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The Spectral Polynomials of Two Joining Graphs: Splices and Links

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ABSTRACT: Energy of a graph, firstly defined by E. Hückel as the sum of absolute values of the eigenvalues of the adjacency matrix, in other words the sum of absolute values of the roots of the characteristic (spectral) polynomials, is an important sub area of graph theory. Symmetry and regularity are two important and desired properties in many areas including graphs. In many molecular graphs, we have a pointwise symmetry, that is the graph corresponding to the molecule under investigation has two identical subgraphs which are symmetrical at a vertex. Therefore, in this paper, we shall study only the vertex joining graphs. In this article we study the characteristic polynomials of the two kinds of joining graphs called splice and link graphs of some well known graph classes.

Key Words: Joining graphs, Splice, Vertex joining, Link, Edge joining, Spectral polynomial.

Contents

1. Significance of the work

In the last seven decades, graphs have been implemented increasingly to model real life situations to obtain numerical data by mathematical ways which can be commented to obtain physical or chemical information normally obtained as a result of time and money consuming laboratory experiments. There are three main ways of transforming such a case to mathematical language: by means of vertex degrees, matrices or distances. In this work, we give an algebraic method for one of the matrices called the sum-edge characteristic polynomials corresponding to graphs.

2. Introduction

We take $G = (V, E)$ to be a simple connected graph, that is G is a graph with no loops nor multiple edges. We call two vertices u and v of G adjacent if there is an edge e of G connecting u to v. If G has n vertices $v_1, v_2, \cdots, v_n$, we can form an $n \times n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent \\ 0, & otherwise \end{cases}$$

This matrix is called the adjacency matrix of the graph G. As well-known, the eigenvalues $\lambda_1, \cdots, \lambda_n$ of a square $n \times n$ matrix A, which will also be called as the eigenvalues of the graph G, are the roots of the equation $|A - \lambda I_n| = 0$. The polynomial on the left hand side of this equation is called the characteristic (or spectral) polynomial of A (and of the graph G). The set of all eigenvalues of the adjacency matrix A is called the spectrum of the graph G, denoted by $S(G)$. As usual, we denote a complete graph by K_n, a star graph by S_n and a path graph by P_n. The spectrum of these graphs are known in literature, [3] and [8]. For more detailed information about the fundamental topics on graphs and spectrums of some well-known graphs, see [1], [3], [4], [5] and [6].

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One of the methods of studying graphs is to make use of the graph operations. There is a large and increasing number of graph operations such as join, corona, cartesian product, union, composition, concatenation, brick product, etc. In this paper we will give two new graph operations by joining two given graphs by two new methods and shall study their spectral polynomials.

3. Joining graphs (splices and links) and their spectral polynomials

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The classical union of these two graphs denoted by $G_1 \cup G_2$ is a graph $G_1 \cup G_2 = (V(V_1 \cup V_2), E(E_1 \cup E_2))$. In this section we rebuilt two new operations similar to the union. We will study the resulting graphs for $G_1 = G_2 = K_n$, S_n and P_n.

3.1. Joining graphs at a vertex: Splices

Doslic defined a new type of graph operation in [7]:

Definition 3.1. *Let G_1, G_2 be two graphs and let us label two vertices, one in $V(G_1)$ and the other in $V(G_2)$, by v. The* **vertex joining graph at v** *or* **the splice** *of these two graphs is denoted by $G_1 \vee_v G_2$ and obtained by identifying the vertices v of the two graphs. The vertex set of $G_1 \vee_v G_2$ is $V(G_1 \vee_v G_2) = V(G_1) \cup V(G_2)$ and the edge set of $G_1 \vee_v G_2$ is $E(G_1 \vee_v G_2) = E(G_1) \cup E(G_2)$.*

If $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$, then $|V(G_1 \vee_v G_2)| = n_1 + n_2 - 1$, and if $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$, then $|E(G_1 \vee_v G_2)| = m_1 + m_2$.

Naturally, if two graphs G_1, G_2 are not labelled, the vertex at which we join these two graphs can be selected in many different ways. So the vertex joining (splice) graph is not unique if we do not identify the vertex at which we obtain the vertex joining graph. Some properties of splice graphs were studied in [2].

Example 3.2. *Let us have the two graphs G_1, G_2 as in Fig. 1.*

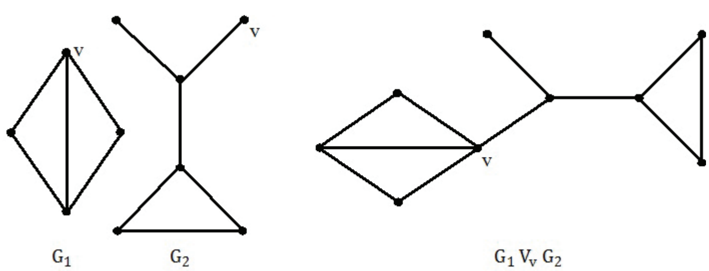

Figure 1: The splice graph of G_1 and G_2.

Vertex joining graph of two graphs is useful in many kinds of calculations with large graphs. We can use the cut vertices of a large graph G to divide G into smaller components and calculate the desired property for these smaller graph pieces to obtain the result for G.

Symmetry and regularity are two important and desired properties in many areas including graphs. In many molecular graphs, we have a pointwise symmetry, that is the graph corresponding to the molecule under investigation has two identical subgraphs which are symmetric at a vertex. Therefore, in this paper, we shall study only the vertex joining graphs $G \vee_v G$, that is we take $G_1 = G_2$, and call the obtained graph as the vertex joining graph at v of G.

Using the well-known characteristic polynomials of K_n, S_n and P_n, we shall formulate the characteristic polynomials of $K_n \vee_v K_n$, $S_n \vee_v S_n$ and $P_n \vee_v P_n$.

We now recall the characteristic polynomials of complete graph K_n, star graph S_n, and path graph P_n which are well-known in literature, see [3]:

$$Pol(G) = \begin{cases} (-1)^n (\lambda+1)^{n-1} (\lambda - n + 1), & if\ G = K_n, \\ (-\lambda)^{n-2} \left(\lambda^2 - n + 1\right), & if\ G = S_n, \\ \sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n-k}{k} \lambda^{n-2k}, & if\ G = P_n,\ n\ is\ even, \\ \sum_{k=0}^{\frac{n-1}{2}} (-1)^{k+1} \binom{n-k}{k} \lambda^{n-2k}, & if\ G = P_n,\ n\ is\ odd. \end{cases} \tag{3.1}$$

We now give the formula for the vertex joining graphs $G \vee_v G$ for $G = K_n, S_n$ and P_n as follows. First we deal with the complete graph K_n:

Theorem 3.3. *The characteristic polynomial $Pol(K_n \vee_v K_n)$ of $K_n \vee_v K_n$ in terms of K_n and K_{n-1} is*

$$\begin{aligned} Pol(K_n \vee_v K_n) &= (1+\lambda)^{n-2} \big[(-1)^{n+1} ((n-2)Pol(K_{n-1}) \\ &\quad +(\lambda - n + 2)Pol(K_n)) - Pol(K_{n-1})\big]. \end{aligned}$$

Proof. Let us consider the vertex joining graph at v of the complete graph K_n. For clearence, we illustrate the $n = 5$ case in Fig. 2:

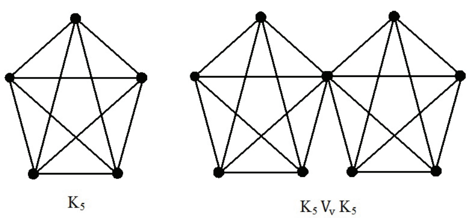

Figure 2: Vertex joining graph of K_5 and K_5.

The adjacency matrix of $K_n \vee_v K_n$ is

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 0 \end{bmatrix}_{(2n-1)\times(2n-1)}.$$

In this adjacency matrix, we can easily see that its elements have some special rule in rows and columns. By dividing this adjacency matrix, the top of left side matrix is K_n and the bottom of right side K_{n-1}.

In literature the characteristic polynomial of $K_n \vee_v K_n$ is given by

$$\begin{aligned} Pol(K_n \vee_v K_n) &= |\lambda I_{2n-1} - A| \\ &= \begin{vmatrix} -\lambda & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & -\lambda & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & -\lambda & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & -\lambda & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & -\lambda & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & -\lambda \end{vmatrix}_{(2n-1)\times(2n-1)}. \end{aligned}$$

By using the elementary column operations $-C_{2n-1} + C_{n+1} \longrightarrow C_{n+1}$, $-C_{2n-1} + C_{n+2} \longrightarrow C_{n+2}$, $-C_{2n-1} + C_{n+3} \longrightarrow C_{n+3}$, $\dots$, $-C_{2n-1} + C_{2n-2} \longrightarrow C_{2n-2}$. This time by using row operations $\frac{1}{\lambda}R_{2n-1}+R_{n+1} \longrightarrow R_{n+1}$, $\frac{1}{\lambda}R_{2n-1}+R_{n+2} \longrightarrow R_{n+2}$, $\frac{1}{\lambda}R_{2n-1}+R_{n+3} \longrightarrow R_{n+3}, \dots,$ $\frac{1}{\lambda}R_{2n-1}+R_{2n-2} \longrightarrow R_{2n-2}$. Taking the $(n+1)-th$, $(n+2)-th$, $\cdots$, $(2n-2)-th$ rows into the paranthesis of $(1+\lambda)/\lambda$, the above determinant becomes

$$= \left(\frac{1+\lambda}{\lambda}\right)^{n-2} \begin{vmatrix} -\lambda & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -\lambda & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & -\lambda & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -\lambda & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1-\lambda & 1 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 & 1-\lambda & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1-\lambda & 0 \\ 1 & 0 & 0 & \dots & 0 & 1+\lambda & 1+\lambda & \dots & 1+\lambda & -\lambda \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Adding the negative of the $(n+1)-th$ row to all the rows below it, and taking the last $n-2$ rows into paranthesis of λ, we get

$$= (1+\lambda)^{n-2} \begin{vmatrix} -\lambda & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -\lambda & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & \dots & -\lambda & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 1-\lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Adding the sum of all the rows between $(n+1)-th$ and $(2n-2)-th$ to the $(n+1)-th$ row, and calculate the obtained determinant according to the last row, we get

$$= (1+\lambda)^{n-2} \left(- \begin{vmatrix} -\lambda & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -\lambda & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & \dots & -\lambda & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & (n-2-\lambda) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}_{(2n-2)\times(2n-2)} \right.$$

$$+(-1)^{3n}\left.\begin{vmatrix}
-\lambda & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 & 1\\
1 & -\lambda & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0\\
1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & 1 & \dots & -\lambda & 0 & 0 & \dots & 0 & 0 & 0\\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & \dots & 0 & 0 & -1 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -1 & 0
\end{vmatrix}_{(2n-2)\times(2n-2)}\right).$$

Adding the columns C_{2n-2}, C_{2n-3}, $\cdots$, C_{n+1} to C_{n+1} in the first determinant and calculating the second determinant according to the last row, we get

$$= (1+\lambda)^{n-2}\left(-\begin{vmatrix}
-\lambda & 1 & \dots & 1 & (n-2) & 1 & 1 & \dots & 1 & 1\\
1 & -\lambda & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
1 & 1 & \dots & -\lambda & 0 & 0 & 0 & \dots & 0 & 0\\
1 & 0 & \dots & 0 & (n-2-\lambda) & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1
\end{vmatrix}_{(2n-2)\times(2n-2)}\right.$$

$$+(-1)^{3n}\left.\begin{vmatrix}
-\lambda & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 & 1\\
1 & -\lambda & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0\\
1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & 1 & \dots & -\lambda & 0 & 0 & \dots & 0 & 0 & 0\\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & \dots & 0 & 0 & -1 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -1 & 0
\end{vmatrix}_{(2n-3)\times(2n-3)}\right).$$

Calculating the first determinant according to the last row, and afterwards, continuing in the same fashion for both determinants until the $(n+2)-th$ row, we obtain

$$= (1+\lambda)^{n-2}\left((-1)^{n-4}\begin{vmatrix}
-\lambda & 1 & 1 & \dots & 1 & 1 & n-2\\
1 & -\lambda & 1 & \dots & 1 & 1 & 0\\
1 & 1 & -\lambda & \dots & 1 & 1 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
1 & 1 & 1 & \dots & -\lambda & 1 & 0\\
1 & 1 & 1 & \dots & 1 & -\lambda & 0\\
1 & 0 & 0 & \dots & 0 & 0 & n-2-\lambda
\end{vmatrix}_{(n+1)\times(n+1)}\right.$$

$$+(-1)^{3n}\left.\begin{vmatrix}
-\lambda & 1 & 1 & \dots & 1 & 1 & 1\\
1 & -\lambda & 1 & \dots & 1 & 1 & 0\\
1 & 1 & -\lambda & \dots & 1 & 1 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
1 & 1 & 1 & \dots & -\lambda & 1 & 0\\
1 & 1 & 1 & \dots & 1 & -\lambda & 0\\
1 & 0 & 0 & \dots & 0 & 0 & 0
\end{vmatrix}_{(n+1)\times(n+1)}\right).$$

Next calculating both determinants first according to the last rows and after according to the last columns, we get

$$= (1+\lambda)^{n-2}\left((-1)^{n+1}(n-2)\begin{vmatrix} -\lambda & 1 & 1 & \dots & 1 \\ 1 & -\lambda & 1 & \dots & 1 \\ 1 & 1 & -\lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & -\lambda \end{vmatrix}_{(n-1)\times(n-1)} \right.$$

$$\left. + (-1)^{n}(n-2-\lambda)\begin{vmatrix} -\lambda & 1 & 1 & \dots & 1 \\ 1 & -\lambda & 1 & \dots & 1 \\ 1 & 1 & -\lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & -\lambda \end{vmatrix}_{n\times n} - \begin{vmatrix} -\lambda & 1 & 1 & \dots & 1 \\ 1 & -\lambda & 1 & \dots & 1 \\ 1 & 1 & -\lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & -\lambda \end{vmatrix}_{(n-1)\times(n-1)} \right).$$

Finally using the statements for $Pol(K_n)$ and $Pol(K_{n-1})$, we get the required result. □

This formula can be stated explicitly in terms of λ as follows:

Theorem 3.4. *The characteristic polynomial of $K_n \vee_v K_n$ is*

$$Pol(K_n \vee_v K_n) \quad = \quad -(\lambda - n + 2)\left(\lambda^2 - (n-2)\lambda - 2n + 2\right)(\lambda+1)^{2n-4}.$$

Proof. Using the formula for $Pol(K_n)$ in Equation 3.1, we obtain the result. □

Secondly, we study the characteristic polynomial of the vertex joining graph at v of the star graph.

Theorem 3.5. *The characteristic polynomial $Pol(S_n \vee_v S_n)$ of $S_n \vee_v S_n$ is*

$$Pol(S_n \vee_v S_n) = \begin{cases} Pol(S_{2n-1}), & if\ v\ is\ the\ central\ vertex. \\ (-\lambda)^{n-3}\left(\lambda^2 - n + 2\right)\left(Pol(S_n) - (-\lambda)^{n-2}\right), & if\ not. \end{cases}$$

Proof. First, let v be the central vertex of S_n and let us consider the vertex joining graph at v of the star graph S_n.

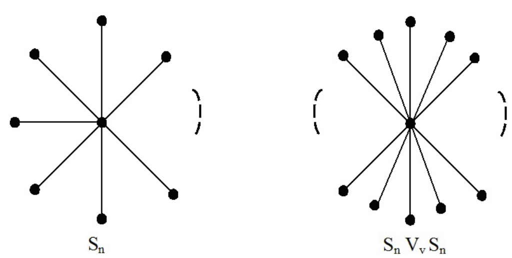

Figure 3: Vertex joining graph of S_n and S_n at the central vertex.

As $S_n \vee_v S_n = S_{2n-1}$, using the Eqn. 3.1 for S_n, we obtain $S_n \vee_v S_n$ as $(-\lambda)^{n-2}\left(\lambda^2 - n + 1\right)$.

Secondly, let v be one of the outer vertices of S_n and let us consider the vertex joining graph at v of the star graph S_n. For clearence, we illustrate the obtained graph in Fig. 4:

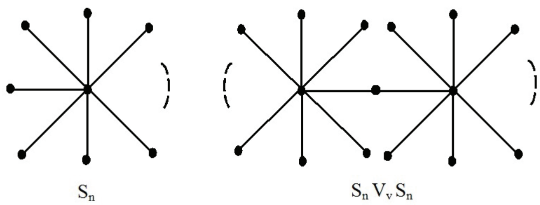

Figure 4: Vertex joining graph of S_n and S_n at an outer vertex.

The adjacency matrix of $S_n \vee_v S_n$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \end{bmatrix}_{(2n-1)\times(2n-1)}.$$

Hence the characteristic polynomial of $S_n \vee_v S_n$ is

$$\begin{aligned} Pol(S_n \vee_v S_n) &= |\lambda I_{2n-1} - A| \\ &= \begin{vmatrix} -\lambda & 1 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & -\lambda & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -\lambda & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & -\lambda & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & -\lambda & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -\lambda & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & -\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & -\lambda \end{vmatrix}_{(2n-1)\times(2n-1)}. \end{aligned}$$

In this determinant, we can easily see that the top left $n \times n$ part is the characteristic polynomial of S_n and the bottom right $(n-1)\times(n-1)$ part is the characteristic polynomial of S_{n-1}. Firstly by using the elementary column operations $\frac{1}{\lambda}C_{2n-1} + C_{n+1} \longrightarrow C_{n+1}$, $\frac{1}{\lambda}C_{2n-1} + C_{n+2} \longrightarrow C_{n+2}$, $\frac{1}{\lambda}C_{2n-1} + C_{n+3} \longrightarrow C_{n+3}$, $\dots$, $\frac{1}{\lambda}C_{2n-1} + C_{2n-2} \longrightarrow C_{2n-2}$, and secondly applying $\frac{1}{\lambda}C_{n+2} + C_{n+1} \longrightarrow C_{n+1}$, $\frac{1}{\lambda}C_{n+3} + C_{n+1} \longrightarrow C_{n+1}$, $\frac{1}{\lambda}C_{n+4} + C_{n+1} \longrightarrow C_{n+1}$, $\dots$, $\frac{1}{\lambda}C_{2n-2} + C_{n+1} \longrightarrow C_{n+1}$, the above determinant becomes

$$= \begin{vmatrix} -\lambda & 1 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & -\lambda & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -\lambda & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & -\lambda & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & \frac{n-2-\lambda^2}{\lambda} & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -\lambda & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -\lambda \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

In the last determinant, it is clearly seen that $a_{ij}=0$ where $i=n+2,\cdots,2n-1$ and $j=1,\cdots,n+1$. These zero terms give a nice way to compute this determinant by using linear algebraic methods. It is equal to the product of its left upper side $n+1\times n+1$ square determinant and right bottom side $n-2\times n-2$ square determinant as follows:

$$= \begin{vmatrix} -\lambda & 1 & 0 & \dots & 0 & 1 \\ 1 & -\lambda & 1 & \dots & 1 & 0 \\ 0 & 1 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & -\lambda & 0 \\ 1 & 0 & 0 & \dots & 0 & \frac{-\lambda^2+(n-3)\lambda+1}{\lambda} \end{vmatrix}_{(n+1)\times(n+1)} \cdot \begin{vmatrix} -\lambda & 0 & \dots & 0 \\ 0 & -\lambda & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\lambda \end{vmatrix}_{(n-2)\times(n-2)}$$

It is clearly seen that the second determinant is diagonal and its value is the product of diagonal entries, $(-\lambda)^{n-2}$. Calculating the first determinant according to the last row, we get

$$= (-\lambda)^{n-2} \left(\frac{-\lambda^2+(n-3)\lambda+1}{\lambda} Pol(S_n) + (-1)^{n+2} \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ -\lambda & 1 & 1 & \dots & 1 & 0 \\ 1 & -\lambda & 0 & \dots & 0 & 0 \\ 1 & 0 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & -\lambda & 0 \end{vmatrix}_{n\times n} \right).$$

From the last determinant, using some elementary row and column operations, we get

$$= (-\lambda)^{n-2} \left(\left(\frac{-\lambda^2+(n-3)\lambda+1}{\lambda}\right) Pol(S_n) + (-1)^{n+2}\left(\frac{-\lambda^2+n-2}{\lambda}\right) \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -\lambda & 0 & \dots & 0 & 0 \\ 1 & 0 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & -\lambda & 0 \end{vmatrix}_{n\times n} \right).$$

Adding the negative of the second row to R_i for $i=3,4,\cdots,n$ and finally calculating the last determinant according to the first row, we get the required result. □

Replacing the formula for the characteristic polynomial of S_n, we obtain the explicit formula for $S_n \vee_v S_n$ as follows:

Corollary 3.6. *Let v be one of the outer vertices of S_n. The characteristic polynomial of $S_n \vee_v S_n$ is*

$$S_n \vee_v S_n \quad = \quad -\lambda^{2n-5}\left(\lambda^2-n+2\right)\left(\lambda^2-n\right).$$

Next we deal with the characteristic polynomial of P_n. In this case, as $P_n \vee_v P_n$ is again a path graph P_{2n-1}, we obtain the following result:

Theorem 3.7. *The characteristic polynomial of $P_n \vee_v P_n$ is*

$$Pol(P_n \vee_v P_n) = Pol(P_{n-1})\left[Pol(P_n) - Pol(P_{n-2})\right]$$

Also it is equal to

$$Pol(P_n \vee_v P_n) = -Pol(P_{n-1})\left[\lambda Pol(P_{n-1}) + 2Pol(P_{n-2})\right].$$

Proof. Let us consider the graph of P_n and $P_n \vee_v P_n$.

Figure 5: Vertex joining graph of P_n and P_n.

Joining two P_n's at one of the end vertices, say v, of both of them, we get a new path graph, which has $2n-1$ vertices. By using Eqn. 3.1, we get the result easily. □

Corollary 3.8. *The characteristic polynomial of $P_n \vee_v P_n$ is*

$$P_n \vee_v P_n \quad = \quad \sum_{k=0}^{n-1}(-1)^{k+1}\binom{2n-1-k}{k}\lambda^{2n-2k-1}.$$

The proof can be done by replacing the formulae for P_n, P_{n-1} and P_{n-2} in Eqn. 3.1.

3.2. Joining graphs by an edge: Links

In this section we recall a new joining operation called the link or edge joining graph which uses an extra edge, actually a bridge, for adding two graphs to each other, see [7].

Definition 3.9. *Let G_1, G_2 be two graphs and let us label two vertices, one in $V(G_1)$ and the other in $V(G_2)$, by v. The* **edge joining graph at v** *or* **the link** *of these two graphs is denoted by $G_1 \vee_v^e G_2$ and obtained by adding a new edge e between the identified vertices v of the two graphs.*

The vertex set of $G_1 \vee_v^e G_2$ is $V(G_1 \vee_v^e G_2) = V(G_1) \cup V(G_2)$ and the edge set of $G_1 \vee_v^e G_2$ is $E(G_1 \vee_v^e G_2) = E(G_1) \cup E(G_2) \cup \{e\}$.

Example 3.10. *Let us have the two graphs G_1, G_2 and their joining graph by a new edge e on v as in Fig. 6.*

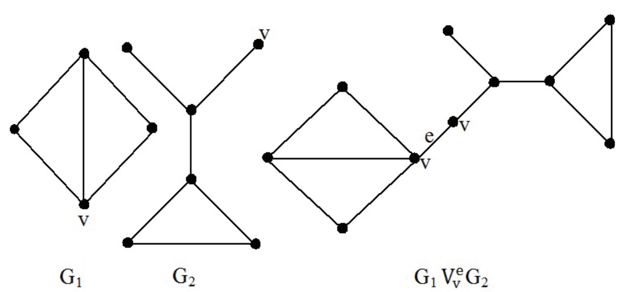

Figure 6: Edge joining graph at v of G_1 and G_2.

Now we are ready to obtain the characteristic polynomials of edge joining graphs at a vertex v for $G_1 = G_2 = K_n$, S_n and P_n:

Theorem 3.11. *The characteristic polynomial of $K_n \vee_v^e K_n$ is*

$$Pol(K_n \vee_v^e K_n) = Pol(K_n) \cdot \frac{(\lambda^2-(n-3)\lambda-(2n-3))(\lambda^2-(n-1)\lambda-1)(\lambda+1)^{n-3}}{(\lambda-n+1)}.$$

Proof. For illustration, the figure of $K_n \vee_v^e K_n$ is given for $n = 5$.

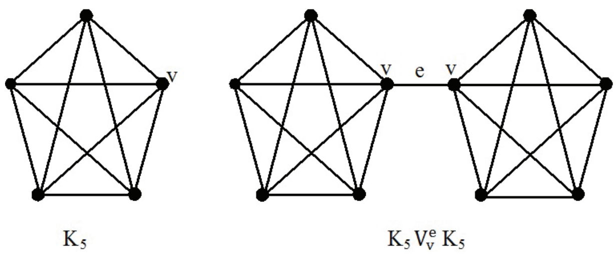

Figure 7: Edge joining graph at v, $K_5 \vee_v^e K_5$.

The adjacency matrix of $K_n \vee_v^e K_n$ is

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}_{2n\times 2n}.$$

In this adjacency matrix, the top left $n \times n$ matrix and the bottom right $n \times n$ matrix are the adjacency matrix of K_n. The characteristic polynomial of $K_n \vee_v^e K_n$ is given by

$$\begin{aligned} Pol(K_n \vee_v^e K_n) &= |A - \lambda I_n| \\ &= \begin{vmatrix} -\lambda & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -\lambda & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -\lambda & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & -\lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & -\lambda & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & -\lambda & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & -\lambda & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & -\lambda \end{vmatrix}_{2n\times 2n}. \end{aligned}$$

Firstly, adding the negative of the $(n+2)-th$ row to all the rows below it, and then adding the sum of all the colums between $(n+3)-th$ and $2n-th$ to the $(n+2)-th$ column, we get

$$= \begin{vmatrix} -\lambda & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -\lambda & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -\lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & -\lambda & (n-1) & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & (n-2-\lambda) & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -(1+\lambda) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -(1+\lambda) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -(1+\lambda) \end{vmatrix}_{2n\times 2n}$$

In the last determinant, $a_{ij}=0$ where $i=n+3,\cdots,2n$ and $j=1\cdots,n+2$. This determinant is equal to the product of its left upper side $(n+2)\times(n+2)$ square determinant and right bottom side $(n-3)\times(n-3)$ square determinant as below:

$$
=\begin{vmatrix}
-\lambda & 1 & 1 & \dots & 1 & 0\\
1 & -\lambda & 1 & \dots & 0 & 0\\
1 & 1 & -\lambda & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 0 & 0 & \dots & -\lambda & n-1\\
0 & 0 & 0 & \dots & 1 & (n-2-\lambda)
\end{vmatrix}_{(n+2)\times(n+2)}
\cdot
\begin{vmatrix}
-(1+\lambda) & 0 & \dots & 0\\
0 & -(1+\lambda) & \dots & 0\\
0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & -(1+\lambda)
\end{vmatrix}_{(n-3)\times(n-3)}
$$

The second determinant is diagonal and its value is the product of diagonal entries, $(-1-\lambda)^{n-3}$. Calculating the first determinant according to the last row, we get

$$
=(-1-\lambda)^{n-3}\left((-1)^{2n+1}\begin{vmatrix}
-\lambda & 1 & 1 & \dots & 1 & 0\\
1 & -\lambda & 1 & \dots & 1 & 0\\
1 & 1 & -\lambda & \dots & 1 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & 1 & \cdots & -\lambda & 0\\
1 & 0 & 0 & \dots & 0 & n-1
\end{vmatrix}_{(n+1)\times(n+1)}\right.
$$

$$
\left.+(n-2-\lambda)\begin{vmatrix}
-\lambda & 1 & 1 & \dots & 1 & 1\\
1 & -\lambda & 1 & \dots & 1 & 0\\
1 & 1 & -\lambda & \dots & 1 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & 1 & \cdots & -\lambda & 0\\
1 & 0 & 0 & \dots & 0 & -\lambda
\end{vmatrix}_{(n+1)\times(n+1)}\right).
$$

Calculating the both determinants according to the last columns, and afterwards, by using suitable editing and the polynomials of K_n and K_{n-1} we get the result.

$\square$

Theorem 3.12. *The characteristic polynomial of $S_n \vee_v^e S_n$ is*

$$Pol(S_n \vee_v^e S_n) = -\lambda^{2n-2} + (-\lambda)^{n-1}\left(\lambda^2-n+1\right)Pol(S_n).$$

Proof. For clearence, we illustrate the edge joining graph at v of S_n, where e joins central vertices of S_n's.

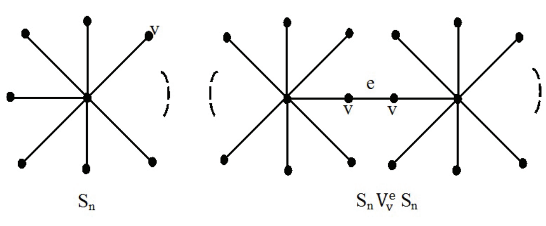

Figure 8: Edge joining graph at v, $S_n \vee_v^e S_n$.

Firstly, getting the characteristic polynomial of $S_n \vee_v^e S_n$, $|S_n \vee_v^e S_n - \lambda I_{2n}|$ and then by using similar operations, we get the result easily. $\square$

Theorem 3.13. *The characteristic polynomial of $P_n \vee_v^e P_n$ is*

$$Pol(P_n \vee_v^e P_n) = -\lambda Pol(P_n)Pol(P_{n-1}) - Pol^2(P_{n-1}) - Pol(P_n)Pol(P_{n-2}).$$

Proof. It is clearly seen at Figure 9 that the edge joining graph at v of P_n's is equal to the graph of P_{2n}.

Figure 9: Edge joining graph at v, $P_n \vee_v^e P_n$.

By Eqn. 3.1, we obtain the result. □

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