}(u_i)\Big)\nabla T_{m+1}(u_i)\,dx\,dt\\
&\quad-\int_Q a\Big(x,t,T_m(u_i),\nabla T_m(u_i)\Big)\nabla T_m(u_i)\,dx\,dt\\
&=\int_{m\leq|u_i|\leq m+1\}} a(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt.
\end{aligned} \tag{5.62}$$

Pass to limit as m tends to $+\infty$ in (5.62) and using (5.23) show that u_i satisfies (4.9).
Now we shown that u_i to satisfy (4.10)and (4.11).
Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\operatorname{supp} S'\subset[-K,K]$. the Pointwise multiplication of the approximate equation (1) by $S'(u_{i,n})$ leads to

$$\begin{aligned}
&\frac{\partial S(u_{i,n})}{\partial t}-\operatorname{div}\Big(S'(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\Big)+S''(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\\
&g_{i,n}(x,t,u_{i,n},\nabla u_{i,n})S'(u_{i,n})-\operatorname{div}\Big(S'(u_{i,n})\Big)=f_{i,n}(x,u_{1,n},u_{1,n})S'(u_{i,n})
\end{aligned} \tag{5.63}$$

in $D'(Q_T)$, for $i=1,2$.
Now we pass to the limit in each term of (5.63).
Limit of $\frac{\partial S(u_{i,n})}{\partial t}$: Since $S(u_{i,n})$ converges to $S(u_i)$ a.e. in Q_T and in $L^\infty(Q_T)$ weak $\star$ and S is bounded and continuous. Then $\frac{\partial S(u_{i,n})}{\partial t}$ converges to $\frac{\partial S(u_i)}{\partial t}$ in $D'(Q_T)$ as n tends to $+\infty$.
Limit of $div\Big(S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\Big)$. Since $\operatorname{supp} S'\subset[-K,K]$, for $n>K$, we have

$$S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})=S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\quad\text{a.e. in } Q_T.$$

Using the pointwise convergence of $u_{i,n}$, (5.39),(5.22) and (5.57),imply that

$$S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big) \rightharpoonup S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)$$

weakly in $(L_\psi(Q_T))^N$, for $\sigma(\Pi L_\psi,\Pi E_\varphi)$ as $n\to+\infty$, since $S'(u_i)=0$ for $|u_i|\geq K$ a.e. in Q_T. And

$$S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big) = S'(u_i)a(x,t,u_i,\nabla u_i) \quad \text{ a.e. in } Q_T.$$

Limit of $S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$. Since $\operatorname{supp} S''\subset[-K,K]$, for $n>K$, we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} = S''(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n}) \quad \text{a.e. in } Q_T.$$

The pointwise convergence of $S''(u_{i,n})$ to $S''(u_i)$ as $n\to+\infty$, (5.39) and (5.58) we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} \rightharpoonup S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i),$$

weakly in $L^1(Q_T)$, as $n\to+\infty$. And

$$S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i) = S''(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i \quad \text{a.e. in } Q_T.$$

Limit of $S'(u_{i,n})g_{i,n}(x,t,u_{i,n})$: We have since $\operatorname{supp} S'\subset[-K,K]$. Using (5.50), it's easy to see that

$$S'(u_{i,n})g_{i,n}(x,t,u_{i,n},\nabla u_{i,n}) \to S'(u_i)g_i(x,t,u_i,\nabla u_i)$$

Limit of $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n})$: Using that f_i belongs to $L^1(Q_T)$, and (5.3) and (5.4), we have $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n})\to f_i(x,u_1,u_2)S'(u_i)$ strongly in $L^1(Q_T)$, as $n\to+\infty$.

It remains to show that for i=1,2 $S(u_i)$ satisfies the initial condition ((4.11)).
To this end, firstly remark that, in view of the definition of S'_φ, we have $B_\varphi(x,u_{i,n})$ is bounded in $L^\infty(Q_T)$.

Secondly, by ((5.42)) we show that $\dfrac{\partial S(u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T)+W^{-1,x}L_\psi(Q_T))$. As a consequence, an Aubin's type Lemma (see e.g., [20], Corollary 4) implies that $S(u_{i,n})$ lies in a compact set of $C^0([0,T];L^1(\Omega))$.

It follows that, on one hand,$Su_{i,n}(t=0)$ converges to $S(u_i)(t=0)$ strongly in $L^1(\Omega)$.

On the order hand, the smoothness of S imply that $S(ui,n)(t=0)$ converges to $S(u_i)(t=0)$ strongly in $L^1(\Omega)$, we conclude that $Su_{i,n})(t=0)=S(u_{i,0n})$ converges to $S(u_i)(t=0)$ strongly in $L^1(\Omega)$, we obtain $S(u_i)(t=0)=S(u_{i,0})$ a.e. in Ω and for all $M>0$, now letting M to $+\infty$, we conclude that $u_i)(t=0)=u_{i,0})$ a.e. in Ω.

As a conclusion, the proof of Theorem (5.1) is complete.

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