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On a Positive Solutions for (p, q)-Laplacian Steklov Problem with Two Parameters

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abstract: We study the existence and non-existence of positive solutions for (p, q)-Laplacian Steklov problem with two parameters. The main result of our research is the construction of a continuous curve in plane, which becomes a threshold between the existence and non-existence of positive solutions.

Key Words: (p, q)-Laplacian, Nonlinear boundary conditions, Indefinite weight, Mountain pass theorem, Global minimizer,super- and sub-solution, Modified Picones identity.

Contents

1. Introduction

In this paper, we prove various existence and non-existence of positive solutions for the following (p, q)-Laplacian Steklov eigenvalue problem:

$$(P_{\alpha,\beta}) \begin{cases} \operatorname{div}[A_{p,q}(|\nabla u|)\nabla u] &= A_{p,q}(u)u & \text{in } \Omega, \\ \langle A_{p,q}(|\nabla u|)\nabla u, \nu\rangle &= \alpha|u|^{p-2}u + \beta|u|^{q-2}u & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\langle ., . \rangle$ is the scalar product of $\mathbb{R}^N$. $\alpha, \beta \in \mathbb{R}$, $A_{p,q}(s) = |s|^{p-2} + |s|^{q-2}$ and $1 < q < p < \infty$. Not that the assumption $q < p$ is taken without loss of generality, due to the symmetry of symbols in $(P_{\alpha,\beta})$; therefore all results of the present work have corresponding counterparts in the case $p > q$. It is easy to see that $\operatorname{div}[A_{p,q}(\nabla u)] = \Delta_p + \Delta_q$, called (p, q)-Laplacian, occurs in quantum field theory, where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

The problem $(P_{\alpha,\beta})$ comes, for example, from a general reaction diffusion system

$$u_t = \operatorname{div}(D(u)\nabla u) + c(x, u), \tag{1.1}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences like chemical reaction design [2], biophysics [8] and plasma physics [17]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.1) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x; u)$ has a polynomial form with respect to the concentration. In the last few years, the (p, q)-Laplace attracts a lot

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of attention and has been studied by many authors (see [12,16,20,23]). However, there are few results one the eigenvalue problems for the (p,q)-Laplacian, we cite [3,7,14,18].

Under the zero Dirichlet boundary condition in Ω, the authors obtained in [15] reasonably complete description of the subsets of (α,β) plane which correspond to the existence/nonexistence of positive solutions to the following problem:

$$(D_{\alpha,\beta})\begin{cases} -\Delta_p u-\Delta_q u &= \alpha m_p(x)|u|^{p-2}u+\beta m_q(x)|u|^{q-2}u \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $m_r\in L^\infty(\Omega)$, $m_r\not\equiv 0$ and $m_r\geq 0$ a.e in Ω for $r=p,q$.
In [21,22] the authors of the present article studied the existence and non-existence results of a positive solution for the Steklov eigenvalue problem $P_{(\lambda,\lambda)}$ ($\alpha=\beta=\lambda$).

Our goal in this paper is to provide a *complete* description of 2-dimensional sets in the (α,β) plane, which correspond to the existence and non-existence of positive solutions for $(P_{\alpha,\beta})$ by generalizing and complementing the research [21,22], and seems more natural, due to the structure of the equation. We restrict ourselves to the case where m_p and m_q are constants, to save transparency and simplicity of presentation. However, we emphasize that all the results of the present article remain valid for the following problem with non-negative weights $(P_{\alpha,\beta,m_p,m_q})$:

$$\begin{cases} \Delta_p u+\Delta_q u &= |u|^{p-2}u+|u|^{q-2}u \quad \text{in } \Omega,\\ \langle|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u,\nu\rangle &= \alpha m_p(x)|u|^{p-2}u+\beta m_q(x)|u|^{q-2}u \quad \text{on } \partial\Omega \end{cases} \tag{1.3}$$

Let $r=p,q$ and let $\frac{N-1}{r-1}<s_r<\infty$ if $r<N$ and $s_r\geq 1$ if $r\geq N$. The function weight $m_r\in\mathbb{M}_r^+$, where $\mathbb{M}_r^+:=\{m_r\in L^{s_r}(\partial\Omega); m_r\not\equiv 0, m_r\geq 0\}$. Hereinafter, $\|u\|_{1,r}:=\|u\|_{W^{1,r}(\Omega)}$ denotes the norm of Sobolev space $W^{1,r}(\Omega)$.
We say that $u\in W^{1,p}(\Omega)$ is a weak solution of $(P_{\alpha,\beta})$ if its holds

$$\begin{aligned}\int_\Omega(|\nabla u|^{p-2}\nabla u\nabla\varphi+|u|^{p-2}u\varphi)dx+&\int_\Omega(|\nabla u|^{q-2}\nabla u\nabla\varphi+|u|^{q-2}u\varphi)dx\\ &=\int_{\partial\Omega}(\alpha|u|^{p-2}u+\beta|u|^{q-2}u)\varphi d\sigma\end{aligned}$$

for all $\varphi\in W^{1,p}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure.

As usual, we say that λ is an eigenvalue of Δ_r with weight function $m_r\in\mathbb{M}_r$ if the problem

$$(P_\lambda)\begin{cases} \Delta_r u &= |u|^{r-2}u \quad \text{in } \Omega,\\ |\nabla u|^{r-2}\frac{\partial u}{\partial\nu} &= \lambda m_r(x)|u|^{r-2}u \quad \text{on } \partial\Omega \end{cases}$$

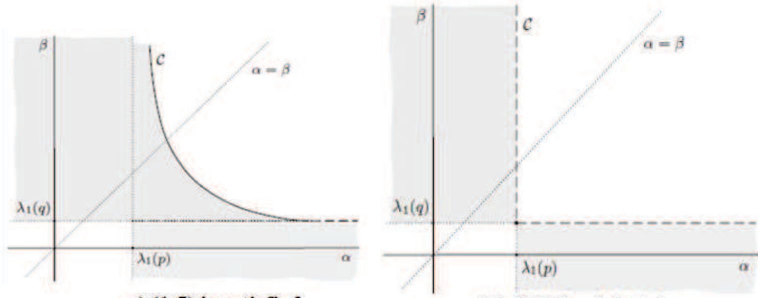
has a non-trivial solution. If the Lebesgue measure of $\{x\in\Omega : m_r(x)>0\}$ is positive, then (P_λ) possesses the first positive eigenvalue $\lambda_1(r,m_r)$ (cf. [11]) that can be obtained by minimizing the Rayleigh quotient:

$$\lambda_1(r,m_r):=\inf\left\{\frac{\Phi(u)}{\Psi(u)}; u\in W^{1,r}(\Omega), \Psi(u)>0\right\}. \tag{1.4}$$

where $\Phi(u):=\int_\Omega|\nabla u|^r dx+\int_\Omega|u|^r dx$ and $\Psi(u):=\int_{\partial\Omega}m_r|u|^r d\sigma$.
Note that $\lambda_1(r,m_r)$ is simple and isolated. It is also worth mentioning that $\lambda_1(r,m_r)$ has positive eigenfunctions $\varphi_1(r,m_r)\in C^{1,\alpha_r}(\overline{\Omega})$ with some $\alpha_r\in(0,1)$ (see [1]). Hereinafter we will also use the notation $\lambda_1(r):=\lambda_1(r,1)$ for the first eigenvalue of Δ_p without weight and φ_r for the corresponding eigenfunction. In what follows, we will say that $\lambda_1(p)$ and $\lambda_1(q)$ have different eigenspaces if the corresponding eigenfunctions φ_p and φ_q are linearly independent, i.e. the following assumption is satisfied:

$$\forall k\neq 0 \text{ its holds } \varphi_p\not\equiv k\varphi_q \text{ in } \overline{\Omega}. \tag{1.5}$$

Let us note that availability or violation of the assumption (1.5) significantly affects the sets of existence of solutions for $P_{(\alpha,\beta)}$, see Fig. 1 and the section 3 for precise statements.



a) (1.5) is satisfied **b) (1.5) is violated.**

Figure 1: Shaded sets correspond to existence, unshaded to non-existence

The rest of this paper is organized as follows. In section 2, we give some preliminary results and definitions which are needed in the proof of the main results. Section 3, we state our main results. Section 4, we prove the existence of solution for $P_{(\alpha,\beta)}$ in the neighborhood of $(\lambda_1(p),\ \lambda_1(q))$ provided (1.5) is satisfied. Section 5, we prove our results stated in section 4.

2. Preliminaries

In this section, we give some preliminary results and definitions which well be used in the following sections. First, we give three results from [21,22], where they were proved using the variational methods.

Theorem 2.1. *([21], Theorem 2.5) One assumes that $m_p \in \mathbb{M}_p$ and $m_q \in \mathbb{M}_q$. If $0 < \lambda < \min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\}$, then the problem $(P_{\lambda,\lambda,m_p,m_q})$ has no non-trivial solutions.*

Theorem 2.2. *([21], Theorem 3.1) One supposes that $m_p \in \mathbb{M}_p$, $m_q \in \mathbb{M}_q$ and $\lambda_1(p,m_p) \neq \lambda_1(q,m_q)$. If*

$$\min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\} < \lambda < \max\{\lambda_1(p,m_p), \lambda_1(q,m_q)\},$$

then the problem $(P_{\lambda,\lambda,m_p,m_q})$ has at least one positive solution.

Theorem 2.3. *([22], Theorem 1.4) One supposes that $m_p \in \mathbb{M}_p$ and $m_q \in \mathbb{M}_q$. If*

$$\lambda = \lambda_1(p,m_p) > \lambda_1(q,m_q) \tag{2.1}$$

and

$$\int_\Omega (|\nabla\varphi_1(p,m_p)|^q + |\varphi_1(p,m_p)|^q)dx - \lambda\int_{\partial\Omega} m_q\varphi_1(p,m_p)^q d\sigma > 0, \tag{2.2}$$

then the problem $(P_{\lambda,\lambda,m_p,m_q})$ has at least one positive solution.

Next, we introduce the super- and sub-solution method for the problem $(P_{\alpha,\beta})$.
We recall the notations of sub- and super-solutions of our problem.

Definition 2.4. *A function $\overline{u} \in W^{1,p}(\Omega)$ is called a super-solution of $(P_{\alpha,\beta})$ if the following holds*

$$\int_\Omega \left(|\nabla\overline{u}|^{p-2} + |\nabla\overline{u}|^{q-2}\right)\nabla\overline{u}\nabla\varphi dx + \int_\Omega \left(|\overline{u}|^{p-2} + |\overline{u}|^{q-2}\right)\overline{u}\varphi dx - \int_{\partial\Omega}\left(\alpha|\overline{u}|^{p-2} + \beta|\overline{u}|^{q-2}\right)\varphi d\sigma \geq 0$$

for any $\varphi \in W^{1,p}(\Omega)_+$.

Definition 2.5. *A function $\underline{u} \in W^{1,p}(\Omega)$ is called a sub-solution of $(P_{\alpha,\beta})$ if the following holds*

$$\int_\Omega \left(|\nabla\underline{u}|^{p-2} + |\nabla\underline{u}|^{q-2}\right)\nabla\underline{u}\nabla\varphi dx + \int_\Omega \left(|\underline{u}|^{p-2} + |\underline{u}|^{q-2}\right)\underline{u}\varphi dx - \int_{\partial\Omega}\left(\alpha|\underline{u}|^{p-2} + \beta|\underline{u}|^{q-2}\right)\varphi d\sigma \leq 0$$

for any $\varphi \in W^{1,p}(\Omega)_+$.

Here, $W^{1,p}(\Omega)_+ := \{\varphi \in W^{1,p}(\Omega) : \varphi \geq 0\}$ stands for all nonnegative functions of $W^{1,p}(\Omega)$. To simplify the notation, we set

$$f^{\alpha,\beta} := \alpha|u|^{p-2}u + \beta|u|^{q-2}u$$

We introduce truncation function $f^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ of $f^{\alpha,\beta}$ defined by two $L^\infty(\partial\Omega)$ functions $\underline{u}$ and $\overline{u}$ with $\overline{u} \geq \underline{u}$ (a.e.in $\partial\Omega$)

$$f^{\alpha,\beta}_{[\underline{u},\overline{u}]} =: \begin{cases} f^{\alpha,\beta}(\overline{u}(x)) \; if \; s \geq \overline{u} \\ f^{\alpha,\beta}(s) \; if \; \underline{u} < s < \overline{u} \\ f^{\alpha,\beta}(\underline{u}) \; if \; s \leq \underline{u} \end{cases}$$

Set $F^{\alpha,\beta}_{[\underline{u},\overline{u}]} := \int_0^{u(x)} f^{\alpha,\beta}_{[\underline{u},\overline{u}]}(x,s)ds$. Then, we define a C^1-functional

$$E^{\alpha,\beta}_{[\underline{u},\overline{u}]} := \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right)dx + \frac{1}{q}\int_\Omega \left(|\nabla u|^q + |u|^q\right)dx - \int_{\partial\Omega} F^{\alpha,\beta}_{[\underline{u},\overline{u}]}(x,u)d\sigma \tag{2.3}$$

It is easily seen that $f^{\alpha,\beta}_{[\underline{u},\overline{u}]}(x,s) = f^{\alpha,\beta}(u(x))$ provided $\underline{u} < s < \overline{u}$. Let us show the following elementary result which implies the existence of global minimizer (cf. [13], Theorem 1.1)

Lemma 2.6. *Let $\underline{u}, \overline{u} \in L^\infty(\partial\Omega)$ satisfy $\overline{u} \geq \underline{u}$ (a.e.on $\partial\Omega$), then $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ defined by (2.3) is weakly lower semi-continuous, bounded from below and coercive.*

Proof. Because $f^{\alpha,\beta}$ is bounded on $\partial\Omega \times [-|\underline{u}|_{L^\infty(\partial\Omega)}, |\overline{u}|_{L^\infty(\partial\Omega)}]$, there exists positive d such that $|f^{\alpha,\beta}_{[\underline{u},\overline{u}]}(x,s)| \leq d$ for every $s \in \mathbb{R}$, a.e, $x \in \partial\Omega$. Thus we obtain

$$\begin{aligned} \frac{E^{\alpha,\beta}_{[\underline{u},\overline{u}]}(u)}{\|u\|_{1,p}} &= \frac{\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|u\|^q_{1,q} - \int_{\partial\Omega} F^{\alpha,\beta}_{[\underline{u},\overline{u}]}(x,u)d\sigma}{\|u\|_{1,p}} \\ &\geq \frac{\frac{1}{p}\|u\|^p_{1,p} - d\|u\|_{L^1(\partial\Omega)}}{\|u\|_{1,p}} \\ &\geq \frac{\frac{1}{p}\|u\|^p_{1,p} - d'\|u\|_{1,p}}{\|u\|_{1,p}} \to \infty \text{ as } \|u\|_{1,p} \to \infty \end{aligned}$$

Where the inclusion $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$ is compact and $d' > 0$. This implies that $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ is bounded from below and coercive. Moreover, its easy to see that $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ is weakly lower semi-continuous. It well known that the properties of $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ started in lemma 2.6, imply the existence of a global minimizer u of $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ (cf. [13], Theorem 1.1), which becomes a solution of $(P_{\alpha,\beta})$. Moreover, $u \in [\underline{u}, \overline{u}]$. Indeed, since $\overline{u}$ is a super-solution, taking $(u - \overline{u})_+$ as a test function, we have

$$\begin{aligned} 0 &\geq \langle (E^{\alpha,\beta}_{[\underline{u},\overline{u}]})'(u), (u-\overline{u})_+\rangle - \langle \Delta_p\overline{u} + \Delta_q\overline{u}, (u-\overline{u})_+\rangle \\ &+ \int_{\partial\Omega} (\alpha|\overline{u}|^{p-2}\overline{u} + \beta|\overline{u}|^{q-2}\overline{u})(u-\overline{u})_+ d\sigma \\ &= \int_{u>\overline{u}} (|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u})(\nabla u - \nabla\overline{u})dx \\ &+ \int_{u>\overline{u}} (|\nabla u|^{q-2}\nabla u - |\nabla\overline{u}|^{q-2}\nabla\overline{u})(\nabla u - \nabla\overline{u})dx \geq 0 \end{aligned}$$

where we take into account that $f^{\alpha,\beta}_{[\underline{u},\overline{u}]})(x,s) = \alpha|\overline{u}|^{p-2}\overline{u} + \beta|\overline{u}|^{q-2}\overline{u}$ provided $s \geq \overline{u}(x)$. This implies that $u \leq \overline{u}$. Similarly, by taking $(u - \underline{u})_-$ as test function, we see that $u \geq \underline{u}$ holds. Therefore, $f^{\alpha,\beta}_{[\underline{u},\overline{u}]}(x,u(x)) = \alpha|u|^{p-2}u + \beta|u|^{q-2}u$, whence u is a solution of $(P_{\alpha,\beta})$. In particular, if a sub-solution

$\underline{u} \geq 0$ and u is not-trivial, then it is known that $u \in \mathrm{int}C^1(\overline{\Omega})_+$(see Remark 2.7). Where $\mathrm{int}C^1(\overline{\Omega})_+$ denotes the interior of the positive cone

$$C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \geq 0 \text{ for every } x \in \Omega\}$$

in the Banach space $C^1(\overline{\Omega})$, given by

$$\mathrm{int}C^1(\overline{\Omega})_+ = \Big\{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\Big\}. \tag{2.4}$$

□

Remark 2.7. *It can be shown that non-trivial critical points of $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ correspond to non-negative solutions of $(P_{\alpha,\beta})$ by taking u_- as test function. Moreover, any non-negative solution of $(P_{\alpha,\beta})$ belongs to $intC^1(\overline{\Omega})_+$. Indeed, we can check that each non-trivial critical point u of $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ belongs to $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ (see [1]), then the maximum principle of Vasquez [19] can be applied to ensure that $u > 0$ in $\overline{\Omega}$. As result, $u \in intC^1(\overline{\Omega})_+$. Here we denote $u_\pm := \max\{\pm u, 0\}$ in $\partial\Omega$.*

Lemma 2.8. *Assume that $\beta > \lambda_1(q)$ and $\overline{u} \in intC^1(\overline{\Omega})_+$ is a positive super-solution of $(P_{\alpha,\beta})$. Then $\min_{W^{1,p}(\Omega)} E^{\alpha,\beta}_{[0,\overline{u}]} < 0$ holds, and hence $(P_{\alpha,\beta})$ has a positive solution belonging to $intC^1(\overline{\Omega})_+$.*

Proof. Let $\beta > \lambda_1(q)$ and $\overline{u} \in intC^1(\overline{\Omega})_+$ be a positive super-solution of $(P_{\alpha,\beta})$. Recall that $E^{\alpha,\beta}_{[0,\overline{u}]}$ has a global minimum point. Since $\overline{u} \in \mathrm{int}C^1(\overline{\Omega})_+$, for sufficiently small $t > 0$ we have $\overline{u} - t\varphi_q \geq 0$ in $\partial\Omega$. This implies that $f^{\alpha,\beta}_{[0,\overline{u}]}(x, t\varphi_q) = \alpha t^{q-1}\varphi_q^{p-1} + \beta t^{p-1}\varphi_q^{q-1}$. Hence, for sufficiently small $t > 0$, we obtain

$$E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q) = \frac{t^p}{p}(\|\varphi_q\|^p_{1,p} - \alpha\|\varphi_q\|^p_{L^p(\partial\Omega)}) - \frac{t^q}{q}(\beta - \lambda_1(q))\|\varphi_q\|^q_{L^q(\partial\Omega)}.$$

Recalling that $q < p$ and $\beta - \lambda_1(q) > 0$, we see that $E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q) < 0$ for sufficiently small $t > 0$, whence $\min_{W^{1,p}(\Omega)} E^{\alpha,\beta}_{[0,\overline{u}]} < 0$. Therefore, $E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q$ has a non-trivial critical point, and our conclusion follows. □

Finally, we give the Picone's identity for (p,q)-Laplacian. We state a proposition that well be used.

Proposition 2.9. *(Proposition A.2, [6]). Let $1 < q < p < \infty$. Then there exists $\rho > 0$ such that for any differentiable functions $u > 0$ and $\varphi \geq 0$ in Ω it holds*

$$(|\nabla u|^{p-2} + |\nabla u|^{q-2})|\nabla u|\nabla u\left(\frac{\varphi^p}{u^{p-1} + u^{q-1}}\right) \leq \frac{|\nabla\varphi|^p + |\nabla(\varphi^{p/q})|^q}{\rho}.$$

3. Main results

In this section we state our main results.

3.1. Case $(\alpha,\beta) \in \mathbb{R}^2\backslash[\lambda_1(p), +\infty) \times [\lambda_1(q), +\infty)$

First, we state the result of non-existence which generalize Theorem 2.1 from [21] for the problem $(P_{\alpha,\beta}, m_p, m_q)$ with non-negative weights.

Proposition 3.1. *If*

$$(\alpha,\beta) \in (-\infty, \lambda_1(p)] \times (-\infty, \lambda_1(q)]\backslash\{(\lambda_1(p), \lambda_1(q))\},$$

then $(P_{\alpha,\beta})$ has non-trivial solutions. Moreover, $(P_{\alpha,\beta})$ with $\alpha = \lambda_1(p)$ and $\beta = \lambda_1(q)$ has a non-trivial (positive) solution if and only if they have the same eigenspace, namely, there exists $k \neq 0$ such that $\varphi_p \equiv k\varphi_q$ in $\overline{\Omega}$ (that is, (1.5) is not satisfied).

Next, we state the existence results of our problem $(P_{\alpha,\beta})$. These results generalize Theorem 2.2 from [21].

Proposition 3.2. *If*

$$(\alpha,\beta) \in (\lambda_1(p),+\infty) \times (-\infty,\lambda_1(q)) \cup (-\infty,\lambda_1(p)) \times (\lambda_1(q),+\infty),$$

then$(P_{\alpha,\beta,m_p,m_q})$ *has at least one positive solution.*

Theorem 3.3. *Assume that* (1.5) *does not hold. Then* $(P_{\alpha,\beta})$ *has at least one positive solution if and only if*

$$(\alpha,\beta) \in (\lambda_1(p),+\infty) \times (-\infty,\lambda_1(q)) \cup (-\infty,\lambda_1(p)) \times (\lambda_1(q),+\infty) \times \{(\lambda_1(p),\lambda_1(q))\}. \tag{3.1}$$

The main novelty of the work is the treatment of the rest part of (α,β) plane, i.e. $(\alpha,\beta) \in [\lambda_1(p),+\infty) \times [\lambda_1(q),+\infty)$, where we construct a threshold curve, which separates the regions of existence and non-existence of positive solutions for $(P_{\alpha,\beta})$.

3.2. Case $(\alpha,\beta) \in [\lambda_1(p),+\infty) \times [\lambda_1(q),+\infty)$

3.2.1. Instruction of the curve. Note first that for any $\alpha,\beta \in \mathbb{R}$ the problem $(P_{\alpha,\beta})$ is equivalent to $(P_{\beta+s,\beta})$, where $s=\alpha-\beta$. Denoting now, for convenience, $\lambda=\beta$, for each $s\in\mathbb{R}$ we consider

$$\lambda^*(s) := \sup\{\lambda \in \mathbb{R} : (P_{\lambda+s,\lambda}) \text{ has a positive solution}\}, \tag{3.2}$$

provided $(P_{\lambda+s,\lambda})$ has a positive solution for some λ. If there are no such λ, we set $\lambda^*(s)=-\infty$. Define also

$$s^* := \lambda_1(p)-\lambda_1(q) \text{ and } s^*_+ := \frac{\|\varphi_q\|^p_{1,p}}{\|\varphi_q\|^p_{L^p(\partial\Omega)}} - \lambda_1(q).$$

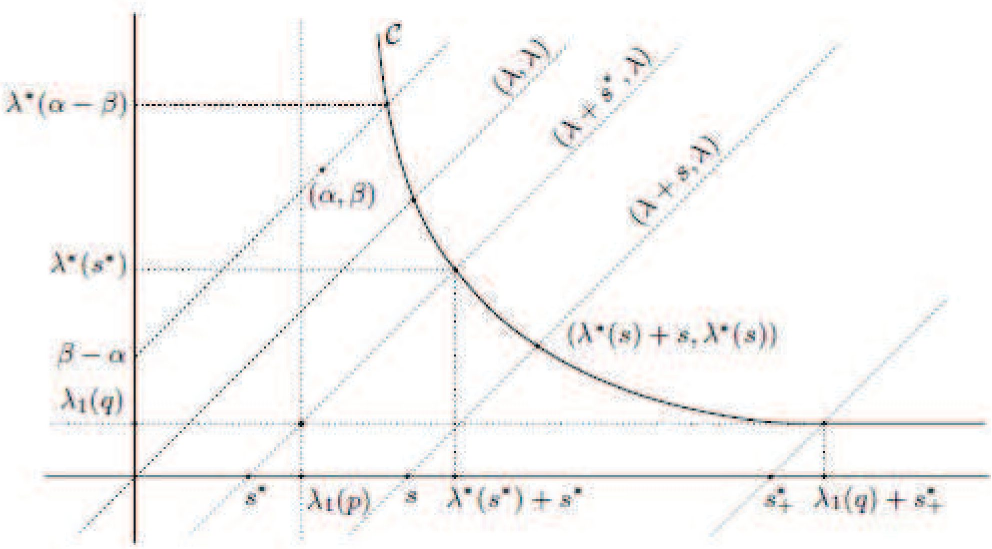


Figure 2: (α,β) plane. Construction of the curve $\mathcal{C}$

Obviously, $s^* \le s^*_+$, and $s^*=s^*_+$ if and only if (1.5) is not satisfied.
In the next proposition we collect the main facts about $\lambda^*(s)$:

Proposition 3.4. *Let* $\lambda^*(s)$ *be defined by* (3.2) *for* $s\in\mathbb{R}$. *Then the following assertions hold: (i)* $\lambda^*(s) < +\infty$ *for all* $s\in\mathbb{R}$;
(ii) $\lambda^*(s)+s \ge \lambda_1(p)$ *and* $\lambda^*(s) \ge \lambda_1(q)$ *for all* $s\in\mathbb{R}$;
(iii) $\lambda^*(s)=\lambda_1(q)$ *for all* $s \ge s^*_+$;
(iv) $\lambda^*(s)+s^*_+ > \lambda_1(p)$ *and* $\lambda^*(s) > \lambda_1(q)$ *if and only if* (1.5) *is satisfied;*
(v) $\lambda^*(s)$ *is continuous on* $\mathbb{R}$;
(vi) $\lambda^*(s)$ *is non-increasing and* $\lambda^*(s)+s$ *is non-decreasing on* $\mathbb{R}$.

Notice that it is still unknown if there is $s^*_- \in \mathbb{R}$, such that $\lambda^*(s)+s=\lambda_1(p)$ for all $s \le s^*_-$, or $\lambda^*(s)+s>\lambda_1(p)$ for all $s\in\mathbb{R}$, whenever (1.5) is satisfied. Now we define the curve $\mathcal{C}$ in(α,β) plane as follows:

$$\mathcal{C} := \{(\lambda^*(s)+s, \lambda^*(s)), s\in\mathbb{R}\}.$$

From proposition 3.4 there directly follow the corresponding conclusion for $\mathcal{C}$, namely, $\mathcal{C}$ is locally finite, $\mathcal{C}\subset[\lambda_1(p),+\infty)\times[\lambda_1(q),+\infty)$, $\mathcal{C}$ is continuous, monotone, and coincides with $[\lambda_1(q)+s^*_+,+\infty)\times\{\lambda_1(q)\}$ for $s\ge s^*_+$(see Figure2).
We especially note that $\lambda_1(s)+s=\lambda_1(p)$ for $s\le s^*$ and $\lambda_1(s)=\lambda_1(q)$ for $s\ge s^*$ if and only if (1.5) doesn't hold. It directly follows from the combination of the criterion (iv), estimations (ii) and monotonicity (vi) from Proposition 3.4. In other words, our curve $\mathcal{C}$ coincides with the polygonal line $\{\lambda_1(p)\}\times[\lambda_1(q),\infty)\cup[\lambda_1(q),\infty)\times\{\lambda_1(p)\}$ if and only if (1.5) is violated.
Let us note that the main disadvantage of characterization 3.2 of $\lambda^*(s)$ is its non-constructive form. However, using the extended functional method (see [5,10]) we provide the equivalent characterization of $\lambda^*(s)$ by an explicit minimax formula, which can be used in further numerical investigations of $(P_{\alpha,\beta})$:

$$\Lambda^*(s) = \sup_{u\in \mathrm{int}C^1(\overline{\Omega})_+}\ \inf_{\varphi\in C^1(\overline{\Omega})_+\setminus\{0\}} \mathcal{L}_s(u;\varphi), \tag{3.3}$$

where

$$\mathcal{L}_s(u;\varphi) = \frac{\Psi_p(u,\varphi)+\Psi_q(u,\varphi) - s\int_{\partial\Omega}|u|^{p-2}u\varphi d\sigma}{\int_{\partial\Omega}|u|^{p-2}u\varphi d\sigma + \int_{\partial\Omega}|u|^{q-2}u\varphi d\sigma}.$$

Where $\Psi_r(u,\varphi) := \int_\Omega |\nabla u|^{r-2}\nabla u\nabla\varphi dx + \int_\Omega |u|^{r-2}u\varphi dx$.
The next proposition shows that (3.2) and (3.3) are in fact, equivalent.

Proposition 3.5. *$\Lambda^*(s)=\lambda^*(s)$ for all $s\in\mathbb{R}$.*

3.2.2. Existence and non-existence results. First, we state our second main theorem, which shows that $(P_{\alpha,\beta})$ possesses a positive solution if (α,β) is below the curve $\mathcal{C}$, and has no positive solutions if (α,β) is above $\mathcal{C}$.

Theorem 3.6. *Assume that (1.5) is satisfied. If one of the following cases holds, then $(P_{\alpha,\beta})$ has at least one positive solution:*
(i) $\lambda_1(q)<\beta$ and $\beta<\lambda^*(\alpha-\beta)$;
(ii) $\lambda_1(p)<\alpha$ and $\beta<\lambda^*(\alpha-\beta)$.
Conversely, if $\beta>\lambda^*(\alpha-\beta)$, then $(P_{\alpha,\beta})$ has no positive solutions.

Next, we provide the results about existence and non-existence on the curve $\mathcal{C}$.

Proposition 3.7. *The following assertion holds:*
if $\lambda^*(s)+s>\lambda_1(p)$ and $\lambda^*(s)>\lambda_1(q)$, then $(P_{\lambda^*(s)+s,\lambda^*(s)})$ has at least one positive solution;
if $s>s^*$ then $(P_{\lambda^*(s)+s,\lambda^*(s)})\equiv(P_{\lambda_1(q)+s,\lambda_1(q)})$ has no positive solutions.

4. Existence result in the neighborhood of $(\lambda_1(p),\lambda_1(q))$

In this section we prove the existence of solution for $\alpha=\lambda_1(p)+\varepsilon, \beta=\lambda_1(q)+\varepsilon$ under the assumption (1.5). First, we define the energy functional corresponding to $(P_{\alpha,\beta})$ by

$$E_{\alpha,\beta}(u) = \frac{1}{p}H_\alpha(u)+\frac{1}{q}G_\beta(u),$$

where for further simplicity we denote

$$H_\alpha(u) := \int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx - \alpha\int_{\partial\Omega}|u|^p d\sigma,$$

$$G_\beta(u) := \int_\Omega |\nabla u|^q dx + \int_\Omega |u|^q dx - \beta\int_{\partial\Omega}|u|^q d\sigma,$$

Note that $E_{\alpha,\beta} \in C^1(W^{1,p}(\Omega), \mathbb{R})$.
Next, we introduce the so-called Nehari manifold (see [4])

$$\mathcal{N}_{\alpha,\beta} := \Big\{ u \in W^{1,p}(\Omega) \backslash \{0\} : \langle E'_{\alpha,\beta}(u), u \rangle = H_\alpha(u) + G_\beta(u) = 0 \Big\}.$$

Proposition 4.1. *Let $u \in W^{1,p}(\Omega)$. If $H_\alpha(u).G_\beta(u) < 0$, then there exists a unique extremum point $t(u) > 0$ of $E_{\alpha,\beta}(tu)$ w.r.t. $t > 0$, and $t(u)u \in \mathcal{N}_{\alpha,\beta}$. In particular, if*

$$G_\beta(u) < 0 < H_\alpha(u), \tag{4.1}$$

then $t(u)$ is the unique minimum point of $E_{\alpha,\beta}(tu)$ w.r.t. $t > 0$, and $E_{\alpha,\beta}(t(u)u) < 0$.

Proof. Fix some non-trivial function $u \in W^{1,p}(\Omega)$ and consider the fibred functional corresponding to $E_{\alpha,\beta}(u)$:

$$\begin{aligned} E_{\alpha,\beta}(tu) &= \frac{1}{p} H_\alpha(tu) + \frac{1}{q} G_\beta(tu) \\ &= \frac{t^p}{p} H_\alpha(tu) + \frac{t^q}{q} G_\beta(tu), \ t > 0 \end{aligned}$$

Under the assumption $H_\alpha(u).G_\beta(u) < 0$ the equation

$$\frac{d}{dt} E_{\alpha,\beta}(tu) = t^{p-1} H_\alpha(u) + t^{q-1} G_\beta(u) = 0 \ t > 0.$$

is satisfied for unique $t > 0$ given by

$$t = t(u) = \left(\frac{-G_\beta(u)}{H_\alpha(u)} \right)^{\frac{1}{p-q}} > 0.$$

This implies that

$$\langle E'_{\alpha,\beta}(t(u)u, t(u)u \rangle = t(u) \frac{d}{dt} E_{\alpha,\beta}(tu) \Big|_{t=t(u)} = 0,$$

and hence $t(u)u \in \mathcal{N}_{\alpha,\beta}$.
Moreover, recalling that $q < p$, if (4.1) holds, then

$$E_{\alpha,\beta}(t(u)u) = \frac{1}{p} H_\alpha(t(u)u) + \frac{1}{q} G_\beta(t(u)u) = \frac{p-q}{pq} G_\beta(t(u)u) < 0,$$

and

$$\begin{aligned} &\frac{d^2}{dt^2} E_{\alpha,\beta}(tu) \Big|_{t=t(u)} = (p-1) t(u)^{p-2} H_\alpha(u) + (q-1) t(u)^{q-2} G_\beta(u) \\ &= \frac{1}{t(u)^2} (p-1) H_\alpha(t(u)u) + (q-1) G_\beta(t(u)u) = \frac{q-p}{t(u)^2} G_\beta(t(u)u) > 0, \end{aligned}$$

which implies that $t(u)$ is minimum point of $E_{\alpha,\beta}(tu)$ w.r.t. $t > 0$. □

Lemma 4.2. *Assume that (1.5) is satisfied. Then there exists $\varepsilon_0 > 0$ such that $\mathcal{N}_{\lambda_1(p)+\varepsilon, \lambda_1(q)+\varepsilon} \neq \emptyset$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, there exists $u \in \mathcal{N}_{\lambda_1(p)+\varepsilon, \lambda_1(q)+\varepsilon}$, such that $E_{\lambda_1(p)+\varepsilon, \lambda_1(q)+\varepsilon} < 0$.*

Proof. Since (1.5) is satisfied and due to the simplicity of $\lambda_1(p)$, we have $H_{\lambda_1(p)}(\varphi_q) \neq 0$, which yields $G_{\lambda_1(p)}(\varphi_q) > 0$. At the same time, $G_{\lambda_1(q)}(\varphi_q) = 0$, by the definition of $\lambda_1(q)$. Hence, there exists sufficiently small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ it still holds $H_{\lambda_1(p)+\varepsilon}(\varphi_q) > 0$. Moreover $G_{\lambda_1(q)+\varepsilon}(\varphi_q) < 0$. Applying now proposition 4.1 we get the desired results. □

Lemma 4.3. *Assume that $H_\alpha(u) \neq 0$ or $G_\beta(u) \neq 0$. If $u \in \mathcal{N}_{\alpha,\beta}$ is a critical point of $E_{\alpha,\beta}$ on $\mathcal{N}_{\alpha,\beta}$, then u is a critical point of $E_{\alpha,\beta}$ on $W^{1,p}(\Omega)$.*

Proof. Let $u \in \mathcal{N}_{\alpha,\beta}$ be a critical point of $E_{\alpha,\beta}$ on $\mathcal{N}_{\alpha,\beta}$. Since we are assuming that $H_\alpha(u) \neq 0$ or $G_\beta(u) \neq 0$, u satisfies

$$\langle (H_\alpha(u) + G_\beta(u))', u\rangle = pH_\alpha(u) + qG_\beta(u) = (p-q)H_\alpha(u) = (q-p)G_\beta(u) \neq 0,$$

where we used the fact that $H_\alpha(u) + G_\beta(u) = 0$ for $u \in \mathcal{N}_{\alpha,\beta}$. This implies that $(H_\alpha(u) + G_\beta(u))' \neq 0$ in $W^{1,p}(\Omega)^*$. Due to the Lagrange multiplier rule (see, e.g.[[24],Theorem 48.B and Corollary 48.10]), there exists $\mu \in \mathbb{R}$ such that

$$\langle E'_{\alpha,\beta}(u), \xi\rangle = \mu\langle (H_\alpha(u) + G_\beta(u))', \xi\rangle$$

for each $\xi \in W^{1,p}(\Omega)$. Taking $\xi = u$ we get

$$0 = \langle E'_{\alpha,\beta}(u), u\rangle = \mu(pH_\alpha(u) + qG_\beta(u)) = \mu(p-q)H_\alpha(u) = \mu(q-p)G_\beta(u),$$

since $H_\alpha(u) + G_\beta(u) = 0$. Therefore $\mu = 0$ and

$$\langle E'_{\alpha,\beta}(u), \xi\rangle = 0 \text{ for all } \xi \in W^{1,p}(\Omega),$$

i.e u is a critical point of $E_{\alpha,\beta}$ on $W^{1,p}$. If $u \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ is a minimizer of $E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon} < 0$ on the Nehari manifold $\mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ and satisfies $H_{\lambda_1(p)+\varepsilon}(u) \neq 0$ (or$G_{\lambda_1(q)+\varepsilon}(u) \neq 0$), then u is a critical point of $E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ by Lemma 4.3, i.e. u is a solution of $(P_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon})$.
Since Lemma 4.2 implies the existence of $\varepsilon_0 > 0$ such that $\mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon} \neq \emptyset$ for every $\varepsilon \in (0, \varepsilon_0)$, we can find a corresponding minimization sequence $\{u_k^\varepsilon\}_{k=1}^\infty \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ namely,

$$E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(u_k^\varepsilon) \to \inf\{E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(u) : u \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}\} =: M_\varepsilon$$

as $k \to \infty$. The following result states that this minimization sequence is bounded for any sufficiently small $\varepsilon > 0$, and so $M_\varepsilon > -\infty$ holds, since $E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ is bounded on bounded sets. ☐

Lemma 4.4. *Assume that (1.5) is satisfied. Then there exist $\varepsilon_1 > 0$ and $C > 0$ such that $\|u_k^\varepsilon\|_{1,p} \leq C$ for all $k \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_1)$.*

Proof. Notice that for $\varepsilon \in (0, \varepsilon_0)$ Lemma 4.2 implies $M_\varepsilon < 0$. Hence, considering sufficiently large $k \in \mathbb{N}$, we may assume that $G_{\lambda_1(q)+\varepsilon}(u_k^\varepsilon) < 0$ by noting that for $u_k^\varepsilon \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ it holds

$$\frac{p-q}{pq} G_{\lambda_1(q)+\varepsilon}(u_k^\varepsilon) = E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(u_k^\varepsilon) \tag{4.2}$$

with $q < p$. Consequently, we also get that $H_{\lambda_1(p)+\varepsilon}(u_k^\varepsilon) > 0$ for such $k \in \mathbb{N}$ there exist $\varepsilon(m) \in (0, 1/m)$ and $k(m) \in \mathbb{N}$ such that for $u_m := u_{k(m)}^{\varepsilon(m)} \in \mathcal{N}_{\lambda_1(p)+\varepsilon(m),\lambda_1(q)+\varepsilon(m)}$ it holds $\|u_m\|_{1,p} > m$. Consider the normalized sequence $\{v_m\}_{m=1}^\infty$, such that $u_m = t_m v_m, t_m = \|u_m\|_{1,p} > m$ and $\|v_m\|_{1,p} = 1$. Then the Sobolev embedding theorem imply the existence of a subsequence of $\{v_m\}_{m=1}^\infty$ (which we denote again $\{v_m\}_{m=1}^\infty$) and $v^* \in W^{1,p}$ such that

$$v_m \rightharpoonup v^* \text{ weakly in } \; W^{1,p}(\Omega) \; \text{ and } \; W^{1,q}(\Omega) \text{ as } m \to \infty,$$

$$v_m \to v^* \text{ strongly in } \; L^p(\partial\Omega) \; \text{ and } \; L^q(\partial\Omega) \text{ as } m \to \infty.$$

Moreover, by weakly lower semicontinuity of the norms of $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ we have

$$\|v^*\|_{1,p} \leq \liminf_{m\to\infty} \|v_m\|_{1,p}, \;\; \|v^*\|_{1,q} \leq \liminf_{m\to\infty} \|v_m\|_{1,q}. \tag{4.3}$$

Since $H_{\lambda_1(p)+\varepsilon(m)}(u_m) = -G_{\lambda_1(q)+\varepsilon(m)}(u_m)$ for all $m \in \mathbb{N}$ we have

$$t_m^{p-q}|H_{\lambda_1(p)+\varepsilon(m)} v_m)| = |G_{\lambda_1(q)+\varepsilon(m)}(v_m)| \leq C_1 < +\infty$$

for some constant C_1 uniformly w.r.t. $m \in \mathbb{N}$, because $G_{\lambda_1(q)+\varepsilon}$ is bounded on bounded sets and $\varepsilon(m) \to 0$. Therefore, taking into account that $t_m \to \infty$, we conclude that $H_{\lambda_1(p)+\varepsilon(m)}(v_m) \to 0$ as $m \to \infty$. Using this fact, (4.3), and recalling that $G_{\lambda_1(q)+\varepsilon(m)}(v_m) < 0$ for all $m \in \mathbb{N}$, we deduce

$$H_{\lambda_1(p)}(v^*) \leq \liminf_{m\to\infty} H_{\lambda_1(p)+\varepsilon(m)}(v_m) = 0, \tag{4.4}$$

$$G_{\lambda_1(q)}(v^*) \leq \liminf_{m\to\infty} G_{\lambda_1(q)+\varepsilon(m)}(v_m) \leq 0. \tag{4.5}$$

Noting that $v_m \to v^*$ in $L^p(\partial\Omega)$ and $H_{\lambda_1(p)+\varepsilon(m)}(v_m) = 1 - (\lambda_1(p) + \varepsilon(m))\|v_m\|^p_{L^p(\partial\Omega)}$, we get

$$\begin{aligned} \lambda_1(p) \int_{\partial\Omega} |v^*|^p d\sigma &= \limsup_{m\to\infty} (\lambda_1(p) + \varepsilon(m)) \int_{\partial\Omega} |v^*|^p d\sigma \\ &= 1 - \liminf_{m\to\infty} H_{\lambda_1(p)+\varepsilon(m)}(v_m) = 1, \end{aligned} \tag{4.6}$$

which implies that $v^* \neq 0$. At the same time, in view of (4.4) and (4.5), using definition of $\lambda_1(r)$ for $(r = p, q)$, we get

$$H_{\lambda_1(p)}(v^*) = 0 \text{ and } G_{\lambda_1(q)}(v^*) = 0,$$

and therefore from the simplicity of the first eigenvalue $\lambda_1(p)$ and $\lambda_1(q)$ we must have $|v^*| = \varphi_p/\|\varphi_p\|$ and $|v^*| = \varphi_q/\|\varphi_q\|$ simultaneously. However, it contradicts (1.5). ∎

From Lemma 4.4 it follows that there exist non-trivial weak limits $u_0^\varepsilon \in W^{1,p}(\Omega)$ of the corresponding minimization subsequence $\{u_k^\varepsilon\}_{k=1}^\infty \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ for any $\varepsilon \in (0, \varepsilon_1)$.

Lemma 4.5. *Assume that (1.5) is satisfied. Then there exists $\varepsilon_2 > 0$ such that*

$$G_{\lambda_1(q)+\varepsilon}(u_0^\varepsilon) < 0 < H_{\lambda_1(p)+\varepsilon}(u_0^\varepsilon) \tag{4.7}$$

for all $\varepsilon \in (0, \varepsilon_2)$.

Proof. Let $\varepsilon_1 > 0$ be given by Lemma 4.4 and $\varepsilon \in (0, \varepsilon_1)$. Note that from (4.2) and weakly lower semicontinuity of the norm of $W^{1,p}(\Omega)$ it follows that $G_{\lambda_1(q)+\varepsilon}(u_0^\varepsilon) < 0$. Therefore, we need to show only that $H_{\lambda_1(p)+\varepsilon}(u_0^\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$.

To obtain a contradiction, suppose that for any $m \in \mathbb{N}$ there exists $\varepsilon(m) < 1/m$ such that $H_{\lambda_1(p)+\varepsilon(m)}(u_0^{\varepsilon(m)}) \leq 0$. We consider the normalized sequence $\{v_m\}_{m=1}^\infty$, where $v_m = u_0^{\varepsilon(m)}/t_m, t_m = \|u_0^{\varepsilon(m)}\|_{1,p}$ and $\|v_m\|_{1,p} = 1$. Hence, proceeding as in the proof of Lemma 4.4, we get a contradiction. ∎

Now we are able to prove the main result of this section.

Proposition 4.6. *Assume that (1.5) is satisfied. Then $u_0^\varepsilon \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ and*

$$M_\varepsilon := \inf\{E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(u) : u \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}\}$$

is attained on u_0^ε for all $\varepsilon \in (0, \varepsilon_2)$, where $\varepsilon_2 > 0$ is given by Lemma 4.5.

Proof. Fix any $\varepsilon \in (0, \varepsilon_2)$. Then there exists a weak limit $u_0^\varepsilon \in W^{1,p}$ of the minimizing sequence $\{u_k\}_{k=1}^\infty \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ and (4.7) is satisfied. Let us show that $u_k^\varepsilon \to u_0^\varepsilon$ strongly in $W^{1,p}(\Omega)$ and $u_0^\varepsilon \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$.

Indeed, contrary to our claim, we suppose that

$$\|u_0^\varepsilon\|_{1,p} < \liminf_{k\to\infty} \|u_k^\varepsilon\|_{1,p}$$

Then

$$H_{\lambda_1(p)}(u_0^\varepsilon) + G_{\lambda_1(q)}(u_0^\varepsilon) < \liminf_{k\to\infty} \left(H_{\lambda_1(p)}(u_k^\varepsilon) + G_{\lambda_1(q)}(u_k^\varepsilon)\right) = 0,$$

which implies that $u_0^\varepsilon \notin \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$. However, according to 4.7, the assumptions of Proposition 4.1 are satisfied. Therefore, there exists a unique minimum point $t(u_0^\varepsilon) \neq 1$ of $E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(tu_0^\varepsilon)$ w.r.t. $t > 0$, such that $t(u_0^\varepsilon)u_0^\varepsilon \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$. Hence,

$$\begin{aligned} M_\varepsilon \leq E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(t(u_0^\varepsilon)u_0^\varepsilon) &< E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(u_0^\varepsilon) \\ &< \liminf_{k\to\infty} E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}(u_k^\varepsilon) = M_\varepsilon, \end{aligned}$$

which leads to a contradiction. Therefore, $u_0^\varepsilon \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ and $u_k^\varepsilon \to u_0^\varepsilon$ strongly in $W^{1,p}(\Omega)$. □

Lemma 4.7. *Assume that (1.5) is satisfied. Then $(P_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon})$ possesses a positive solution for all $\varepsilon \in (0,\varepsilon_2)$.*

Proof. According to Lemma 4.5 and Proposition 4.6, $u_0^\varepsilon \in \mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ satisfies (4.7) and it is a minimizer of $E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ on $\mathcal{N}_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ for all $\varepsilon \in (0,\varepsilon_2)$. Since the functional $E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ is even, we may assume that $u_0^\varepsilon \geq 0$. Hence, due to Lemma 4.3 and noting (4.7), u_0^ε is a non-trivial and non-negative critical point of $E_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon}$ on $W^{1,p}(\Omega)$. This ensures that u_0^ε is a positive solution of $(P_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon})$ (see Remark 2.7). □

5. Proofs of main results

In this section, we collect the proofs of our results stated in section 3.

Proof. Proof of Proposition 3.1. Let $\alpha \leq \lambda_1(p)$ and $\beta \leq \lambda_1(q)$. Assume that $u \in W^{1,p}(\Omega)$ is a non-trivial solution of $(P_{\alpha,\beta})$. Taking u as a test function we have

$$\begin{aligned} 0 \leq (\lambda_1(p)-\alpha)\|u\|^p_{L^p(\partial\Omega)} &\leq \|u\|^p_{1,p} - \alpha\|u\|^p_{L^p(\partial\Omega)} \\ &= \beta\|u\|^q_{L^q(\partial\Omega)} - \|u\|^q_{1,q} \\ &\leq (\beta - \lambda_1(q))\|u\|^q_{L^q(\partial\Omega)} \leq 0. \end{aligned}$$

This chain of inequalities is satisfied if and only if $\alpha = \lambda_1(p), \beta = \lambda_1(q)$ and u is the eigenfunction corresponding to $\lambda_1(p)$ and $\lambda_1(q)$ simultaneously. As a result, our conclusion is shown.
□

To prove Proposition 3.2 we introduce functional $I_{\alpha,\beta}$ on $W^{1,p}(\Omega)$ by

$$I_{\alpha,\beta}(u) := \frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|u\|^q_{1,q} - \frac{\alpha}{p}\|u_+\|^p_{L^p(\partial\Omega)} - \frac{\beta}{q}\|u_+\|^q_{L^q(\partial\Omega)}. \tag{5.1}$$

Proof. Proof of Proposition 3.2. **Case(i):** $\alpha > \lambda_1(p)$ and $0 < \beta < \lambda_1(q)$. In this case, we note that

$$\lambda_1(p, \frac{\alpha}{\beta}) = \frac{\lambda_1(p)\beta}{\alpha} < \beta < \lambda_1(q) = \lambda_1(q,1)$$

and

$$\alpha|u|^{p-2}u + \beta|u|^{q-2} = \beta\left(\frac{\alpha}{\beta}|u|^{p-2} + |u|^{q-2}\right)$$

Thus, our conclusion follows from application of Theorem 2.2 to the problem (1.3) with $\lambda = \beta, m_p = \frac{\alpha}{\beta}$ and $m_q = 1$.
Case(ii): $0 < \alpha < \lambda_1(p)$ and $\beta > \lambda_1(q)$. We proceed as above, applying theorem 2.2 to (1.3) with $\lambda = \alpha, m_p = 1$ and $m_q = \frac{\beta}{\alpha}$
Case(iii): $\alpha > \lambda_1(p)$ and $\beta \leq 0$. By the same argument as in [[21] lemma3.3], it can be shown that $I_{\alpha,\beta}$ satisfies the Palais-Smail condition. Moreover, it is proved in [[21] Theorem3.1] that for functional J on $W^{1,p}(\Omega)$ defined by

$$J(u) = \frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|u\|^q_{1,q} - \frac{\alpha}{p}\|u_+\|^p_{L^p(\partial\Omega)}$$

there exist $\delta > 0$ and $\rho > 0$ such that

$$J(u) \geq \delta \text{ provided } \|u\|_{L^p(\partial\Omega)} = \rho.$$

Since $\beta \leq 0$, this implies that $I_{\alpha,\beta} \geq J(u) \geq \beta$ provided $\|u\|_{L^p(\partial\Omega)} = \rho$. For the positive eigenfunction φ_p corresponding to $\lambda_1(p)$ and sufficiently large $t > 0$, we have

$$I_{\alpha,\beta}(t\varphi_p) = \frac{t^p}{p}(\lambda_1(p) - \alpha)\|\varphi_p\|^p_{L^p(\partial\Omega)} + \frac{t^q}{q}\big(\|\varphi_p\|^q_{1,q} - \beta\|\varphi_p\|^q_{L^q(\partial\Omega)}\big) < 0,$$

since $\lambda_1(p) - \alpha < 0$ and $q < p$. Consequently, by applying the mountain pass theorem, $I_{\alpha,\beta}$ has a positive critical value (see [[21] Theorem 3.1])

Case(iv): $\alpha \leq 0$ and $\beta > \lambda_1(q)$. In this case, it can be easily shown that $I_{\alpha,\beta}$ is coercive and bounded from below, due to $q < p$ and the inequality

$$I_{\alpha,\beta}(u) \geq \frac{1}{p}\|u\|_{1.p} - \frac{\beta}{q}\|u_+\|^q_{L^q(\partial\Omega)} \geq \frac{1}{p}\|u\|^p_{1,p} - C\|u_+\|^p_{1,p},$$

where $C > 0$ is independent of $u \in W^{1,p}(\Omega)$. Moreover, $I_{\alpha,\beta}$ is weakly lower semi-continuous by the compactness of the embedding $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$ and $L^q(\partial\Omega)$, and therefore $I_{\alpha,\beta}$ has a global minimizer $u \in W^{1,p}(\Omega)$ (cf. [13] Theorem 1.1). On the other hand, for the positive eigenfunction φ_q corresponding to $\lambda_1(q)$ and sufficiently small $t > 0$, we have

$$I_{\alpha,\beta}(t\varphi_q) = \frac{t^q}{q}(\lambda_1(q) - \beta)\|\varphi_q\|^q_{L^q(\partial\Omega)} + \frac{t^p}{p}\big(\|\varphi_q\|^q_{1,q} - \alpha\|\varphi_q\|^p_{L^p(\partial\Omega)}\big) < 0,$$

whence $I(u) = \min_{W^{1,p}(\Omega)} I_{\alpha,\beta} < 0$, and therefore u is a non-trivial solution of $(P_{\alpha,\beta})$. □

Proof of Proposition 3.4. Here we prove Properties of $\lambda^*(s)$

Part (i). Fix any $s \in \mathbb{R}$ and let $u \in W^{1,p}(\Omega)$ be a positive solution of $(P_{\lambda+s,\lambda})$ for some $\lambda \in \mathbb{R}$. Then $u \in \text{int}C^1(\overline{\Omega})_+$ (see Remark 2.7), Replacing $E^{\alpha,\beta}_{[\underline{u},\overline{u}]}$ with $I_{\alpha,\beta}$. Choose any $\varphi \in \text{int}C^1(\overline{\Omega})_+$. Then, $\varphi/u \in L^\infty(\partial\Omega)$, and hence we can take

$$\xi = \frac{\varphi^p}{u^{p-1} + u^{q-1}} \in W^{1,p}(\Omega)$$

as a test function. Therefore, from Proposition 2.9, there follows the existence of $\rho > 0$ independent of u and λ such that

$$\begin{aligned}&\int_\Omega (|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)\nabla\Big(\frac{\varphi^p}{u^{p-1} + u^{q-1}}\Big)dx + \int_\Omega \varphi^p dx\\ &= \lambda\int_{\partial\Omega}\varphi^p d\sigma + s\int_{\partial\Omega}\frac{u^{p-1}\varphi^p}{u^{p-1} + u^{q-1}}d\sigma \leq \frac{1}{\rho}\Big(\int_\Omega (|\nabla\varphi|^p dx + \int_\Omega |\nabla\varphi^{p/q}|^q dx + \int_\Omega \rho\varphi|^p dx\Big).\end{aligned}$$

Combining this inequality with the estimation

$$s\int_{\partial\Omega}\frac{u^{p-1}\varphi^p}{u^{p-1} + u^{q-1}}d\sigma \geq \min\{0, s\int_{\partial\Omega}\varphi^p d\sigma\},\ \ s \in \mathbb{R},$$

we conclude that

$$\lambda\int_{\partial\Omega}\varphi^p d\sigma + \min\{0, s\int_{\partial\Omega}\varphi^p d\sigma\} \leq \frac{1}{\rho}\Big(\int_\Omega (|\nabla\varphi|^p dx + \int_\Omega |\nabla\varphi^{p/q}|^q dx + \int_\Omega \rho\varphi|^p dx\Big). \tag{5.2}$$

Since $\int_{\partial\Omega}\varphi^p d\sigma$, $\int_\Omega (|\nabla\varphi|^p dx$, $\int_\Omega |\nabla\varphi^{p/q}|^q dx$, $\int_\Omega \rho\varphi|^p dx$ and ρ are positive constants independent of u and λ, λ satisfying (5.2) is bounded from above.Therefore, $\lambda^*(s) < +\infty$, which completes the proof of Part (i).

Part (iv). Assume first that (1.5) holds. Then Lemma 4.7 implies that

$(P_{\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon})$ possesses a positive solution for sufficiently small $\varepsilon>0$. Noting that $(\lambda_1(p)+\varepsilon,\lambda_1(q)+\varepsilon)=\lambda_1(q)+\varepsilon+s^*,\lambda_1(q)+\varepsilon)$, by definition of $\lambda^*(s^*)$ we have $\lambda^*(s^*\geq\lambda_1(q)+\varepsilon$, and so $\lambda^*(s^*)+s^*\geq\lambda_1(q)+\varepsilon+s^*=\lambda_1(p)+\varepsilon$, which is the desired conclusion.

Assume that (1.5) is violated, i.e. $\varphi_p=k\varphi_q$ in $\overline{\Omega}$ for some $k\neq 0$. Let u be a positive weak solution of $(P_{\alpha,\beta})$ for some $\alpha,\beta\in\mathbb{R}$. Then, due to the regularity of φ_p and u (see Remark 2.7), the Proposition 2.9 implies

$$\begin{aligned}\int_\Omega(|\nabla u|^{p-2}\nabla u\nabla\left(\frac{\varphi_p^p}{u^{p-1}}\right)dx+\int_\Omega(|u|^{p-2}u\left(\frac{\varphi_p^p}{u^{p-1}}\right)dx&\leq\int_\Omega(|\nabla\varphi_p|^p+|\varphi_p|^p)dx\\&=\lambda_1(p)\int_{\partial\Omega}\varphi_p^pd\sigma\end{aligned}\tag{5.3}$$

At the same time, generalized Picone's identity from [[9]. Lemma 1, p.536] yields

$$\begin{aligned}&\int_\Omega(|\nabla u|^{q-2}\nabla u\nabla\left(\frac{\varphi_p^p}{u^{p-1}}\right)dx+\int_\Omega(|u|^{q-2}u\left(\frac{\varphi_p^p}{u^{p-1}}\right)dx\\&\leq\int_\Omega(|\nabla\varphi_p|^{q-2}\nabla\varphi_p\nabla\left(\frac{\varphi_p^{p-q+1}}{u^{p-q}}\right)dx+\int_\Omega|\varphi_p|^{p-2}\varphi_p\left(\frac{\varphi_p^{p-q+1}}{u^{p-q}}\right)dx\\&=\lambda_1(q)\int_{\partial\Omega}\varphi_p^pu^{q-p}d\sigma\end{aligned}\tag{5.4}$$

where the last equality is valid because φ_p is an eigenfunction of Δ_p, by assumption.

Hence, using (5.3) and (5.4), we obtain for the solution u of $(P_{\alpha,\beta})$ the following inequality:

$$\begin{aligned}&\int_\Omega\left(|\nabla u|^{p-2}+|\nabla u|^{q-2}\right)\nabla u\left(\frac{\varphi_p^p}{u^{p-1}}\right)dx+\int_\Omega\left(|u|^{p-2}+|u|^{q-2}\right)\frac{\varphi_p^p}{u^{p-1}}dx\\&=\alpha\int_{\partial\Omega}\varphi_p^pd\sigma+\int_{\partial\Omega}\varphi_p^pu^{q-p}d\sigma\leq\lambda_1(p)\int_{\partial\Omega}\varphi_p^pd\sigma+\lambda_1(q)\int_{\partial\Omega}\varphi_p^pu^{q-p}d\sigma\end{aligned}$$

which is impossible if $\alpha>\lambda_1(p)$ and $\beta>\lambda_1(q)$ simultaneously, and the proof is complete.

Part(ii). Assume that $s\neq s^*$. Then taking $\alpha=\lambda+s$ and $\beta=\lambda$, Proposition 3.2 implies that $\lambda^*(s)+s\geq\lambda_1(p)$ and $\lambda^*(s)\geq\lambda_1(q)$. If now $s=s^*$ and $\lambda_1(p)$ and $\lambda_1(q)$ have the same eigenspaces, i.e. there exists $k\neq 0$ such that $\varphi_p\equiv k\varphi_q$ in $\overline{\Omega}$, then from Proposition 3.1 it follows that $(P_{\lambda_1(p),\lambda_1(q)})$ possesses a positive solution, i.e. $\lambda^*(s)+s\geq\lambda_1(p)$ and $\lambda^*(s^*)\geq\lambda_1(q)$. Finally, if $\lambda_1(p)$ and $\lambda_1(q)$ have different eigenspaces, that is, (1.5) is satisfied, then Part (iv) of Proposition 3.4 yields desired result.

Part(vi). Let $s<s'$. Part (ii) of Proposition 3.4 implies that $\lambda^*(s),\lambda^*(s')\geq\lambda_1(q)$. Thus, in order to prove $\lambda^*(s)\geq\lambda^*(s')$, it sufficient to consider only the case $\lambda^*(s')>\lambda_1(q)$.

Fix any $\varepsilon>0$ such that $\lambda^*(s')-\varepsilon>\lambda_1(q)$. Then, by definition of $\lambda^*(s')$, there exists μ satisfying $\lambda^*(s')>\mu>\lambda^*(s')-\varepsilon$ such that $(P_{\mu+s',\mu})$ has a positive solution $\overline{u}_\mu\in\mathrm{int}C^1(\overline{\Omega})_+$. It is easy to see that $\overline{u}_\mu$ is a positive super-solution of $(P_{\mu+s,\mu})$, since $s<s'$. Hence, Lemma 2.8 ensures the existence of a positive solution of $(P\mu+s,\mu)$(note $\mu>\lambda^*(s')-\varepsilon>\lambda_1(q)$). Hence, $\lambda^*(s)\geq\mu>(\lambda^*(s')-\varepsilon)$. Since ε is arbitrary, we have $\lambda^*(s)>\mu>\lambda^*(s')$.

Next, we show that $\lambda^*(s)+s\leq\lambda^*(s')+s'$ for $s<s'$. If $\lambda^*(s)+s-s'\leq\lambda_1(q)$, then $\lambda^*(s)+s\leq\lambda_1(q)+s'\leq\lambda^*(s')+s'$, due to the fact that $\lambda_1(q)\leq\lambda^*(s')$. So, we may suppose that $\lambda^*(s)+s-s'>\lambda_1(q)$. Fix any $\varepsilon>0$ such that $\lambda^*(s)+s-s'-\varepsilon>\lambda_1(q)$. By the definition of $\lambda^*(s)$, there exists $\mu>\lambda^*(s)-\varepsilon$ such that $(P_{\mu+s,\mu})$ has a positive solution $\overline{u}_\mu$. Putting $\beta=\mu+s-s',\overline{u}_\mu$ is the positive solution of $(P_{\beta+s',\beta+s'-s})$. Noting that $\beta+s'-s-\varepsilon>\beta,\overline{u}_\mu$ is a positive super-solution of $(P_{\beta+s',\beta})$. Since $\beta>\lambda^*(s)+s-s'-\varepsilon>\lambda_1(q)$ by the same argument above, we get $\lambda^*(s')\geq\lambda^*(s)+s-s'$, whence $\lambda^*(s)+s\leq\lambda^*(s')+s'$.

Part(iii). Assume first that (1.5) doesn't hold. Then $s^*=s^*_+$ and, due to Part (iv) of Proposition 3.4, $\lambda^*(s)\geq\lambda_1(q)$ for all $s\in\mathbb{R}$ by Part (ii). Hence, $\lambda^*(s^*)=\lambda_1(q)$ and noting that $\lambda^*(s)$ is non-increasing by Part (vi) we get the desired result.

Let now (1.5) hold and suppose, by contradiction, that exists $s>s^*_+$ such that $\lambda^*(s)>\lambda_1(q)$. Since $\lambda^*(s^*)+s^*>\lambda_1(p)$ by Part (iv) of Proposition 3.4, using Part (vi) and recalling that s, we get

$$\lambda^*(s)+s\geq\lambda^*(s^*)+s^*>\lambda_1(p).$$

By definition of $\lambda^*(s)$, for any $\varepsilon_0 > 0$ there exists $\varepsilon \in [0, \varepsilon_0)$ such that $(P_{\lambda^*(s)+s-\varepsilon, \lambda^*(s)-\varepsilon})$ possesses a positive solution. Let us take ε_0 small enough to satisfy

$$\lambda^*(s) + s - \varepsilon_0 > \lambda_1(p), \ \lambda^*(s) - \varepsilon_0 > \lambda_1(q), \tag{5.5}$$

and let u be a corresponding solution of $(P_{\lambda^*(s)+s-\varepsilon, \lambda^*(s)-\varepsilon})$, where $\varepsilon \in [0, \varepsilon_0)$.
Using the Picone Identities (5.3) and (5.4) implied to φ_p, we obtain the following inequality:

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u|^{p-2} + |\nabla u|^{q-2}\Big)\nabla u \Big(\frac{\varphi_q^p}{u^{p-1}}\Big)dx + \int_\Omega \Big(|u|^{p-2} + |u|^{q-2}\Big)\frac{\varphi_p^q}{u^{p-1}}dx \\
&= (\lambda^*(s) + s - \varepsilon)\int_{\partial\Omega} \varphi_q^p d\sigma + (\lambda^*(s) - \varepsilon)\int_{\partial\Omega} \varphi_q^p u^{q-p} d\sigma \\
&\leq \int_\Omega |\nabla \varphi_q|^p dx + \int_\Omega |\varphi_q|^p dx + \lambda_1(q)\int_{\partial\Omega} \varphi_q^p u^{q-p} d\sigma,
\end{aligned} \tag{5.6}$$

on the other hand, since $\varepsilon < \varepsilon_0$, from (5.5) it follows that

$$\begin{aligned}
&(\lambda_1(q) + s)\int_{\partial\Omega} \varphi_q^p d\sigma + \lambda_1(q)\int_{\partial\Omega} \varphi_q^p u^{q-p} d\sigma \\
&\qquad < (\lambda^*(s) + s - \varepsilon)\int_{\partial\Omega} \varphi_q^p d\sigma + (\lambda^*(s) - \varepsilon)\int_{\partial\Omega} \varphi_q^p u^{q-p} d\sigma
\end{aligned} \tag{5.7}$$

Finally, combining (5.6) and (5.7) we conclude that

$$s < \frac{\int_\Omega |\nabla \varphi_q|^p dx + \int_\Omega |\varphi_q|^p dx}{\int_{\partial\Omega} \varphi_q^p d\sigma} - \lambda_1(q) = s_+^*,$$

which contradicts our assumption $s \geq s_+^*$.
Part(v). Since $\lambda^*(s)$ is bounded for any $s \in \mathbb{R}$ by Part (i) of Proposition 3.4 and non-increasing by Part(vi), for every $s' \in \mathbb{R}$ there exist one-sided limits of $\lambda^*(s)$ and

$$\lim_{s \to s'-0} \lambda^*(s) \geq \lambda^*(s') \geq \lim_{s \to s'+0} \lambda^*(s). \tag{5.8}$$

On the other hand, $\lambda^*(s) + s$ is non-decreasing by Part(vi) of Proposition 3.4, and hence

$$\lim_{s \to s'-0} (\lambda^*(s) + s) \leq \lambda^*(s') + s' \leq \lim_{s \to s'+0} \lambda^*(s) + s,$$

which yields

$$\lim_{s \to s'-0} \lambda^*(s) \leq \lambda^*(s') \leq \lim_{s \to s'+0} \lambda^*(s). \tag{5.9}$$

Combining (5.8) with (5.9) we conclude that the one-sided limits are equal to $\lambda^*(s')$, which establishes the desired continuity, due to the arbitrary choice of $s' \in \mathbb{R}$. □

Proof. Proof of Proposition 3.5. We prove that (3.2) and (3.3) are in fact, equivalent.
Fix any $s \in \mathbb{R}$. Since $\lambda^*(s)$ is bounded from below by Part(ii) of Proposition 3.4, the definition (3.2) implies the existence of a sequence of solutions $\{u_n\}_{n=1}^\infty \in \mathrm{int}C^1(\overline{\Omega})_+$ (see Remark 2.7) for $(P_{\lambda_n+s,\lambda})$ such that $\lambda_n \to \lambda^*(s)$ as $n \to \infty$ and each $\lambda_n \leq \lambda^*(s)$ (note that there we allow $\lambda_n = \lambda^*(s)$ for all $n \in \mathbb{N}$.)
Using u_n as admissible function for (3.3) and noting that for any $0 \neq \varphi \in C^1(\overline{\Omega})_+$ the denomination of $\mathcal{L}_s(u_n; \varphi)$ is positive , namely,

$$\int_\Omega (u_n^{p-1} + u_n^{q-1})\varphi dx > 0,$$

we get

$$\Lambda^*(s) \geq \inf_{\varphi \in C^1(\overline{\Omega})_+ \setminus \{0\}} \mathcal{L}_s(u_n; \varphi) = \lambda_n \to \lambda^*(s)$$

and therefore $\Lambda^*(s) \geq \lambda^*(s)$ for any $s \in \mathbb{R}$.
Assume now that there exists $s_0 \in \mathbb{R}$ such that $\Lambda^*(s_0) > \lambda^*(s_0)$. Then, by the definition of $\Lambda^*(s)$, there exist $\overline{u} \in C^1(\overline{\Omega})_+$ which we have

$$\Lambda^*(s_0) \geq \mu := \inf_{\varphi \in C^1(\overline{\Omega})_+ \setminus \{0\}} \mathcal{L}_s(\overline{u};\varphi) > \lambda^*(s_0).$$

However, this implies that $\overline{u}$ is a positive super-solution of $(P_{\mu+s_0,\mu})$. Indeed

$$\mathcal{L}_s(\overline{u};\varphi) \geq \mu > \lambda^*(s_0) \text{ for all } \varphi \in C^1(\overline{\Omega})_+ \setminus \{0\},$$

and therefore

$$\int_\Omega \left(|\nabla \overline{u}|^{p-2} + |\nabla \overline{u}|^{q-2}\right)\nabla \overline{u} \nabla \varphi dx + \int_\Omega \left(|\overline{u}|^{p-2} + |\overline{u}|^{q-2}\right)\overline{u}\varphi dx$$
$$- \int_{\partial\Omega} \left((\mu + s_0)|\overline{u}|^{p-2} + \mu|\overline{u}|^{q-2}\right)\overline{u}\varphi d\sigma \geq 0$$

for any $\varphi \in W^{1,p}(\Omega)_+$, due to approximation arguments. Hence, recalling that $\mu > \lambda^*(s_0) \geq \lambda_1(q)$, Lemma 2.8 guarantees the existence of a positive solution for $(P_{\mu+s_0,\mu})$, however it contradicts the definition of $\lambda^*(s_0)$. $\square$

Proof. Proof of Theorem 3.3. Note first that from Proposition 3.1 and 3.2 it directly follows that if (3.1) is satisfied, then $(P_{\alpha,\beta})$ has at least one positive solution.
Conversely, if $(P_{\alpha,\beta})$ has at least one positive solution, then by the definition of $\lambda^*(s)$, Part(iv) of Proposition 3.4 and Proposition 3.1, it has to satisfy

$$(\alpha,\beta) \in (\lambda_1(p), +\infty) \times (-\infty, \lambda_1(q)] \cup (-\infty, \lambda_1(p)) \times (\lambda_1(q), +\infty) \cup \{(\lambda_1(p), \lambda_1(q))\}.$$

To prove (3.1), it is sufficient to show that $(\alpha,\beta) \notin \{\lambda_1(p)\} \times (\lambda_1(q), +\infty)$ and $(\alpha,\beta) \notin (\lambda_1(p), +\infty) \times \{\lambda_1(q)\}$. Suppose that $(P_{\alpha,\beta})$ has a positive solution u for $\alpha = \lambda_1(p)$ and $\beta \geq \lambda_1(q)$ (res. $\alpha \geq \lambda_1(p)$ and $\beta = \lambda_1(q)$). Then, as in the proof of Proposition 3.4, Part(iv), from (5.3) and (5.4) we have

$$\int_\Omega \left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right)\nabla u \left(\frac{\varphi_q^p}{u^{p-1}}\right) dx + \int_\Omega \left(|u|^{p-2} + |u|^{q-2}\right)\frac{\varphi_p^q}{u^{p-1}} dx$$
$$= \alpha \int_{\partial\Omega} \varphi_q^p d\sigma + \beta \int_{\partial\Omega} \varphi_q^p u^{q-p} d\sigma \leq \lambda_1(p) \int_{\partial\Omega} \varphi_q^p d\sigma + \lambda_1(q) \int_{\partial\Omega} \varphi_q^p u^{q-p} d\sigma$$

which implies that $\beta = \lambda_1(q) (res. \alpha = \lambda_1(p))$. Hence, we get the desired result. $\square$

Proof. Proof of Theorem 3.6. Consider first the non-existence result. Let $\beta > \lambda^*(\alpha - \beta)$. Then the definition of $\lambda^*(\alpha - \beta)$ implies that $(P_{\alpha,\beta})$ has no positive solutions.
(i) Assume that $\lambda_1(q) < \beta < \lambda^*(s)$ with $s = \alpha - \beta$. Then, by the definition of $\lambda^*(s)$, there exists $\mu \in (\beta, \lambda^*(s))$ such that $(P_{\mu+s,\mu})$ has a positive solution $\overline{u} \in \mathrm{int} C^1(\overline{\Omega})_+$(see Remark 2.7). Moreover, $\overline{u}$ is a positive super-solution of $(P_{\alpha,\beta}) \equiv (P_{\beta+s,\beta})$, since $\mu > \beta$. Hence, the assumptions of Lemma 2.8 are satisfied, which guarantees the existence of a positive solution of $(P_{\alpha,\beta})$.
(ii) Assume now that $\lambda_1(p) < \alpha$ and $\beta < \lambda^*(s)$ with $s = \alpha - \beta$. Note that if $\beta > \lambda_1(q)$, then Part (i) gives the claim. If $\beta < \lambda_1(q)$, then Proposition 3.2 implies the desired result. Therefore, it remains to consider the case $\beta = \lambda_1(q)$.
Let us divide the proof into tree cases:
Case 1. φ_q satisfied $\|\varphi_q\|_{1,p}^p - \alpha\|\varphi_q\|_{L^p(\partial\Omega)}^p > 0$. Note that

$$\lambda_1\big(q, \frac{\lambda_1(q)}{\alpha}\big) = \frac{\alpha\lambda_1(q)}{\lambda_1(q)} = \alpha > \lambda_1(p) = \lambda_1(p,1).$$

This yields (2.1) with $r = q, r' = p, \lambda = \alpha, m_p \equiv \frac{\lambda_1(q)}{\alpha}$. Moreover, since $\lambda_1(r,c)$ and $\lambda_1(r,1) = \lambda_1(r)$ have the same eigenspace for any constant $c > 0$, namely, $\varphi_1(r,c) = t\varphi_1(r,1) = \varphi_r$ for same $t > 0$, the

hypothesis of Case 1 ensures (2.2). Hence, Theorem 2.3 guarantees our conclusion.
Case 2. φ_q satisfies $\|\varphi_q\|^p_{1,p} - \alpha\|\varphi_q\|^p_{L^p(\partial\Omega)} < 0$. Since $\beta < \lambda^*(s)$, by definition of $\lambda^*(s)$ there exists $\mu \in (\beta, \lambda^*(s)]$ such that $(P_{\mu+s,\mu})$ possesses a positive solution $\overline{u} \in \mathrm{int}C^1(\overline{\Omega})_+$. As in proof of Part(i) it is easy to see that $\overline{u}$ is a positive super-solution of $(P_{\alpha,\beta}) \equiv (P_{\beta+s,\beta})$.
Let $E^{\alpha,\beta}_{[0,\overline{u}]}$ be the functional defined by (2.3) with a positive super-solution $\overline{u}$ and sub-solution 0. Since $\overline{u}$ and φ_q belong to $\mathrm{int}C^1(\overline{\Omega})_+$, for sufficiently small $t > 0$ we get $t\varphi_q \le \overline{u}$ in Ω, whence $f^{\alpha,\beta}_{[0,\overline{u}]}(x, t\varphi_q) = \alpha t^{p-1}\varphi_q^{p-1} + \beta t^{q-1}\varphi_q^{q-1}$. Therefore, noting that $\beta = \lambda_1(q)$, for such small $t > 0$ we have

$$E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q) = \frac{t^p}{p}(\|\varphi_q\|^p_{1,p} - \alpha\|\varphi_q\|^p_{L^p(\partial\Omega)}) < 0.$$

This ensures that $\inf_{W^{1,p}(\Omega)} E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q) < 0$. Hence, $(P_{\alpha,\beta})$ has a positive solution (refer to Lemma 2.8).
Case 3. φ_q satisfied $\|\varphi_q\|^p_{1,p} - \alpha\|\varphi_q\|^p_{L^p(\partial\Omega)}) = 0$. Similarly to Case 2, we know that $E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q) = 0$ for sufficiently small $t > 0$. $\min_{W^{1,p}(\Omega)} E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q) < 0$. Holds, then $(P_{\alpha,\beta})$ has a positive solution. On the other hand, if $\min_{W^{1,p}(\Omega)} E^{\alpha,\beta}_{[0,\overline{u}]}(t\varphi_q) = 0$, then $t\varphi_q$ is a global minimizer of $E^{\alpha,\beta}_{[0,\overline{u}]}$, whence $t\varphi_q$ is a positive solution of $(P_{\alpha,\beta})$. Consequently, the proof is complete. □

Proof. Proof of Proposition 3.7. For the proof of Proposition 3.7, we prepare two lemmas. The following lemma is needed to prove the boundedness of approximate solutions.

Lemma 5.1. *Let u_n be a positive solution of (P_{α_n,β_n}) with $\alpha_n \to \alpha$ and $\beta_n \to \beta$. If $\|u_n\|1, p \to \infty$ as $n \to \infty$, then $\alpha = \lambda_1(p)$.*

Proof. Let u_n be a positive solution of (P_{α_n,β_n}) with $\alpha_n \to \alpha$ and $\beta_n \to \beta$ and $\|u_n\|_{1,p} \to \infty$ as $n \to \infty$. Setting $w_n := u_n/\|u_n\|_{1,p}$, we my admit, up to subsequence, that $w_n \to w_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\partial\Omega)$ and $L^q(\partial\Omega)$ for some $w_0 \in W^{1,p}(\Omega)$. By taking $(w_n - w_0)/\|u_n\|^{p-1}_{1,p}$ as a test function, we obtain

$$\begin{aligned}
0 &= \int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla(w_n - w_0)dx + \int_\Omega |w_n|^{p-2}w_n(w_n - w_0)dx \\
&+ \frac{1}{\|u_n\|^{p-q}_{1,p}}\int_\Omega |\nabla w_n|^{q-2}\nabla w_n\nabla(w_n - w_0)dx + \frac{1}{\|u_n\|^{p-q}_{1,p}}\int_\Omega |w_n|^{q-2}w_n(w_n - w_0)dx \\
&- \alpha_n\int_{\partial\Omega} w_n^{p-1}(w_n - w_0)d\sigma - \frac{\beta_n}{\|u_n\|^{p-q}_{1,p}}\int_{\partial\Omega} w_n^{p-1}(w_n - w_0)d\sigma \\
&= \int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla(w_n - w_0)dx + \int_\Omega |w_n|^{p-2}w_n(w_n - w_0)dx + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. Due to the (S_+) property of Δ_p implies that $w_n \to w_0$ strongly in $W^{1,p}(\Omega)$. Then, for any $\varphi \in W^{1,p}(\Omega)$, by taking $\varphi/\|u_n\|^{p-1}_{1,p}$ as function test we have

$$\begin{aligned}
0 &= \int_\Omega (|\nabla w_n|^{p-2}\nabla w_n\nabla\varphi dx + \int_\Omega (|w_n|^{p-2}w_n\varphi dx \\
&+ \frac{1}{\|u_n\|^{p-q}_{1,p}} int_\Omega(|\nabla w_n|^{q-2}\nabla w_n\nabla\varphi dx + \frac{1}{\|u_n\|^{p-q}_{1,p}}\int_\Omega (|w_n|^{q-2}w_n\varphi dx \\
&- \alpha_n\int_{\partial\Omega}(w_n^{p-1}\varphi d\sigma - \frac{\beta_n}{\|u_n\|^{p-q}_{1,p}}\int_{\partial\Omega}(w_n^{p-1}\varphi d\sigma.
\end{aligned}$$

Letting $n \to \infty$ we conclude that w_0 is a non-negative, non-trivial solution of $(P_{\alpha,\beta})$ (not $w_0 \ge 0$ and $\|w_0\|_{1,p} = 1$). According to the strong maximum principle (see Remark 2.7), we have $w_0 > 0$ in Ω. This yields that w_0 is a positive eigenfunction corresponding to α and $\alpha = \lambda_1(p)$, since any eigenvalue other than $\lambda_1(p)$ has no positive eigenfunctions.

Lemma 5.2. *If u is a positive solution of $(P_{\alpha,\beta})$, then*

$$\int_\Omega |\nabla\varphi|^q|\nabla u|^{p-q}dx + \int_\Omega \varphi^q u^{p-q}dx \int_\Omega |\nabla\varphi|^q dx + \int_\Omega \varphi^q dx \geq \int_{\partial\Omega}(\alpha u^{p-q}+\beta)\varphi^q d\sigma$$

for every $\varphi \in intC^1(\overline{\Omega})_+$.

Proof. Let u be a positive solution of $(P_{\alpha,\beta})$. Then, $u \in \mathrm{int}C^1(\overline{\Omega})_+$. (see Remark 2.7). Choose any $\varphi \in \mathrm{int}C^1(\overline{\Omega})_+$. Then, $\varphi/u \in L^\infty(\partial\Omega)$, and hence we can take

$$\xi = \frac{\varphi^q}{u^{q-1}} \in W^{1,p}(\Omega)$$

as a test function, By the similar estimation as in the proof in[[6], Proposition A.2], we have

$$\begin{aligned}|\nabla u|^{p-2}\nabla u\nabla\left(\frac{\varphi^q}{u^{q-1}}\right) &= q|\nabla u|^{p-2}\nabla u\nabla\varphi\left(\frac{\varphi}{u}\right)^{q-1} - (q-1)|\nabla u|^p\left(\frac{\varphi}{u}\right)^q \\ \leq q|\nabla u|^{p-1}\nabla u|\nabla\varphi|\left(\frac{\varphi}{u}\right)^{q-1} &- (q-1)|\nabla u|^p\left(\frac{\varphi}{u}\right)^q \leq \nabla\varphi|^q|\nabla u|^{p-q}\end{aligned} \tag{5.10}$$

in $\partial\Omega$, where we use the standard Young's inequality

$$ab \leq \frac{a^q}{q} + \frac{(q-1)b^{q/(q-1)}}{q}$$

with $a = |\nabla\varphi||\nabla u|^{p-1-d}, b = (\varphi/u)^{q-1}|\nabla u|^d$ and $d = (q-1)p/q = p - p/q$.
T the same time, the standard Picone identity (Proposition 2.9 implies

$$|\nabla u|^{p-2}\nabla u\nabla\left(\frac{\varphi^q}{u^{q-1}}\right) \leq |\nabla\varphi|^q \text{ in } \Omega. \tag{5.11}$$

Applying now estimation (5.10) and (5.11) to the definition of a weak solution, we obtain the desired result.
Proof of Proposition 3.7. **Part (i).** Put $\alpha = \lambda^*(s) + s > \lambda_1(p)$ and $\beta = \lambda^*(s) > \lambda_1(q)$ for some $s \in \mathbb{R}$. By the definition of $\lambda^*(s)$, there exists $\beta_n > \lambda_1(q)$ such that $\beta_n \to \beta = \lambda^*(s)$ and (P_{α_n,β_n}) has a positive solution u_n, where $\alpha_n = \beta_n + s$. Since $\alpha_n \to \beta + s = \lambda^*(s) + s > \lambda_1(p)$, Lemma 5.1 guarantees the boundedness of $\{u_n\}$ in $W^{1,p}(\Omega)$.
The $\{u_n\}$ is a bounded Palais-Smale sequence for the functional $I_{\alpha,\beta}$ defined by (5.1). Indeed, $I'_{\alpha_n,\beta_n} = 0$ and so

$$\begin{aligned}\|I'_{\alpha,\beta}(u_n)\|_{W^{1,p}(\Omega)^*} &= \|I'_{\alpha_n,\beta_n}(u_n) - I'_{\alpha,\beta}(u_n)\|_{W^{1,p}(\Omega)^*} \\ &\leq \frac{|\alpha_n - \alpha|}{p\lambda_1(p)^{1/p}}\|u_n\|^{p-1}_{L^p(\partial\Omega)} + \frac{|\beta_n - \beta|}{q\lambda_1(p)^{1/q}}\|u_n\|^{q-1}_{L^q(\partial\Omega)}|\partial\Omega|^{1/q-1/p}\end{aligned}$$

On the other hand, by a standard argument based on the (S_+) property of $-\Delta_p$, it can be readily shown that $I_{\alpha,\beta}$ satisfies the bounded Palais-Smale condition. Hence, $\{u_n\}$ has a subsequence converging to some critical point u_0 of $I_{\alpha,\beta}$. Thus, if we show that $u_0 \neq 0$, then u_0 is a positive solution of $(P_{\alpha,\beta})$, whence the proof is complete.
Now, we will prove that $u_0 \neq 0$ by way of contradiction. Assume that u_n strongly converges to $0 \in W^{1,p}(\Omega)$. Applying Lemma 5.2 with $\varphi = \varphi_q$, we see that any u_n satisfies the inequality

$$\int_\Omega |\nabla\varphi_q|^q_q|\nabla u|^{p-q}dx + \int_\Omega \varphi_q u^{p-q}dx + \int_\Omega |\nabla\varphi_q|^q dx + \int_\Omega \varphi_q^q dx \geq \int_{\partial\Omega}(\alpha u^{p-q}+\beta)\varphi_q^q d\sigma.$$

Letting $n \to \infty$, we have $\|\varphi_q\|^q_{1,q} \geq \beta\|\varphi_q\|^q_{L^q(\partial\Omega)}$. However, this is a contradiction, since $\lambda_1(q)\|\varphi_q\|^q_{L^q(\partial\Omega)} = \|\varphi_q\|^q_{1,q} \geq \beta\|\varphi_q\|^q_{L^q(\partial\Omega)}$ and $\beta > \lambda_1(q)$.

Parti (ii). From Parti (iii) of Proposition 3.4 it follows that $(P_{\lambda^*(s)+s,\lambda^*(s)}) \equiv (P_{\lambda_1(q)+s,\lambda_1(q)})$ for all $s \geq s^*_+$. Suppose, contrary to our claim, that $(P_{\lambda_1(q)+s,\lambda_1(q)})$ possesses a positive solution u for some $s > s^*_+$. As in the proof of Part (iii), Proposition 3.4, we replace φ_p by φ_q in Picone's identities (5.3) and (5.4), and get

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\frac{\varphi_q^p}{u^{p-1}}\right)dx + \int_\Omega u^{p-2}u\left(\frac{\varphi_q^p}{u^{p-1}}\right)dx \\
&+ \int_\Omega |\nabla u|^{q-2}\nabla u \nabla\left(\frac{\varphi_q^p}{u^{p-1}}\right)dx + \int_\Omega u^{q-2}u\left(\frac{\varphi_q^p}{u^{p-1}}\right)dx \\
&= (\lambda_1(q)+s)\int_{\partial\Omega} \varphi_q^q d\sigma + \lambda_1(q)\int_{\partial\Omega} \varphi_q^q u^{q-p} d\sigma \\
&\leq \int_\Omega |\nabla \varphi_q|^p dx + \int_\Omega \varphi_q^p dx + \lambda_1(q)\int_{\partial\Omega} \varphi_q^q u^{q-p} d\sigma,
\end{aligned}$$

which implies that

$$s \leq \frac{\int_\Omega |\nabla \varphi_q|^p dx + \int_\Omega \varphi_q^p dx}{\int_{\partial\Omega} \varphi_q^q d\sigma} - \lambda_1(q) = s^*_+.$$

However, it is a contradiction, since $s > s^*_+$. □

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