Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–9.
ISSN-0037-8712 in press
doi:10.5269/bspm.47828

Yamabe Solitons on Some Types of Generalized Sasakian Space Forms

A. Sarkar and Gour Gopal Biswas

ABSTRACT: The object of the present paper is to study Yamabe solitons on three dimensional generalized Sasakian space forms with quasi-Sasakian metric and Kenmotsu metric. Illustrative examples have been given.

Key Words: Yamabe soliton, Generalized Sasakian space forms, Quasi-Sasakian metric, Kenmotsu metric.

Contents

1. Introduction

In differential geometry an interesting problem is that whether a compact connected Riemannian manifold is conformally equivalent to a manifold of constant scalar curvature. This problem was formulated by Yamabe in 1960 [15]. Yamabe himself gave the affirmative answer, though there were some lacuna in his arguments. Later Trudinger [14], Aubin [2] and Schoen [12] solved the problem satisfactorily.

Another important topic of differential geometry is Ricci flow which was devolved by Richerd Hamilton [6] in order to solve the century long open problem 'Poincare conjecture'. The notion of Yamabe flow also arose parallelly from the work of Hamilton.

A Yamabe flow on Riemannian manifold is a heat type parabolic partial differential equation of the form

$$\frac{\partial}{\partial t}g = -rg,\ g(0) = g_0, \tag{1.1}$$

where g is Riemannian metric and r is the scalar curvature of the matric.

Self similar solutions of the geometric flows are known as solitons. A Yamabe soliton on a Riemannian manifold is defined by

$$\pounds_U g = (c - r)g, \tag{1.2}$$

where $\pounds_U$ denotes the Lie-derivative operator along the vector field U and the constant $c = -\dot{\sigma}(g_0)$, where σ is a scaling function.

A generalized Sasakian space form is an almost contact metric manifold whose Riemannian curvature is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.3}$$

where f_1, f_2 and f_3 are C^∞-functions on the manifold.

2010 *Mathematics Subject Classification:* 53C15, 53D25.

Submitted May 07, 2019. Published September 17, 2019

The concept of generalized Sasakian space forms was introduced by P. Algree, D. E. Blair and A. Carrizo subsequently. The first author of the present paper has studied generalized Sasakian space forms [3], [9], [10], [11].

P. Algree and A. Carrizo gave some more characterizations of such space forms in the paper [1]. In that paper they also have introduced generalized Sasakian space forms with trans-Sasakian structure and developed some of its properties. In the paper [4] U. C. De and A. Sarkar studied generalized Sasakian space forms with quasi-Sasakian metric.

Yamabe solitons on contact and almost contact manifolds have been studied by several authors [5], [8]. Motivated by these works, in the present paper we would like to study Yamabe soliton on generalized Sasakian space forms with quasi-Sasakian metric. We also, would like to study generalized Sasakian space forms with Kenmotsu metric.

The paper is organized as follows. After the introduction we mention some required preliminaries in Section 2. In Sections 3 and 4 we deduce some characteristic properties of three dimensional generalized Sasakian space forms with quasi-Sasakian metric and Kenmotsu metric respectively admitting Yamabe soliton. The last section contains examples.

2. Preliminaries

A smooth odd dimensional manifold (M, g) is said to be an almost contact metric manifold if it admits a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1. \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

For such manifolds, we also have the following :

$$\phi\xi = 0,\ g(X, \xi) = \eta(X),\ \eta \circ \phi = 0. \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ g(\phi X, X) = 0. \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y). \tag{2.5}$$

An almost contact metric manifold is called contact metric manifold if $d\eta(X, Y) = \Phi(X, Y) = g(X, \phi Y)$.

Φ is called the fundamental two form of the manifold. For a three dimensional generalized Sasakian space form we know the following [4]

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ &+ 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}. \end{aligned} \tag{2.6}$$

$$S(X, Y) = (2f_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + f_3)\eta(X)\eta(Y). \tag{2.7}$$

$$r = 6f_1 + 6f_2 - 4f_3. \tag{2.8}$$

Definition 2.1. A vector field U in an n-dimensional Riemannian manifold (M, g) is said to be conformal if

$$\pounds_U g = 2\rho g, \tag{2.9}$$

for a smooth function ρ on M. Moreover, a conformal vector field satisfies

$$(\pounds_U S)(X, Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X, Y) \tag{2.10}$$

and

$$\pounds_U r = -2\rho r + 2(n-1)\Delta\rho, \tag{2.11}$$

where D is the gradient operator and $\Delta = -\text{div}D$ is the Laplacian operator of g. For details, we refer to Yano [16].

Lemma 2.1. In an almost contact metric manifold, the following relations hold

(i) $\eta(\pounds_U \xi) = \dfrac{r-c}{2}$.

(ii) $(\pounds_U \eta)\xi = \dfrac{c-r}{2}$.

(iii) $\rho = \dfrac{c-r}{2}$.

If a generalized Sasakian space form $M(f_1, f_2, f_3)$ admits quasi-Sasakian metric [4] then

$$(\nabla_X \phi)Y = \beta(g(X,Y)\xi - \eta(Y)X),\ \forall\, X, Y \in \chi(M), \tag{2.12}$$

where β is a C^∞-function on the manifold.
As a consequence, it follows that

$$\nabla_X \xi = -\beta\phi X. \tag{2.13}$$

$$(\nabla_X \eta)Y = -\beta g(\phi X, Y). \tag{2.14}$$

For details about quasi-Sasakian generalized Sasakian space forms see [4].

If the space form admits Kenmotsu metric, then

$$(\nabla_X \phi)Y = \alpha(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.15}$$

where α is a C^∞-function on the manifold.

It follows that

$$\nabla_X \xi = \alpha(X - \eta(X)\xi). \tag{2.16}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, \phi Y). \tag{2.17}$$

3. Yamabe solitons on generalized Sasakian space forms with quasi-Sasakian metric

In this section we like to characterize three dimensional generalized Sasakian space forms with quasi-Sasakian metric admitting Yamabe soliton.

Putting the value of ρ from Lemma 2.1 in (2.10) and (2.11), respectively, we have

$$(\pounds_U S)(X,Y) = \frac{1}{2}\left[g(\nabla_X Dr, Y) - (\Delta r)g(X,Y)\right] \tag{3.1}$$

and

$$\pounds_U r = -2\Delta r - r(c-r). \tag{3.2}$$

Taking Lie-derivative of (2.7) in the direction of U and using (3.1) we get

$$\begin{aligned} g(\nabla_X Dr, Y) &= [U(4f_1 + 6f_2 - 2f_3) + (4f_1 + 6f_2 - 2f_3)(c-r) + \Delta r]\, g(X,Y) \\ &- \{U(6f_2 + 2f_3)\}\eta(X)\eta(Y) - (6f_2 + 2f_3)[\eta(Y)\,(\pounds_U\eta)X \\ &+ \eta(X)\,(\pounds_U\eta)Y]. \end{aligned} \tag{3.3}$$

As ξ is killing, we have $\xi r = 0$. Differentiating it covariantly along the arbitrary vector field X and using (2.13) we get $g(\nabla_X Dr, \xi) = \beta(\phi X)r$.
Substituting ξ in place of Y in (3.3) and using the above equation and Lemma 2.1, we get

$$\begin{aligned} \beta(\phi X)r &= [U(4f_1 - 4f_3) + (4f_1 + 3f_2 - 3f_3)(c-r) + \Delta r]\,\eta(X) \\ &- (6f_2 + 2f_3)(\pounds_U\eta)X. \end{aligned} \tag{3.4}$$

Substituting ξ for X and using Lemma 2.1 we obtain from above

$$\Delta r = -U(4f_1 - 4f_3) - (4f_1 - 4f_3)(c - r). \tag{3.5}$$

From (3.4) and (3.5), we have

$$(6f_2 + 2f_3)(\pounds_U \eta)X = (3f_2 + f_3)(c - r)\eta(X) - \beta(\phi X)r. \tag{3.6}$$

Using (3.5) and (3.6) in (3.3), we get

$$\begin{aligned} \nabla_X Dr &= [U(6f_2 + 2f_3) + (6f_2 + 2f_3)(c - r)]\,\{X - \eta(X)\xi\} \\ &+ \beta((\phi X)r)\xi - \beta(\phi\, Dr)\eta(X). \end{aligned} \tag{3.7}$$

Differentiating the above equation covariantly along the direction of Y, we obtain

$$\begin{aligned} &\nabla_Y \nabla_X Dr \\ = \;& [Y(U(6f_2 + 2f_3)) + (Y(6f_2 + 2f_3))(c - r) - (6f_2 + 2f_3)Yr]\,\{X - \eta(X)\xi\} \\ + \;& [U(6f_2 + 2f_3) + (6f_2 + 2f_3)(c - r)]\,\{\nabla_Y X - \eta(\nabla_Y X)\xi + \beta g(\phi Y, X)\xi \\ + \;& \beta\eta(X)\phi Y\} + (Y\beta)((\phi X)r)\xi + \beta\, Y((\phi X)r)\,\xi - \beta^2((\phi X)r)\,\phi Y \\ - \;& (Y\beta)(\phi\, Dr)\eta(X) - \beta\{\phi(\nabla_Y Dr) + \beta\,(Yr)\xi\}\eta(X) \\ - \;& \beta(\phi\, Dr)\{\eta(\nabla_Y X) - \beta\, g(\phi Y, X)\}. \end{aligned} \tag{3.8}$$

Interchanging X and Y in the above, we get

$$\begin{aligned} &\nabla_X \nabla_Y Dr \\ = \;& [X(U(6f_2 + 2f_3)) + (X(6f_2 + 2f_3))(c - r) - (6f_2 + 2f_3)Xr]\,\{Y - \eta(Y)\xi\} \\ + \;& [U(6f_2 + 2f_3) + (6f_2 + 2f_3)(c - r)]\,\{\nabla_X Y - \eta(\nabla_X Y)\xi + \beta g(\phi X, Y)\xi \\ + \;& \beta\eta(Y)\phi X\} + (X\beta)((\phi Y)r)\xi + \beta\, X((\phi Y)r)\,\xi - \beta^2((\phi Y)r)\,\phi X \\ - \;& (X\beta)(\phi\, Dr)\eta(Y) - \beta\{\phi(\nabla_X Dr) + \beta\,(Xr)\xi\}\eta(Y) \\ - \;& \beta(\phi\, Dr)\{\eta(\nabla_X Y) - \beta\, g(\phi X, Y)\}. \end{aligned} \tag{3.9}$$

Again from (3.7), we obtain

$$\begin{aligned} \nabla_{[Y,X]} Dr &= [U(6f_2 + 2f_3) + (6f_2 + 2f_3)(c - r)]\,\{\nabla_Y X - \nabla_X Y - \eta(\nabla_Y X)\xi \\ &+ \eta(\nabla_X Y)\xi\} + \beta((\phi\, \nabla_Y X)r)\xi - \beta((\phi\, \nabla_X Y)r)\xi \\ &- \beta(\phi\, Dr)\eta(\nabla_Y X) + \beta(\phi\, Dr)\eta(\nabla_X Y). \end{aligned} \tag{3.10}$$

From (3.8), (3.9) and (3.10) we get

$$\begin{aligned} &R(Y, X)Dr \\ = \;& [Y(U(6f_2 + 2f_3)) + (Y(6f_2 + 2f_3))(c - r) - (6f_2 + 2f_3)Yr]\,\{X - \eta(X)\xi\} \\ - \;& [X(U(6f_2 + 2f_3)) + (X(6f_2 + 2f_3))(c - r) - (6f_2 + 2f_3)Xr]\,\{Y - \eta(Y)\xi\} \\ + \;& [U(6f_2 + 2f_3) + (6f_2 + 2f_3)(c - r)]\,\{2\beta g(\phi Y, X)\xi + \beta\eta(X)\phi Y - \beta\eta(Y)\phi X\} \\ + \;& (Y\beta)((\phi X)r)\xi - (X\beta)((\phi Y)r)\xi + \beta\, Y((\phi X)r)\,\xi - \beta\, X((\phi Y)r)\,\xi \\ - \;& \beta^2((\phi X)r)\,\phi Y + \beta^2((\phi Y)r)\,\phi X - (Y\beta)(\phi\, Dr)\eta(X) + (X\beta)(\phi\, Dr)\eta(Y) \\ - \;& \beta\{\phi(\nabla_Y Dr) + \beta\,(Yr)\xi\}\eta(X) + \beta\{\phi(\nabla_X Dr) + \beta\,(Xr)\xi\}\eta(Y) \\ + \;& 2\beta^2(\phi\, Dr)g(\phi Y, X) - \beta((\phi\, \nabla_Y X)r)\xi + \beta((\phi\, \nabla_X Y)r)\xi. \end{aligned} \tag{3.11}$$

The above equation gives us

$$\begin{aligned} S(X, Dr) &= -XU(6f_1 + 2f_3) - X(6f_1 + 2f_3)(c - r) + (6f_1 + 2f_3)Xr \\ &- \xi\, U(6f_2 + 2f_3)\,\eta(X) - \xi(6f_2 + 2f_3)(c - r)\eta(X) \\ &- (e_1\beta)(e_2 r)\eta(X) + (e_2\beta)(e_1 r)\eta(X) - 2\beta^2 Xr. \end{aligned} \tag{3.12}$$

Here $\{e_i\}$, $i = 1, 2, 3$ is an orthonormal ϕ-basis with $e_3 = \xi$.

Putting $X = \xi$ in (3.12) we get

$$\begin{aligned} S(\xi, Dr) &= -2[\xi(U(6f_2 + 2f_3)) + (\xi(6f_2 + 2f_3))(c - r)] - (e_1\beta)(e_2 r) \\ &+ (e_2\beta)(e_1 r). \end{aligned} \tag{3.13}$$

Putting $X = \xi$ and $Y = Dr$ in (2.7), we get

$$S(\xi, Dr) = 0.$$

Suppose that $3f_2 + f_3 =$ constant. Then from (3.13), we have

$$\frac{e_1\beta}{e_1 r} = \frac{e_2\beta}{e_2 r}.$$

Since $e_3 r = 0$ and $e_3\beta = 0$, from above we get

$$\frac{e_1\beta}{e_1 r} = \frac{e_2\beta}{e_2 r} = \frac{k_1 e_1\beta + k_2 e_2\beta + k_3 e_3\beta}{k_1 e_1 r + k_2 e_2 r + k_3 e_3 r},$$

where $k_1.k_2, k_3$ are C^∞-functions on M.

If $X = k_1 e_1 + k_2 e_2 + k_3 e_3$ is an arbitrary vector field, we get

$$X\beta = \sigma\, Xr,$$

where $\sigma = \dfrac{e_1\beta}{e_1 r}$ is a C^∞-function. From above we get, $\text{grad}\,\beta = \sigma \text{grad}\, r$.

Hence $\text{grad}\,\beta$ and $\text{grad}\, r$ are linearly dependent. Hence we can state the following:

Theorem 3.1. *If a three dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ with quasi-Sasakian metric admits Yamabe soliton, then grad β and grad r are linearly dependent, provided $f_3 + 3f_2$ is constant.*

From (3.13), we see that

$$S(\xi, Dr) = -2[\xi(U(6f_2 + 2f_3)) + (\xi(6f_2 + 2f_3))(c - r)] - g(D\beta, \phi Dr).$$

If $3f_2 + f_3 =$ constant, the above equation gives $g(D\beta, \phi Dr) = 0$. This helps us to state the following:

Theorem 3.2. *If a three dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ with quasi-Sasakian metric admits Yamabe soliton, then the vector fields grad β and ϕ(grad r) are orthogonal to each other, provided $f_3 + 3f_2$ is constant.*

4. Yamabe solitons on generalized Sasakian space forms with Kenmotsu metric

As $\xi r = 0$, the equation (2.16) gives $g(\nabla_X Dr, \xi) = -\alpha X r$.

As the previous section, we have

$$\begin{aligned} \nabla_X Dr &= [U(6f_2 + 2f_3) + (6f_2 + 2f_3)(c - r)]\,\{X - \eta(X)\xi\} - \alpha(Xr)\xi \\ &- \alpha(Dr)\eta(X). \end{aligned} \tag{4.1}$$

Differentiating the above equation covariantly along Y, we obtain

$$\begin{aligned} &\nabla_Y\nabla_X Dr \\ =\ & [Y(U(6f_2 + 2f_3)) + (Y(6f_2 + 2f_3))(c - r) - (6f_2 + 2f_3)Yr]\,\{X - \eta(X)\xi\} \\ +\ & [U(6f_2 + 2f_3) + (6f_2 + 2f_3)(c - r)]\,\{\nabla_Y X - \eta(\nabla_Y X)\xi - \alpha g(\phi X, \phi Y)\xi \\ -\ & \alpha\eta(X)Y + \alpha\eta(X)\eta(Y)\xi\} - (Y\alpha)(Xr)\xi - \alpha\,(Y(Xr))\,\xi \\ -\ & \alpha^2(Xr)\{Y - \eta(Y)\xi\} - (Y\alpha)(Dr)\eta(X) - \alpha(\nabla_Y Dr)\eta(X) \\ -\ & \alpha(Dr)\{\eta(\nabla_Y X) + \alpha g(\phi X, \phi Y)\}. \end{aligned} \tag{4.2}$$

Interchanging X and Y in the above equation, we see that

$$\begin{aligned} & \nabla_X \nabla_Y Dr \\ = & \left[X(U(6f_2+2f_3)) + (X(6f_2+2f_3))(c-r) - (6f_2+2f_3)Xr\right]\{Y - \eta(Y)\xi\} \\ + & \left[U(6f_2+2f_3) + (6f_2+2f_3)(c-r)\right]\{\nabla_X Y - \eta(\nabla_X Y)\xi - \alpha g(\phi Y, \phi X)\xi \\ - & \alpha\eta(Y)X + \alpha\eta(Y)\eta(X)\xi\} - (X\alpha)(Yr)\xi - \alpha\left(X(Yr)\right)\xi \\ - & \alpha^2(Yr)\{X - \eta(X)\xi\} - (X\alpha)(Dr)\eta(Y) - \alpha(\nabla_X Dr)\eta(Y) \\ - & \alpha(Dr)\{\eta(\nabla_X Y) + \alpha g(\phi Y, \phi X)\}. \end{aligned} \tag{4.3}$$

From (4.1), it follows that

$$\begin{aligned} \nabla_{[Y,X]}Dr &= \left[U(6f_2+2f_3) + (6f_2+2f_3)(c-r)\right]\{\nabla_Y X - \nabla_X Y - \eta(\nabla_Y X)\xi \\ &+ \eta(\nabla_X Y)\xi\} - \alpha([Y,X]r)\xi - \alpha(Dr)\eta(\nabla_Y X) \\ &+ \alpha(Dr)\eta(\nabla_X Y). \end{aligned} \tag{4.4}$$

from (4.2), (4.3) and (4.4), we have

$$\begin{aligned} & R(Y,X)Dr \\ = & \left[Y(U(6f_2+2f_3)) + (Y(6f_2+2f_3))(c-r) - (6f_2+2f_3)Yr\right]\{X - \eta(X)\xi\} \\ - & \left[X(U(6f_2+2f_3)) + (X(6f_2+2f_3))(c-r) - (6f_2+2f_3)Xr\right]\{Y - \eta(Y)\xi\} \\ + & \left[U(6f_2+2f_3) + (6f_2+2f_3)(c-r)\right]\{-\alpha\eta(X)Y + \alpha\eta(Y)X\} \\ - & \alpha^2(Xr)\{Y - \eta(Y)\xi\} + \alpha^2(Yr)\{X - \eta(X)\xi\} - (Y\alpha)(Dr)\eta(X) \\ + & (X\alpha)(Dr)\eta(Y) - (Y\alpha)(Xr)\xi + (X\alpha)(Yr)\xi - \alpha(\nabla_Y Dr)\eta(X) \\ + & \alpha(\nabla_X Dr)\eta(Y). \end{aligned} \tag{4.5}$$

From the above equation

$$\begin{aligned} S(X,Dr) &= -XU(6f_1+2f_3) - X(6f_1+2f_3)(c-r) \\ &+ (6f_1+2f_3)Xr - (\xi U(6f_2+2f_3))\eta(X) \\ &- \xi(6f_2+2f_3)(c-r)\eta(X) - g(D\alpha, Dr)\eta(X) - 2\alpha^2 Xr \\ &- 4\alpha\left[U(6f_2+2f_3) + (6f_2+2f_3)(c-r)\right]\eta(X). \end{aligned} \tag{4.6}$$

Putting $X = \xi$ in the above equation and using $\xi r = 0$ we obtain

$$\begin{aligned} S(\xi,Dr) &= -2[\xi(U(6f_2+2f_3)) + (\xi(6f_2+2f_3))(c-r)] - g(D\alpha, Dr) \\ &- 4\alpha\left[U(6f_2+2f_3) + (6f_2+2f_3)(c-r)\right]. \end{aligned} \tag{4.7}$$

Suppose that $f_3 = -3f_2$. Then $S(\xi, Dr) = -g(D\alpha, Dr)$.
Hence in view of (2.7), $g(D\alpha, Dr) = 0$. Thus, we are in a position to state the following:

Theorem 4.1. *If a three dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ with Kenmotsu metric admits Yamabe soliton, then the vector fields $\operatorname{grad}\alpha$ and $\operatorname{grad} r$ are orthogonal to each other, provided $f_3 = -3f_2$.*

5. Examples

Example 5.1. We consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard co-ordinates of $\mathbb{R}^3$.
Define the almost contact structure (ϕ, ξ, η) on M by

$$\phi(E_1) = -E_2, \quad \phi(E_2) = E_1, \quad \phi(E_3) = 0, \quad \xi = E_3, \quad \eta = dz + y dx,$$

where $E_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}$.
Let g be the Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Here $i, j = 1, 2, 3$.
It is easy to verify that, (ϕ, ξ, η, g) defines an almost contact metric structure on M.
The Riemannian connection ∇ is given by the Koszul formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) \\ &- g([Y, Z], X) + g([Z, X], Y). \end{aligned}$$

By the above formula

$$\begin{array}{ccc} \nabla_{E_1} E_1 = 0, & \nabla_{E_1} E_2 = \frac{1}{2} E_3, & \nabla_{E_1} E_3 = -\frac{1}{2} E_2 \\ \nabla_{E_2} E_1 = -\frac{1}{2} E_3, & \nabla_{E_2} E_2 = 0, & \nabla_{E_2} E_3 = \frac{1}{2} E_1 \\ \nabla_{E_3} E_1 = -\frac{1}{2} E_2, & \nabla_{E_3} E_2 = \frac{1}{2} E_1, & \nabla_{E_3} E_3 = 0. \end{array}$$

Here $(\nabla_X \phi)Y = -\frac{1}{2}(g(X, Y)\xi - \eta(Y)X)$ for all $X, Y \in \chi(M)$. Hence the structure is quasi-Sasakian.
The components of the curvature tensor $R(X, Y)Z$ are

$$\begin{array}{lll} R(E_1, E_2)E_1 = \frac{3}{4}E_2, & R(E_1, E_3)E_1 = -\frac{1}{4}E_3, & R(E_2, E_3)E_1 = 0 \\ R(E_1, E_2)E_2 = -\frac{3}{4}E_1, & R(E_1, E_3)E_2 = 0, & R(E_2, E_3)E_2 = -\frac{1}{4}E_3 \\ R(E_1, E_2)E_3 = 0, & R(E_1, E_3)E_3 = \frac{1}{4}E_1, & R(E_2, E_3)E_3 = \frac{1}{4}E_2. \end{array}$$

From the above components of curvature tensor, we obtain

$$\begin{aligned} & S(E_1, E_1) = -\frac{1}{2}, \quad S(E_2, E_2) = -\frac{1}{2}, \quad S(E_3, E_3) = \frac{1}{2} \\ \text{and} \quad & S(E_1, E_2) = S(E_2, E_3) = S(E_3, E_1) = 0. \end{aligned}$$

The scalar curvature given by $r = -\frac{1}{2}$.

We see that the components Riemannian curvature calculated here satisfy

$$\begin{aligned} R(X, Y)Z &= f\{g(Y, Z)X - g(X, Z)Y\} + \left(\frac{-4f - 3}{12}\right)\{g(X, \phi Z)\phi Y \\ &- g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + \left(f - \frac{1}{4}\right)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned}$$

where f is any C^∞-function on M.

Hence M is a generalized Sasakian space form with the functions

$$f_1 = f, f_2 = \frac{-4f-12}{12}, f_3 = f - \frac{1}{4}.$$

It is seen that $(\pounds_\xi g)(X,Y) = (c-r)g(X,Y) = 0$ for all $X, Y \in \chi(M)$ and $c = -\frac{1}{2}$, the constructed metric is Yamabe soliton.

Example 5.2. Consider $M^3 = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \neq (0,0,0)\}$ with standard Cartesian coordinates (x,y,z). Define the almost contact structure (ϕ, ξ, η) on M^3 by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0, \quad \xi = e_3, \quad \eta = dz,$$

where $e_1 = \dfrac{\partial}{\partial x}, \quad e_2 = \dfrac{\partial}{\partial y}, \quad e_3 = \dfrac{\partial}{\partial z}.$

Let the metric g be defined by

$$g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0, \quad g(e_1,e_1) = g(e_2,e_2) = \exp(2z), \quad g(e_3,e_3) = 1.$$

We see that (ϕ, ξ, η, g) defines an almost contact metric structure on M^3.
Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. By Koszul formula, we have

$$\nabla_{e_1} e_1 = -\exp(2z)e_3, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_3 = e_1.$$

$$\nabla_{e_2} e_1 = 0, \qquad \nabla_{e_2} e_2 = -\exp(2z)e_3, \qquad \nabla_{e_2} e_3 = e_2.$$

$$\nabla_{e_3} e_1 = e_1, \qquad \nabla_{e_3} e_2 = e_2, \qquad \nabla_{e_3} e_3 = 0.$$

We see that $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for all $X, Y \in \chi(M^3)$. Hence the structure is Kenmotsu.

The non-vanishing components of the curvature tensor are

$$R(e_1,e_2)e_1 = \exp(2z)e_2, \qquad R(e_1,e_2)e_2 = \exp(2z)e_1$$

$$R(e_1,e_3)e_1 = \exp(2z)e_3, \qquad R(e_1,e_3)e_3 = -e_1$$

$$R(e_2,e_3)e_2 = \exp(2z)e_3, \qquad R(e_2,e_3)e_3 = -e_2.$$

The non-vanishing components of the Ricci tensor are

$$S(e_1,e_1) = S(e_2,e_2) = -2\exp(2z), \quad S(e_3,e_3) = -2.$$

The scalar curvature is given by, $r = -6$.
We see that

$$\begin{aligned} R(X,Y)Z &= f\{g(Y,Z)X - g(X,Z)Y\} + \frac{-f-1}{3}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + (f+1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

where f is any C^∞-function on M^3. So, M^3 is a generalized Sasakian space form with functions

$$f_1 = f, f_2 = \frac{-f-1}{3}, f_3 = f+1.$$

Also $f_3 = -3f_2$. We choose $U = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$.

We have $(\pounds_U g)(X,Y) = (c-r)g(X,y) = 0$ for all $X, Y \in \chi(M^3)$ and $c = -6$. So, the constructed metric is Yamabe soliton.

Acknowledgments

The second author is financially supported by UGC, Ref. ID. 423044.

References

1. Alegre, P. and Carriazo, A., Structures on generalized Sasakian space forms, *Differential Geom. Appl.,* 26(2008), 656-666.

2. Aubin, T., Equations differentielles non-lineaires et probleme de Yamabe concernant la courbure scalaire, *J. Math. Pure Appl.,* 55(1976), 269-296.

3. De, U. C. and Sarkar, A., On projective curvature tensor of generalized Sasakian space forms, *Quaest Math.,* 33(2010), 245-252.

4. De, U. C. and Sarkar, A., Some results on Generalized Sasakian space forms, *Thai J. of Math.*, 8(2010), 1-10.

5. Deshmukh, S. and Chen, B. Y., A note on Yamabe solitons, *Balkan J. Geom. and Appl.,* 23(2018), 37-43.

6. Hamilton, R. S., The Ricci flow on surfaces, Mathematics and general Relativity, *Contemp. Math.,* 71(1988), 237-262.

7. Hui, S. K. and Sarkar, A., On W_2 curvature tensor of generalized Sasakian space form, *Math. Panonica,* 23(2012), 113-124.

8. Kundu, S., On Yamabe soliton, *Irish Mathematical Society Bulletin*, 77(2016), 51-60.

9. Sarkar, A. and De, U. C., Some curvature properties of generalized Sasakian space forms, *Lobachevskii J. Math.,* 33(2012), 22-27.

10. Sarkar, A. and Sen, M., Locally ϕ-symmetric generalized Sasakian space forms, *Ukrainian Math. J.,* 65(2014), 1588-1597.

11. Sarkar, A. and Sen, M., On ϕ-recurrent generalized Sasakian space forms, *Lobachevskii J. Math.,* 33(2012), 244-248.

12. Schoen, R., Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Diff. Geom.,* 20(1984), 479-495.

13. Sharma, R., A 3-dimensional Sasakian metric as a Yamabe soliton, *Int. J. Geom. Methods in Mod. Phys.,* 09(2012), 1220003, 5pp.

14. Trudinger, S. N., Remarks on the deformation of Riemannian structure on compact manifolds, *Ann. Scu. Norm. Sup. Pisa*, 22(1968), 265-274.

15. Yamabe, H., On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.,* 12(1960), 21-37.

16. Yano, K., Integrals formulas on Riemannian geometry, *Marcel Dekker*, 1970.

A. Sarkar,
Department of Mathematics,
University of Kalyani,
West Bengal,
741235,
India.
E-mail address: avjaj@yahoo.co.in

and

Gour Gopal Biswas,
Department of Mathematics,
University of Kalyani,
West Bengal,
741235,
India.
E-mail address: ggbiswas6@gmail.com