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A Multiplicity Result to a Class of Schrödinger Equations with Multi-singular Points

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ABSTRACT: In this paper, using variational method, we study the existence and mutiplicity of the solutions to the following multi-singular critical elliptic problem

$$\begin{cases} -\Delta u - \displaystyle\sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u = f_\lambda(x,u) & x \in \Omega\backslash\{a_1,...,a_k\}, \\ u(x) > 0 & x \in \Omega\backslash\{a_1,...,a_k\}, \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain such that $a_i \in \Omega, i = 1,2,...,k$, for $k \geq 2$ are different points, $0 \leq \mu_i \in \mathbb{R}$. In this class of nonlinear elliptic Dirichlet boundary value problems the combination effects of a sublinear and a superlinear term enable us to establish some existence and multiplicity results.

Key Words: Multiple positive solutions, Multi-singular, Concave-convex terms, Asymptotic behavior, Moser iteration.

Contents

1. Introduction

The existence of standing waves solutions to the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta\psi + V(x)\psi - f(x, |\psi|), \quad \text{in} \ \mathbb{R}^N \times \mathbb{R}_+ \setminus \{0\},$$

has been intensively studied in the last decades. The Schrödinger equation plays a central role in quantum mechanic as it predicts the future behavior of a dynamic system. Indeed, the wave function $\psi(x,t)$ represents the quantum mechanical probability amplitude for a given unit-mass particle to have position x at time t. Such equation appears in several fields of physics, from Bose-Einstein condensates and nonlinear optics to plasma physics (see for instance [10,13] and reference therein).
A Lyapunov-Schmidt type reduction, i.e., a separation of variables of the type $\psi(x,t) = u(x) e^{-i\frac{E}{\hbar}t}$, leads to the following semilinear elliptic equation

$$-\Delta u + V(x) u = f(x,u), \quad \text{in} \ \mathbb{R}^N.$$

With the aid of variational methods, the existence and multiplicity of nontrivial solutions to such problems have been extensively studied in the literature over the last decades. For instance, the existence of positive solutions where the potential V is coercive and f satisfies standard mountain pass assumptions, is well-known after the seminal paper of Rabinowitz [29]. Moreover, in the class of potentials bounded from below, several attempts have been made to find general assumptions on V in order to obtain existence

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and multiplicity results (see for instance [5,6,21,33,31]). In these papers the nonlinearity f is required to satisfy the well-known Ambrosetti-Rabinowitz condition; thus, it is superlinear at infinity. For a sublinear growth of f see also [26]. In the past several decades, the equations, containing the multi-singular inverse square potentials, have been studied extensively. This class of operators arises in nonrelativistic molecular physics. For example, in crystalline matter, the presence of many dipoles leads to considering multi-singular Schrödinger operators of the form

$$-\Delta u-\sum_{i=1}^{k}\frac{\mu_i\,(x-a_i)\,.d_i}{|x-a_i|^3}u,$$

where $k\in\mathbb{N},\ \{a_1,\ \ldots\ ,\ a_k\}\in\mathbb{R}^{kN},\ N\geq 3,\ a_i\neq a_j$ for $i\neq j, \mu_i\in\mathbb{R},\ (d_1,\ \ldots,\ d_k)\in\mathbb{R}^{kN}, \mu_i>0$ and $|d_i|=1$ for any $i=1,\ldots,\ k$ ([19]).
The authors in [28] studied the elliptic equation with a multi-singular inverse square potential where $\mu_i=\mu\left(>\frac{(N-2)^2}{4k}\right)$ on the whole $\mathbb{R}^N$.
Existence and multiplicity of solutions to these problems where $\mu_i=0, 1\leq i\leq k$, i.e.,

$$\begin{cases} -\Delta u=f_\lambda\,(x,u) & x\in\Omega,\\ u(x)=0 & x\in\partial\Omega. \end{cases} \tag{1.1}$$

have been extensively investigated. For example, for the sublinear $f_\lambda=\lambda u^q$, subsolutions and supersolutions yield the existence of a unique positive solution to the problem (1.1) for all $\lambda>0$. While for the sublinear $f_\lambda=\lambda|u|^{q-1}u$, variational methods provide the existence of infinitely many solutions to the problem (1.1)(See [1]). In the case f_λ is superlinear, variational tools, such as min-max arguments could be convenient to investigate the existence and multiplicity of solutions. (See [3,32]).
In recent years, much attention has been paid to the existence of nontrivial solutions to the following problem:

$$\begin{cases} -\Delta u-\mu\frac{u}{|x-a|^2}=\lambda u+|u|^{2^*-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

where $a\in\Omega,\ \mu\in(0,\bar{\mu}),\ \ \bar{\mu}=(\frac{N-2}{2})^2$ and $\lambda\in\mathbb{R}$.
Jannelli [25] considered problem (1.2) and proved that if $0<\mu\leq\bar{\mu}-1$, then the problem (1.2) admits a positive solution for all $\lambda\in(0,\lambda_1(\mu))$. If $\bar{\mu}-1<\mu<\bar{\mu}$, and $\Omega=B_1(0)$, then there exists $\lambda^*\in(0,\lambda_1(\mu))$, such that the problem (1.2) admits a positive solution if and only if $\lambda\in(\lambda^*,\lambda_1(\mu))$, where $\lambda_1(\mu)$ is the first eigenvalue of the positive operator $-\Delta-\frac{\mu}{|x|^2}$ with Dirichlet boundary condition. Cao and Peng in [14] considered problem (1.2) and proved that for $N\geq 7,\ \mu\in[0,\bar{\mu}-4)$, problem (1.2) possesses at least a pair of sign-changing solutions for any $\lambda\in(0,\lambda_1(\mu))$. Cao and Han [11] proved that if $\mu\in[0,\bar{\mu}-(\frac{N+2}{N})^2)$, then problem (1.2) admits a nontrivial solution for all $\lambda>0$.
Other relevant papers see [4,9,17,18,20,22,30], and the references therein. The asymptotic behavior of positive solutions to the problem (1.2) had been studied by Chen in [16], by using Moser's iteration method.
Motivated by this large interest, we study here the existence and multiplicity of weak solutions to the following problem

$$\begin{cases} -\Delta u-\displaystyle\sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}u=f_\lambda\,(x,u) & x\in\Omega\backslash\{a_1,\ldots,a_k\},\\ u(x)>0 & x\in\Omega\backslash\{a_1,\ldots,a_k\},\\ u(x)=0 & x\in\partial\Omega. \end{cases} \tag{1.3}$$

where $\Omega\subset\mathbb{R}^N(N\geq 3)$ is a smooth bounded domain such that $a_i\in\Omega, i=1,2,\ldots,k$ are different points $(k\geq 2)$. $0\leq\mu_i,\ f_\lambda:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\displaystyle\sum_{i=1}^{k}\mu_i<\bar{\mu}:=(\frac{N-2}{2})^2$ which $\bar{\mu}$ is the best constant in the Hardy inequality.
Problem (1.3) has a variational nature; hence, its weak solutions can be found as critical points of a

suitable functional J_λ defined on the Sobolev space $H_0^1(\Omega)$, whose analytic construction is recalled in Section 2.
Thanks to this fact, the main approach is based on the direct methods of calculus of variation [16] and [12]. More precisely, under a suitable condition on the nonlinear term f_λ , we are able to prove the existence of at least one (non-trivial) weak solution to problem (1.3) provided that λ belongs to a precise bounded interval of positive parameters.
The main novelty of this new framework is that, the nonlinear term f_λ is the sum of a sublinear and superlinear term. The combined effects of these two types of nonlinearities change the structure of the solution set. See ([2]). In this paper we investigate

$$f_\lambda = Q(x)\, u^{2^*-1} + \lambda u^{q-1},$$

where $Q(x)$ is a positive bounded function on $\bar{\Omega},\ \lambda > 0,\ 1 \le q < 2$, and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.
The plan of the paper is as follows. Section 2 is devoted to our Preliminaries and main results. Next, in Section 3, Theorem 2.1 and some preparatory results (see Lemma 3.1 and Propositions 3.2 and 3.3) are presented. In Section 4 we will use local minimizer method to establish the existence of the first positive solution u_λ to the problem (1.3). In the last section, we will use the Mountain Pass theorem to establish the existence of the second solution to the problem (1.3).

2. Preliminaries

Let $H_0^1(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\| u \| = \| u \|_{H_0^1(\Omega)} = \left(\int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u^2 \right) dx \right)^{1/2}.$$

By using Hardy inequality [4], this norm is equivalent to the usual norm

$$\left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}.$$

The corresponding energy functional of the problem (1.3) is defined by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega Q(x) \left(u^+\right)^{2^*} dx - \frac{\lambda}{q}\int_\Omega \left(u^+\right)^q dx,$$

where $u^+ = \max\{u, 0\}$. Then $J_\lambda(u)$ is well defined and of class C^1 on $H_0^1(\Omega)$.
The function $u \in H_0^1(\Omega)$ is said to be a weak solution to the problem (1.3), if u satisfies

$$\int_\Omega \left(\nabla u \nabla \nu - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u\nu - f_\lambda\left(x, u^+\right) \nu \right) dx = 0, \tag{2.1}$$

where

$$f_\lambda\left(x, u^+\right) = Q(x)\,(u^+)^{2^*-1} - \lambda (u^+)^{q-1},$$

for any $\nu \in H_0^1(\Omega)$. Then the standard elliptic regularity argument yields that

$$u \in C^2(\Omega \backslash \{a_1, ..., a_k\}) \cap C^1(\bar{\Omega} \backslash \{a_1, ..., a_k\}).$$

For $0 \le \mu_i < \bar{\mu}$ and $a_i \in \Omega, i = 1, 2, ..., k$, define the best constant:

$$S_{\mu_i} := \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla u|^2 - \frac{\mu_i}{|x-a_i|^2} u^2 \right) dx}{\left(\displaystyle\int_\Omega |u|^{2^*} dx \right)^{2/2^*}},$$

which is independent of Ω. (See [20,25]). Let

$$\gamma_i := \sqrt{\bar\mu} + \sqrt{\bar\mu - \mu_i}, \quad \gamma_i' := \sqrt{\bar\mu} - \sqrt{\bar\mu - \mu_i}.$$

Catrina and Wang [15] proved that S_{μ_i} is attained by the function

$$U_{\mu_i, a_i}(x) = \frac{\left(4N(\bar\mu - \mu_i)/(N-2)\right)^{\frac{N-2}{4}}}{\left(|x-a_i|^{\frac{\gamma_i'}{\sqrt{\bar\mu}}} + |x-a_i|^{\frac{\gamma_i}{\sqrt{\bar\mu}}}\right)^{\sqrt{\bar\mu}}},$$

and for all $\epsilon > 0$, the function

$$V^{a_i}_{\mu_i,\epsilon}(x) := \epsilon^{-\frac{2-N}{2}} U_{\mu_i,a_i}(\frac{x}{\epsilon}),$$

solves the equation

$$-\Delta u - \frac{\mu_i}{|x-a_i|^2} u = |u|^{2^*-2} u \ \text{ in } \ \mathbb{R}^N \backslash \{a_i\}.$$

In fact we have

$$\int_\Omega \left(|\nabla V^{a_i}_{\mu_i,\epsilon}|^2 - \mu_i \frac{|V^{a_i}_{\mu_i,\epsilon}|^2}{|x-a_i|^2} \right) dx = \int_\Omega |V^{a_i}_{\mu_i,\epsilon}|^{2^*} dx = (S_{\mu_i})^{\frac{N}{2}}.$$

Without loss of generality, we assume that:

$(\mathcal{H}_1)$ $\quad 0 \le \mu_1 \le \mu_2 \le ... \le \mu_k < \bar\mu \quad$ and $\displaystyle\sum_{i=1}^k \mu_i < \bar\mu$.

$(\mathcal{H}_2)$ There is an integer index l, $0 < l \le k$, such that

$$\min \left\{ \frac{S_{\mu_j}^{\frac{N}{2}}}{Q(a_j)^{\frac{N-2}{2}}};\ 0 < j \le k \right\} = \frac{S_{\mu_l}^{\frac{N}{2}}}{Q(a_l)^{\frac{N-2}{2}}},$$

and

$$Q(x) = Q(l) + o\left(|x-a_l|^2\right) \ \text{ as } x \longrightarrow a_l.$$

$(\mathcal{H}_3)$ There exists an $x_0 \in \Omega$, such that $Q(x_0)$ is a strict local maximum satisfying

$$Q(x_0) = Q_M = \max_{\bar\Omega} Q(x)$$

and

$$Q(x) - Q(x_0) = o\left(|x-x_0|^2\right) \ \text{ as } x \longrightarrow x_0.$$

Moreover

$$\sum_{i=1}^k \frac{\mu_i}{|a_i - x_0|^2} > 0 \ \text{ if } \ x_0 \ne a_i \ \ (1 \le i \le k).$$

$(\mathcal{H}_4)$ $0 < \mu_l \le \bar\mu - 1$ and

$$\sum_{j=1, j\ne l}^k \frac{\mu_i}{|a_i - x_0|^2} > 0,$$

where l is given in $(\mathcal{H}_2)$.
Set

$$S := \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u^2 \right) dx}{\left(\displaystyle\int_\Omega |u|^{2^*} dx \right)^{2/2^*}}.$$

Our main results in this paper are the following:

Theorem 2.1. *Suppose that the conditions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_3)$ *hold , then for any solutions* $u \in C^2(\Omega\backslash\{a_1,...,a_k\}) \cap C^1(\bar{\Omega}\backslash\{a_1,...,a_k\})$ *to the problem* (1.3)*, there exist positive constants* N_1, N_2 *such that*

$$N_1|x-a_i|^s \leq u(x) \leq N_2|x-a_i|^s,$$

for any $x \in B_r(a_i)\backslash\{a_i\}$*, r sufficiently small and* $s = -(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu_i})$.

Theorem 2.2. *Suppose that the conditions* $(\mathcal{H}_1)$*-*$(\mathcal{H}_4)$ *hold, then there exists* $\Lambda > 0$*, such that the problem* (1.3) *has at least two solutions in* $H_0^1(\Omega)$*, for any* $\lambda \in (0,\Lambda)$.

We emphasize that in the present paper the functional J_λ does not satisfy $(P.S.)$ condition, leading to lack of compactness in the embeddings

$$H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega) \ \text{ and } \ H_0^1(\Omega) \hookrightarrow L^2(\Omega, |\, x-a\,|^{-2}\, dx),$$

with $a \in \Omega$.
So the standard variational method is not applicable directly. We use Moser iteration method to prove theorem 2.1 and critical point theorem to prove theorem 2.2.
More preciesely we use local minimizer method and Mountain Pass theorem to establish the existence of the first and second solutions to the problem (1.3).

3. main results

Before giving the proof of theorem 2.1, we introduce a preliminary lemma.

Lemma 3.1. *Let* $u(x) = |x-a_i|^s\nu(x)$ *where* $s = -(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu_i})$*, if* $u \in H_0^1(\Omega)$ *is a solution to the problem* (1.3)*, then* $\nu(x) \in C^2(\Omega\backslash\{a_1,...,a_k\}) \cap C^1(\bar{\Omega}\backslash\{a_1,...,a_k\})$ *and satisfies*

$$\begin{cases} -\operatorname{div}\left(|x-a_i|^{2s}\nabla\nu(x)\right) = Q\,(x)\,|x-a_i|^{2^*s}\nu^{2^*-1} & \\ +\lambda|x-a_i|^{qs}\nu^{q-1} + \displaystyle\sum_{j=1,j\neq i}^{k} \frac{\mu_j}{|x-a_j|^2}|x-a_i|^{2s}\nu & \text{in } ,\Omega\backslash\{a_1,...,a_k\}, \\ \nu(x) > 0 & \text{in } \Omega\backslash\{a_1,...,a_k\}, \\ \nu(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Proof: First note that if u is a solution to the problem (1.3) then, as we mentioned in the introduction, $u \in C^2(\Omega\backslash\{a_1,...,a_k\}) \cap C^1(\bar{\Omega}\backslash\{a_1,...,a_k\})$, and so

$$\frac{-\operatorname{div}\Big(|x-a_i|^{2s}\nabla\nu(x)\Big)}{|x-a_i|^s} = -\Delta u - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}u + \sum_{j=1,j\neq i}^{k}\frac{\mu_j}{|x-a_j|^2}u,$$

where $s = -\Big(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu_i}\Big)$. Now we claim that

$$\frac{\operatorname{div}\Big(|x-a_i|^{2s}\nabla\nu\Big)}{|x-a_i|^s} - \operatorname{div}\Big(\nabla(|x-a_i|^s\nu\Big) - \frac{\mu_i}{|x-a_i|^2}|x-a_i|^s\nu = 0.$$

In fact

$$\begin{aligned} \operatorname{div}\Big(|x-a_i|^{2s}\nabla\nu\Big) &= \nabla\Big(|x-a_i|^{2s}\Big).\nabla\nu + |x-a_i|^{2s}\operatorname{div}\Big(\nabla\nu\Big) \\ &= 2s\big(x-a_i\big)|x-a_i|^{2s-2}.\nabla\nu + |x-a_i|^{2s}\Delta\nu, \end{aligned}$$

and

$$
\begin{aligned}
\operatorname{div}\Big(\nabla\big(|x-a_i|^s\nu\big)\Big) &= \operatorname{div}\Big(s\big(x-a_i\big)|x-a_i|^{s-2}\nu+|x-a_i|^s\nabla\nu\Big)\\
&= s\operatorname{div}\Big(\big(x-a_i\big)|x-a_i|^{s-2}\nu\Big)+\operatorname{div}\Big(|x-a_i|^s\nabla\nu\Big)\\
&= s\nabla\Big(|x-a_i|^{s-2}\nu\Big).\big(x-a_i\big)+s|x-a_i|^{s-2}\nu\operatorname{div}\Big(x-a_i\Big)\\
&\quad+\nabla\Big(|x-a_i|^s\Big).\nabla\nu+|x-a_i|^s\operatorname{div}\Big(\nabla\nu\Big)\\
&= s\big(s-2\big)\nu|x-a_i|^{s-4}\big(x-a_i\big).\big(x-a_i\big)\\
&\quad+s|x-a_i|^{s-2}\nabla\nu.\big(x-a_i\big)+s|x-a_i|^{s-2}\nu N\\
&\quad+s\big(x-a_i\big)|x-a_i|^{s-2}.\nabla\nu+|x-a_i|^s\Delta\nu\\
&= s\big(s-2\big)\nu|x-a_i|^{s-2}+2s|x-a_i|^{s-2}\nabla\nu.\big(x-a_i\big)\\
&\quad+|x-a_i|^s\Delta\nu+s|x-a_i|^{s-2}\nu N.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|x-a_i|^s\left(\operatorname{div}\Big(\nabla\big(|x-a_i|^s\nu\big)\Big)+\frac{\mu_i}{|x-a_i|^2}|x-a_i|^s\nu\right)-\operatorname{div}\Big(|x-a_i|^{2s}\nabla\nu\Big)\\
&=s\big(s-2\big)\nu|x-a_i|^{2s-2}+2s|x-a_i|^{2s-2}\nabla\nu.\big(x-a_i\big)+|x-a_i|^{2s}\Delta\nu\\
&+s|x-a_i|^{2s-2}\nu N+\frac{\mu_i}{|x-a_i|^2}|x-a_i|^{2s}\nu-2s\big(x-a_i\big)|x-a_i|^{2s-2}\nabla\nu\\
&-|x-a_i|^{2s}\Delta\nu=|x-a_i|^{2s-2}\nu\Big(s\big(s-2\big)+sN+\mu_i\Big).
\end{aligned}
$$

Inserting s and $N-2=2\sqrt{\bar\mu}$ the claim is proved.

Proposition 3.2. *If $\nu\in C^2(\Omega\backslash\{a_1,...,a_k\})\cap C^1(\bar\Omega\backslash\{a_1,...,a_k\})$ is positive and satisfies (3.1), then there exists a small number $r_0>0$, such that $a_j\notin B_{r_0}(a_i)$ for $j\neq i$ and*

$$\nu(x)\geq\min_{|x-a_i|=r_0}\nu(x)=C_0>0\quad \textit{for any } x\in B_{r_0}(a_i)\backslash\{a_i\}.$$

Proof: Let

$$\phi(t)=\min_{|x-a_i|=t}\nu(x),\ \ 0<t_1\leq t\leq t_2<r_0,$$

we define a comparison function

$$g(x)=A\mid x-a_i\mid^{-2\sqrt{\bar\mu-\mu_i}}+B,$$

where A and B are such that

$$g(x)=\phi(t_j)\ \ \text{for } |x-a_i|=t_j,\ \ j=1,2.$$

More precisely, we have

$$A=\frac{\phi(t_2)-\phi(t_1)}{t_2^{-2\sqrt{\bar\mu-\mu_i}}-t_1^{-2\sqrt{\bar\mu-\mu_i}}},$$

and

$$B=\frac{\phi(t_2)t_1^{-2\sqrt{\bar{\mu}-\mu_i}}-\phi(t_1)t_2^{-2\sqrt{\bar{\mu}-\mu_i}}}{t_1^{-2\sqrt{\bar{\mu}-\mu_i}}-t_2^{-2\sqrt{\bar{\mu}-\mu_i}}},$$

where

$$\phi(t_1)=At_1^{-2\sqrt{\bar{\mu}-\mu_i}}+B \ \text{ and } \phi(t_2)=At_2^{-2\sqrt{\bar{\mu}-\mu_i}}+B.$$

Since

$$\operatorname{div}(\mid x-a_i\mid^{2s}\nabla\nu(x))\leq 0,\ \forall x\ \in\Omega\backslash\{a_i\},$$

where s as in Lemma 3.1, we have

$$\operatorname{div}(\mid x-a_i\mid^{2s}\nabla(\nu(x)-g(x)))\leq 0.$$

By the choice of A and B, we have

$$\nu(x)\geq g(x)\ \text{ in } \partial\big(B_{t_2}(a_i)\backslash B_{t_1}(a_i)\big).$$

Therefore, by the maximum principle, we obtain

$$\begin{aligned}\nu(x) &\geq g(x)=A\mid x-a_i\mid^{-2\sqrt{\bar{\mu}-\mu_i}}+B\\ &= \frac{\mid x-a_i\mid^{-2\sqrt{\bar{\mu}-\mu_i}}-t_2^{-2\sqrt{\bar{\mu}-\mu_i}}}{t_1^{-2\sqrt{\bar{\mu}-\mu_i}}-t_2^{-2\sqrt{\bar{\mu}-\mu_i}}}\phi(t_1)\\ &\quad +d\frac{t_1^{-2\sqrt{\bar{\mu}-\mu_i}}-\mid x-a_i\mid^{-2\sqrt{\bar{\mu}-\mu_i}}}{t_1^{-2\sqrt{\bar{\mu}-\mu_i}}-t_2^{-2\sqrt{\bar{\mu}-\mu_i}}}\phi(t_2)\\ &\geq \frac{\mid x-a_i\mid^{2\sqrt{\bar{\mu}-\mu_i}}-t_1^{2\sqrt{\bar{\mu}-\mu_i}}}{\mid x-a_i\mid^{2\sqrt{\bar{\mu}-\mu_i}}-t_1^{2\sqrt{\bar{\mu}-\mu_i}}t_2^{-2\sqrt{\bar{\mu}-\mu_i}}\mid x-a_i\mid^{2\sqrt{\bar{\mu}-\mu_i}}}\phi(t_2),\end{aligned}$$

for all $x\in B_{t_2}(a_i)\backslash B_{t_1}(a_i)$. Letting $t_1\to 0$, we get

$$\nu(x)\geq\phi(t_2)=\min_{|x-a_i|=t_2}\nu(x)>0,\ \forall x\in B_{t_2}(a_i)\backslash\{a_i\}.$$

Proposition 3.3. *Let* $\mathcal{H}_3$ *and* $\mathcal{H}_4$ *hold. If*

$$\nu\in C^2\left(\Omega\backslash\{a_1,...,a_k\}\right)\cap C^1\left(\bar{\Omega}\backslash\{a_1,...,a_k\}\right)$$

is positive and satisfies (3.1)*, then* $\nu\in L^\infty(B_r(a_i))$ *for* $r>0$ *small enough.*

Proof: Let $\eta_i\in C_0^\infty\left(B_{r_0}(a_i)\right)$ be a cut-off function in $B_R(a_i)$ with $R<r_0$ and

$$\varphi_i=\eta_i^2\nu\nu_L^{2\gamma},\ \text{ for }\ \gamma,\ L>1\ \text{ and }\ \nu_L=\min\{\nu,L\}\ \text{ for }\ i=1,...,k.$$

Multiply (3.1) by φ_i and integrate, we have

$$\begin{aligned}-\int_\Omega\operatorname{div}\left(|x-a_i|^{2s}\nabla\nu\right)\varphi_i dx &= \int_\Omega Q(x)\,|x-a_i|^{2^*s}\nu^{2^*-1}\varphi_i dx\\ &\quad+\lambda\int_\Omega|x-a_i|^{qs}\nu^{q-1}\varphi_i dx\\ &\quad+\sum_{j=1,j\neq i}^{k}\int_\Omega\frac{\mu_j}{|x-a_j|^2}|x-a_i|^{2s}\nu\varphi_i dx.\end{aligned}$$

After a direct calculation, we deduce that

$$\begin{aligned}\int_\Omega |x-a_i|^{2s}\nabla\nu\nabla\varphi_i dx &= \int_\Omega Q(x)\,|x-a_i|^{2^*s}\nu^{2^*-1}\varphi_i dx\\ &+\lambda\int_\Omega |x-a_i|^{qs}\nu^{q-1}\varphi_i dx\\ &+\sum_{j=1,j\neq i}^{k}\int_\Omega \frac{\mu_j}{|x-a_j|^2}|x-a_i|^{2s}\nu\varphi_i dx.\end{aligned}$$

Note that

$$\nabla\varphi_i = 2\eta_i\nu\nu_L^{2\gamma}\nabla\eta_i + \eta_i^2\nu_L^{2\gamma}\nabla\nu + 2\gamma\eta_i^2\nu_L^{2\gamma}\nabla\nu_L,$$

holds on the set $\{x;\ \ \nu\leq L\}$. Taking $\xi_i(x)=\eta_i\nu\nu_L^{\gamma}$, we get

$$\nabla\xi_i = \nu\nu_L^{\gamma}\nabla\eta_i + \eta_i\nu_L^{\gamma}\nabla\nu + \gamma\eta_i\nu_L^{\gamma}\nabla\nu_L.$$

Then

$$\int_\Omega |x-a_i|^{2s}|\nabla\xi_i|^2 dx \leq C_0(\gamma+1)\int_\Omega |x-a_i|^{2s}\Big(\nu^2\nu_L^{2\gamma}|\nabla\eta_i|^2 + 2\gamma\eta_i^2\nu_L^{2\gamma}\nabla\nu\nabla\nu_L \tag{3.2}$$

$$+2\nu\nu_L^{2\gamma}\eta_i\nabla\nu.\nabla\eta_i + \eta_i^2\nu_L^{2\gamma}\mid\nabla\nu\mid^2\Big)dx \tag{3.3}$$

$$\leq C_0(\gamma+1)\Big(\int_\Omega |x-a_i|^{2s}\nu^2\nu_L^{2\gamma}|\nabla\eta_i|^2 dx \tag{3.4}$$

$$+\int_\Omega Q(x)\,|x-a_i|^{2^*s}\nu^{2^*}\nu_L^{2\gamma}\eta_i^2 dx \tag{3.5}$$

$$+\lambda\int_\Omega |x-a_i|^{qs}\eta_i^2\nu^q\nu_L^{2\gamma}dx \tag{3.6}$$

$$+\sum_{j=1,j\neq i}^{k}\int_\Omega \frac{\mu_j}{|x-a_j|^2}|x-a_i|^{2s}\eta_i^2\nu^2\nu_L^{2\gamma}dx\Big) \tag{3.7}$$

$$= C_0(\gamma+1)(I+II+III+IV). \tag{3.8}$$

By the choice of η_i, and using Proposition 3.2, we have

$$\begin{aligned}III &\leq \lambda\int_{B_R}|x-a_i|^{2s}\xi_i^2 dx\\ &\leq C_1\Big(\int_{B_R}|x-a_i|^{2^*s}\xi_i^{2^*}dx\Big)^{2/2^*}|B_R|^{2/N}.\end{aligned} \tag{3.9}$$

On the other hand

$$\int_\Omega |x-a_i|^{2^*s}\nu^{2^*}\eta_i^2\nu_L^{2\gamma}dx = \int_\Omega |x-a_i|^{2s}\xi_i^2|x-a_i|^{(2^*-2)s}\nu^{2^*-2}dx.$$

Again using Holder inequality, $(\mathcal{H}_3)$ and the properties of η_i, we obtain

$$II \leq Q_M\Big(\int_{B_R}|x-a_i|^{2^*s}\xi_i^{2^*}dx\Big)^{2/2^*}\Big(\int_{B_R}|x-a_i|^{2^*s}\nu^{2^*}dx\Big)^{2/N}. \tag{3.10}$$

In the sequal by the choice of the cut-off function η_i, we have

$$IV \leq C_1 \int_{B_R} |x-a_i|^{2s}\xi_i^2 dx \leq C_1\Big(\int_{B_R} |x-a_i|^{2^*s}\xi_i^{2^*}dx\Big)^{2/2^*}|B_R|^{2/N}, \tag{3.11}$$

which we use Holder inequality. Now let R be small enough such that

$$\Big(\int_{B_R} |x-a_i|^{2^*}\nu^{2^*}dx\Big)^{2/N} < \frac{1}{C(\gamma+1)} \quad \text{and} \quad |B_R|^{2/N} < \frac{1}{C(\gamma+1)}, \tag{3.12}$$

by Caffarelli-Kohn-Nirenberg inequality [15], we derive

$$\Big(\int_{\Omega} |x-a_i|^{2^*s}\xi_i^{2^*}dx\Big)^{2/2^*} \leq \int_{\Omega}|x-a_i|^{2s}|\nabla\xi_i|^2 dx. \tag{3.13}$$

Inserting (3.12), (3.13) in (3.9)-(3.11) and (3.2), we obtain that

$$\int_{\Omega}|x-a_i|^{2s}|\nabla\xi_i|^2 dx \leq C_2(\gamma+1)\Big(\int_{B_R}|x-a_i|^{2s}\nu^2\nu_L^{2\gamma}|\nabla\eta_i|^2dx\Big)$$

and

$$\Big(\int_{\Omega}|x-a_i|^{2^*s}\xi_i^{2^*}dx\Big)^{2/2^*} \leq C_2(\gamma+1)\Big(\int_{B_R}|x-a_i|^{2s}\nu^2\nu_L^{2\gamma}|\nabla\eta_i|^2dx\Big). \tag{3.14}$$

Choosing $\gamma+1 = 2^*/2$ and η_i to be constants near zero and letting i go to infinity, we obtain that

$$\nu \in L^{2^*}(B_R(a_i), |x-a_i|^{2^*s}).$$

Now let η_i be a cut-off function in $B_{r+\rho}$ for r sufficiently small and $r+\rho \leq R$ and such that $|\nabla\eta_i| < 1/\rho$, $\eta_i \equiv 1$ on $B_r(a_i)$. Taking $0 < t < 2^*-2$ and using the Holder inequality, we have

$$\begin{aligned} I &= \int_{B_{r+\rho}} |x-a_i|^{2s}\nu^{2(\gamma+1)}|\nabla\eta_i|^2dx \\ &\leq \frac{C_3}{\rho^2}\int_{B_{r+\rho}}|x-a_i|^{2s}\nu^{2(\gamma+1)}dx \\ &= \frac{C_3}{\rho^2}\int_{B_{r+\rho}}|x-a_i|^{(2+t)s}\nu^{2(\gamma+1)}|x-a_i|^{-ts} \\ &\leq \frac{C_3}{\rho^2}\Big(\int_{B_{r+\rho}}(|x-a_i|^{(2+t)s}\nu^{2(\gamma+1)})^{2^*/(2+t)}dx\Big)^{(2+t)/2^*} \\ &\quad \times\Big(\int_{B_{r+\rho}}(|x-a_i|^{-ts})^{2^*/(2^*-2-t)}dx\Big)^{(2^*-2-t)/2^*} \\ &\leq \frac{C_3}{\rho^2}\Big(\int_{B_{r+\rho}}\big(|x-a_i|^{(2+t)s}\nu^{2(\gamma+1)}\big)^{2^*/(2+t)}dx\Big)^{(2+t)/2^*}. \end{aligned} \tag{3.15}$$

Let

$$\gamma+1 = \mathcal{X}^j, \mathcal{X} = (2+t)/2, \rho = (2R_0)^{-j}, j = 1,2,\ldots\,.$$

Then (3.14) leads us to

$$\Big(\int_{B_{r+\rho}}|x-a_i|^{2^*s}\nu^{\mathcal{X}^j.2^*}dx\Big)^{\frac{2}{2^*}} \leq \frac{C_4(\gamma+1)}{\rho^2}\times\Big(\int_{B_{r+\rho}}|x-a_i|^{2^*s}\nu^{\mathcal{X}^{j-1}.2^*}dx\Big)^{\mathcal{X}2/2^*}. \tag{3.16}$$

Therefore, replacing $\gamma+1$ and ρ by $\mathcal{X}^j$ and $(2R_0)^{-j}$, respectively, we get

$$\begin{aligned} \Big(\int_{B_r}|x-a_i|^{2^*s}\nu^{\mathcal{X}^j.2^*}dx\Big)^{1/(\mathcal{X}^j 2^*)} &\leq C_4^{\sum_{k=1}^{j}(\frac{1}{\mathcal{X}^k})}\mathcal{X}^{\sum_{k=1}^{j}(\frac{k}{2\mathcal{X}^k})}(2R_0)^{\sum_{k=1}^{j}(\frac{k}{\mathcal{X}^k})} \\ &\times \Big(\int_{B_{r+R_0/2}}|x-a_i|^{2^*s}\nu^{2^*}dx\Big)^{(2+t)/(2^*)}. \end{aligned}$$

Since the infinite sum in the right-hand side converges, we obtain that $\nu(x)$ is bounded in $B_r(a_i)$ by letting j go infinity. (See (2.24) in [12]).

Proof of Theorem 2.1:

Proof: By the proposition 3.2 we have

$$\begin{aligned} u(x)=|x-a_i|^s\nu(x) &\geq |x-a_i|^s \min_{|x-a_i|=r_0} \nu(x)=|x-a_i|^s C_i \\ &\geq |x-a_i|^s \min_{i=1,\dots,k} C_i=|x-a_i|^s N, \end{aligned}$$

for any $x \in B_{r_0}(a_i)\backslash\{a_i\}$. On the other hand, by using the proposition (2.2), we have

$$u(x)=|x-a_i|^s\nu(x)\leq M|x-a_i|^s \quad \text{for } x \in B_r(a_i)\backslash\{a_i\},$$

where $r \leq r_0$ is sufficiently small and $M=\max\{\| \nu \|_{L^\infty\left(B_R(a_i)\right)}| \quad 1 \leq i \leq k\}$.

Remark 3.4. *A generalization of Brezis-Kato's theorem [7] can be obtained by choosing $a(x)\equiv 1,\ \lambda=\mu$, in Lemma (2.3) of [12] as following:*
Let Ω_0 be an open-bounded region in $\mathbb{R}^N$ $(N\geq 3)$, $0 \in \Omega_0$, $\mu \geq 0$. Assume that $u \in H^1(\Omega_0)$ satisfies $-\Delta u+\mu\frac{u}{|x|^2}=u$ in Ω_0, in the weak sense of

$$\int_{\Omega_0}\Big(\nabla u.\nabla\varphi+\mu\frac{u\varphi}{|\ x\ |^2}-u\varphi\Big)dx \quad \forall\varphi \in H_0^1(\Omega_0).$$

Then $u \in L^q(\Omega_0)$ for all $1\leq q<\infty$.

4. Existence of the first solution

In this section, we will use the local minimizer method to establish the existence of a positive solution u_λ to the problem (1.3) In order to obtain the minimizer u_λ, our functional J_λ has to satisfy Palais-Smale compactness condition. We recall that a functional I on a Banach Space X satisfies the Palais-Smale condition on the level c (shortly $(P.S)_c$) if any sequence $\{\nu_n\}$ in X such that $I\big(\nu_n\big)\to c$ and $I'\big(\nu_n\big)\to 0$ in X^{-1} as $n\to\infty$, has a convergent (in the norm of X) subsequence.

Lemma 4.1. *Assume that the $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold. The functional J_λ satisfies $(P.S)_c$ condition for all $c<0$.*

Proof: Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(P.S)_c$ sequence. We have

$$J_\lambda(u_n)\to c<0,\ J'_\lambda(u_n)\to 0 \quad \text{as} \quad n\to\infty \tag{4.1}$$

From $(\mathcal{H}_3)$ and (4.1) for n large enough we get

$$-c\geq\frac{-Q_M}{N}\|u_n^+\|_{2^*}^{2^*}+\lambda\left(\frac{-1}{2}+\frac{1}{q}\right)\|u_n^+\|_q^q. \tag{4.2}$$

By Hardy inequality, u_n is bounded in $H_0^1(\Omega)$. Therefore, up to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u_0 &&\text{in } H_0^1(\Omega),\\ &u_n \to u_0 &&\text{a.e. in } \Omega,\\ &u_n \to u_0 &&\text{in } L^t(\Omega),\ \text{for } 1<t<2^*. \end{aligned}$$

It follows from the expression of $\langle J'_\lambda(u_n),\varphi\rangle$ that if u_0 is a weak solution to the problem (1.3), then we have

$$J_\lambda(u_0)=\frac{1}{N}\int_\Omega Q\,(x)\,u_0^{2^*}dx-\lambda(\frac{1}{q}-\frac{1}{2})\int_\Omega u_0^q dx, \tag{4.3}$$

Let

$$\Lambda = \frac{\frac{1}{2N}S^{N/2} + \frac{Q_M}{N}\int_\Omega u_0^{2^*}dx}{\left(\frac{1}{q}-\frac{1}{2}\right)\int_\Omega u_0^q dx}.$$

Then $\Lambda > 0$ and $J_\lambda(u_0) \geq -\frac{1}{2N}S^{N/2}$ for any $\lambda \in (0,\Lambda)$. Now fix $\lambda \in (0,\Lambda)$. We have

$$J_\lambda(u_n) = \frac{1}{2}\int_\Omega \left(|\nabla u_n|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}u_n^2\right)dx - \frac{1}{2^*}\int_\Omega Q(x)\left(u_n^+\right)^{2^*}dx - \frac{\lambda}{q}\int_\Omega \left(u_n^+\right)^q dx.$$

Let $u_n = u_0 + \omega_n$. Since u_0 is a weak solution to the problem (1.3) then $\omega_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. Hence

$$\omega_n \to 0 \text{ strongly in } L^t(\Omega) \text{ for all } 1<t<2^* \text{ and } \int_\Omega \left(\omega_n^+\right)^t dx \to 0. \tag{4.4}$$

Since $u_n \to u_0$ pointwise almost everywhere in Ω, from (4.2) and using the Brezis-Lieb Lemma [8] we have

$$\int_\Omega Q(x)\left(\omega_n^+ + u_0\right)^{2^*}dx = \int_\Omega Q(x)\left(\omega_n^+\right)^{2^*}dx + \int_\Omega Q(x)\, u_0^{2^*}dx + o(1), \tag{4.5}$$

and

$$\int_\Omega \left(\omega_n^+ + u_0\right)^q dx = \int_\Omega \left(\omega_n^+\right)^q dx + \int_\Omega u_0^q dx + o(1). \tag{4.6}$$

From (4.4), (2.1) and Holder inequlity, we have

$$\int_\Omega \left(\nabla\omega_n\nabla u_0 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}\omega_n u_0\right)dx \to 0. \tag{4.7}$$

$$\begin{aligned}
J_\lambda(u_n) &= \frac{1}{2}\int_\Omega \left(|\nabla(\omega_n+u_0)|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}(\omega_n+u_0)^2\right)dx \\
&\quad -\frac{1}{2^*}\left(\int_\Omega Q(x)\left(\omega_n^+\right)^{2^*}dx + \int_\Omega Q(x)\, u_0^{2^*}dx + o(1)\right) \\
&\quad -\frac{\lambda}{q}\left(\int_\Omega \left(\omega_n^+\right)^q dx + \int_\Omega u_0^q dx + o(1)\right) \\
&= \frac{1}{2}\int_\Omega \left(|\nabla\omega_n|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}\omega_n^2\right)dx \\
&\quad + \int_\Omega \left(\nabla\omega_n\nabla u_0 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}\omega_n u_0\right)dx \\
&\quad + \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}u_0^2\right)dx - \frac{1}{2^*}\int_\Omega Q(x)\, u_0^{2^*}dx \\
&\quad - \frac{\lambda}{q}\int_\Omega u_0^q dx - \frac{1}{2^*}\int_\Omega Q(x)\left(\omega_n^+\right)^{2^*} - \frac{\lambda}{q}\int_\Omega \left(\omega_n^+\right)^q dx + o(1),
\end{aligned}$$

where $\omega_n := u_n - u_0$. From (4.4), (4.5), (4.6) and (4.7) we have

$$\begin{aligned}
J_\lambda(u_n) = \;& J_\lambda(u_0) + \tfrac{1}{2}\int_\Omega \left(|\nabla\omega_n|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}\omega_n^2\right) \\
& -\tfrac{1}{2^*}\int_\Omega Q(x)\left(\omega_n^+\right)^{2^*}dx - \frac{\lambda}{q}\int_\Omega \left(\omega_n^+\right)^q + o(1).
\end{aligned} \tag{4.8}$$

So

$$\begin{aligned}\langle J'_\lambda(u_n), u_n\rangle &= \int_\Omega \Big(|\nabla \omega_n|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\omega_n^2\Big)dx - \int_\Omega Q(x)\left(\omega_n^+\right)^{2^*}dx \\ &\quad -\lambda \int_\Omega \left(\omega_n^+\right)^q dx + \int_\Omega \Big(|\nabla u_0|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}u_0^2\Big)dx \\ &\quad - \int_\Omega Q(x)\, u_0^{2^*} dx - \lambda \int u_0^q dx + o(1) \\ &= \int_\Omega \Big(|\nabla \omega_n|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\omega_n^2\Big)dx - \int_\Omega Q(x)\left(\omega_n^+\right)^{2^*}dx + o(1).\end{aligned}$$

We may assume that

$$\int_\Omega \Big(|\nabla \omega_n|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\omega_n^2\Big)dx \to b,$$

and

$$\int_\Omega Q(x)\left(\omega_n^+\right)^{2^*} \to b \geq 0.$$

It follows from the definition of S that

$$\int_\Omega \Big(|\nabla \omega_n|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\omega_n^2\Big)dx \geq S\bigg(\int_\Omega \left(\omega_n^+\right)^{2^*}dx\bigg)^{2/2^*},$$

and so $b \geq S b^{2/2^*}$. Assume $b \neq 0$, then $b \geq S^{N/2}$. From (4.3) and (4.8) we get

$$\begin{aligned}0 > c + o(1) &= J_\lambda(u_0) + \frac{1}{2}\int_\Omega \Big(|\nabla \omega_n|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\omega_n^2\Big)dx \\ &\quad -\frac{1}{2^*}\int_\Omega \left(\omega_n^+\right)^{2^*}dx + o(1) \\ &= J_\lambda(u_0) + \frac{1}{N}b + o(1) \geq \frac{1}{2N}S^{N/2}\end{aligned}$$

But $\frac{1}{2N}S^{N/2} > 0$ and this is a contradiction. So $b = 0$, i.e., $u_n \to u_0$ in $H_0^1(\Omega)$.

Existence of the first positive solution:

Let $\phi \in H_0^1(\Omega)$ such that $\| \phi \| = 1$. Then for $t > 0$, we have

$$J_\lambda(t\phi) = \frac{t^2}{2}\| \phi \|^2 - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)\left(\phi^+\right)^{2^*}dx - \frac{\lambda t^q}{q}\int_\Omega \left(\phi^+\right)^q dx.$$

Using the auxiliary function $f(t) = at^2 - bt^{2^*} - ct^q$, one can obtain

$$\text{there is } t_0 > 0 \text{ such that for } 0 < t < t_0,\ J_\lambda(t\phi) < 0. \tag{4.9}$$

$J_\lambda(u)$ is of class C^1 in $H_0^1(\Omega)$ and bounded from bellow for $\lambda \in (0, \Lambda)$. So $c_\lambda := \inf_{u \in \bar{B}_\rho} J_\lambda(u)$ is a critical value of J_λ and (4.9) implies that

$$c_\lambda := \inf_{u \in \bar{B}_\rho} J_\lambda(u) < 0, \text{ for } 0 < \rho < t_0 \text{ and } \lambda \in (0, \Lambda).$$

Since J_λ satisfies the $(PS)_c$ condition for $c < 0$, thus it can achieve its minimum c_λ at u_λ, i.e, $c_\lambda = J_\lambda(u_\lambda)$. Moreover, u_λ, satisfies the problem (1.3).

5. Existence of the second solution

Due the previous section, u_λ is a local minimizer of functional J_λ. We may assume that it is an isolated minimizer. In this section, we will use the Mountain Pass theorem to establish the existence of the second solution to the problem (1.3) of the form $u = u_\lambda + \nu$, where u_λ is the solution obtained in the previous section and $0 < \nu \in \Omega\backslash\{a_1, ..., a_k\}$. To prove this, we show that the assumption $\nu = 0$ leads to contradiction.
Let $u = u_\lambda + \nu$. The corresponding equation for ν is

$$\begin{aligned} -\Delta\nu - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\nu &= Q(x)\left(u_\lambda+\nu\right)^{2^*-1} \\ &\quad +\lambda\left(u_\lambda+\nu\right)^{q-1} - Q(x)\,u_\lambda^{2^*-1} - \lambda u_\lambda^{q-1}. \end{aligned} \tag{5.1}$$

Define

$$g(x,t) = \begin{cases} Q(x)\left(u_\lambda+t\right)^{2^*-1} + \lambda\left(u_\lambda+t\right)^{q-1} - Q(x)\,u_\lambda^{2^*-1} - \lambda u_\lambda^{q-1} & t \geq 0, \\ 0 & t < 0, \end{cases}$$

and

$$G(\nu) = \int_0^{\nu} g(x,t)\,dt.$$

Then

$$I_\lambda(\nu) = \frac{1}{2}\int_\Omega \Big(|\nabla\nu|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}\nu^2\Big)dx - \int_\Omega G(\nu)\,dx.$$

Clearly there is one-to-one correspondence between critical points of I_λ in $H_0^1(\Omega)$ and weak solutions to the problem 5.1. Being u_λ the critical point of J_λ in $H_0^1(\Omega)$ concluds that $\nu = 0$ be a critical point of I_λ in $H_0^1(\Omega)$.

Lemma 5.1. *$\nu = 0$ is a local minimum of I_λ in $H_0^1(\Omega)$.*

Proof: For any $\nu \in H_0^1(\Omega)$, write $\nu = \nu^+ - \nu^-$, $\nu^\pm = \max\{\pm\nu, 0\}$. We have

$$\begin{aligned} I_\lambda(\nu) &= \frac{1}{2}\int_\Omega \Big(|\nabla(\nu^+ - \nu^-)|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}\left(\nu^+ - \nu^-\right)^2\Big)dx \\ &\quad - \int_\Omega G\left(\nu^+ - \nu^-\right)dx. \end{aligned}$$

Using definition of g, one has

$$\begin{aligned} G\left(\nu^+ - \nu^-\right) &= \int_0^{\nu^+ - \nu^-} g(x,t)\,dt \\ &= \int_0^{\nu^+} g(x,t)\,dt + \int_{\nu^+}^{\nu^+ - \nu^-} g(x,t)\,dt \\ &= \int_0^{\nu^+} g(x,t)\,dt + \int_0^{-\nu^-} g(x,t)\,dt \\ &= \int_0^{\nu^+} g(x,t)\,dt. \end{aligned}$$

Note that

$$\begin{aligned} \|\,\nu\,\|^2 &= \int_\Omega \Big(|\nabla\nu|^2 - \sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}\nu^2\Big)dx \\ &= \|\,\nu^+\,\|^2 + \|\,\nu^-\,\|^2, \end{aligned}$$

which we used the inner product of ν^+ and ν^-. Then

$$\begin{aligned} I_\lambda(\nu) &= \frac{1}{2}\int_\Omega \Big(|\nabla \nu|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\nu^2\Big)dx \\ &\quad -\frac{1}{2^*}\int_\Omega Q(x)\Big(\left(u_\lambda+\nu^+\right)^{2^*} - u_\lambda^{2^*} - 2^* u_\lambda^{2^*-1}\nu^+\Big)dx \\ &\quad -\frac{\lambda}{q}\int_\Omega \Big(\left(u_\lambda+\nu^+\right)^{q} - u_\lambda^{q} - q u_\lambda^{q}\nu^+\Big)dx. \end{aligned}$$

A direct computation shows that

$$\begin{aligned} I_\lambda(\nu) &= \frac{1}{2}\int_\Omega \Big(|\nabla \nu^-|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\nu^{-^2}\Big)dx + J_\lambda(u_\lambda+\nu^+) - J_\lambda(u_\lambda) \\ &= \frac{1}{2}\|\nu^-\|^2 + J_\lambda(u_\lambda+\nu^+) - J_\lambda(u_\lambda). \end{aligned}$$

Since u_λ is a local minimizer of J_λ in $H_0^1(\Omega)$, $J_\lambda(u_\lambda+\nu^+) - J_\lambda(u_\lambda) > 0$ for ϵ small enough hence $I_\lambda(\nu) \geq \frac{1}{2} \parallel \nu^- \parallel^2$ as long as $\parallel \nu \parallel \leq \epsilon$.

We will prove the existence of the second solution to the problem (1.3) by contradiction.

Lemma 5.2. *I_λ satisfies the $(P.S)_c$ condition for any*

$$c < c^* = \frac{1}{N}\min\left\{ \frac{S_{\mu_l}^{N/2}}{Q(a_l)^{\frac{N-2}{2}}},\ \frac{S_0^{N/2}}{Q(M)^{\frac{N-2}{2}}} \right\}.$$

Proof: Let $\{\nu_n\} \subset H_0^1(\Omega)$ be such that

$$I_\lambda(\nu_n) \to c < c^*, \qquad I'_\lambda(\nu_n) \to 0 \ \text{ in } H^{-1}(\Omega).$$

Recall that

$$\begin{aligned} I_\lambda(\nu_n) &= \frac{1}{2}\int_\Omega \Big(|\nabla \nu_n|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}\nu_n^2\Big)dx \\ &\quad -\frac{1}{2^*}\int_\Omega Q(x)\Big(\left(u_\lambda+\nu_n^+\right)^{2^*} - u_\lambda^{2^*} - 2^* u_\lambda^{2^*-1}\nu_n^+\Big)dx \\ &\quad -\frac{\lambda}{q}\int_\Omega \Big(\left(u_\lambda+\nu_n^+\right)^{q} - u_\lambda^{q} - q u_\lambda^{q}\nu_n^+\Big)dx. \end{aligned}$$

From Holder inequality, we have

$$\begin{aligned} \int_\Omega (u_\lambda+\nu_n^+)^q dx &\leq \Big(\int_\Omega \left(|u_\lambda+\nu_n^+|^q\right)^{2/q}dx\Big)^{q/2}\Big(\int_\Omega 1dx\Big)^{(2-q)/2} \\ &\leq C_1\Big(\int_\Omega |u_\lambda+\nu_n^+|^2 dx\Big)^{q/2} \\ &\leq C\Big(\int_\Omega |\nabla(u_\lambda+\nu_n^+)|^2 dx\Big)^{q/2} \\ &= C \parallel u_\lambda+\nu_n^+ \parallel^q \leq C\left(\parallel u_\lambda \parallel + \parallel \nu_n^+ \parallel\right)^q \\ &\leq 2^{q-1}C\left(\parallel u_\lambda \parallel^q + \parallel \nu_n^+ \parallel^q\right). \end{aligned} \tag{5.2}$$

Using Holder inequality, (2.1) and (5.2), one has

$$\begin{aligned}
2^* I_\lambda(\nu_n) - \langle I'_\lambda(\nu_n), u_\lambda + \nu_n\rangle &= \frac{2}{N-2}\|\nu_n\|^2 - \int_\Omega \left(\nabla u_\lambda \nabla \nu_n - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u_\lambda \nu_n\right) dx \\
&+ (2^*+1) \int_\Omega \left(Q(x)\, u_\lambda^{2^*-1}\nu_n^+ + \lambda u_\lambda^q \nu_n^+\right) dx \\
&+ \lambda \left(\frac{q-2^*}{q}\right) \int_\Omega (u_\lambda + \nu_n^+)^q\, dx \\
&+ \int_\Omega Q(x) \left(u_\lambda + \nu_n^+\right)^{2^*-1} \nu_n^- dx + \lambda \int_\Omega \left(u_\lambda + \nu_n^+\right)^{q-1} \nu_n^- dx \\
&+ \int_\Omega Q(x)\, u_\lambda^{2^*-1} \nu_n^- dx + \lambda \int_\Omega u_\lambda^q \nu_n^- dx + \lambda \int_\Omega u_\lambda^{q+1} dx \\
&\geq \frac{2}{N-2}\|\nu_n\|^2 - \lambda \frac{2^*-q}{q} \int_\Omega (u_\lambda + \nu_n^+)^q\, dx \\
&\geq \frac{2}{N-2}\|\nu_n\|^2 - 2^{q-1} C\lambda \frac{2^*-q}{q} \left(\| u_\lambda \|^q + \| \nu_n^+ \|^q\right).
\end{aligned}$$

Thus for n large enough

$$\begin{aligned}
2^* c + 1 \quad &+ o(1) \,\| u_\lambda + \nu_n \| \\
&\geq 2^* I_\lambda(\nu_n) - \langle I'_\lambda(\nu_n), u_\lambda + \nu_n\rangle \\
&\geq \frac{2}{N-2}\|\nu_n\|^2 - 2^{q-1} C\lambda \frac{2^*-q}{q} \left(\| u_\lambda \|^q + \| \nu_n^+ \|^q\right).
\end{aligned} \tag{5.3}$$

From (5.3) we get that $\{\nu_n\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, there exists $\nu_\infty \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&\nu_n \rightharpoonup \nu_\infty \quad \text{weakly in } H_0^1(\Omega), \\
&\nu_n \rightharpoonup \nu_\infty \quad \text{weakly in } L^2(\Omega, |\, x - a_i \,|^2) \text{ for } 1 \leq i \leq k, \\
&\nu_n \rightharpoonup \nu_\infty \quad \text{weakly in } L^{2^*}(\Omega), \\
&\nu_n \to \nu_\infty \quad \text{a.e. in } \Omega, \\
&\nu_n \to \nu_\infty \quad \text{in } L^t(\Omega) \text{ for } 1 < t < 2^*.
\end{aligned} \tag{5.4}$$

Now we will prove $\nu_n \to \nu_\infty$ strongly in $H_0^1(\Omega)$. By Brezis-Lieb Lemma, we obtain that

$$I_\lambda(\nu_n) = \frac{1}{2}\int_\Omega \left(|\nabla \nu_n|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2}\nu_n^2\right) dx - \frac{1}{2^*}\int_\Omega Q(x) \left(\nu_n^+\right)^{2^*} dx + o(1). \tag{5.5}$$

Using the concentration compactness principle [27], there exists a subsequence, still denoted by ν_n, at most countable set $\mathcal{J}$, a set of different points $\{x_j\}_{j\in\mathcal{J}} \subset \Omega \backslash \{a_1,\ a_2,\ \ldots,\ a_k\}$, real numbers $\tilde{\mu}_{x_j},\ \tilde{\nu}_{x_j},\ j \in \mathcal{J}$ and $\tilde{\mu}_{a_i},\ \tilde{\nu}_{a_i},\ \tilde{\gamma}_{a_i},\ (1 \leq i \leq k)$ such that

$$\begin{aligned}
&|\nabla \nu_n|^2 \rightharpoonup d\tilde{\mu} \geq |\nabla \nu_\infty|^2 + \textstyle\sum_{j\in\mathcal{J}} \tilde{\mu}_{x_j}\delta_{x_j} + \sum_{i=1}^k \tilde{\mu}_{a_i}\delta_{a_i}, \\
&|\nu_n|^{2^*} \rightharpoonup d\tilde{\nu} = |\nu_\infty|^{2^*} + \textstyle\sum_{j\in\mathcal{J}} \tilde{\nu}_{x_j}\delta_{x_j} + \sum_{i=1}^k \tilde{\nu}_{a_i}\delta_{a_i}, \\
&\frac{|\nu_n|^2}{|x-a_i|^2} \rightharpoonup d\tilde{\gamma} = \frac{|\nu_\infty|^2}{|x-a_i|^2} + \tilde{\gamma}_{a_i}\delta_{a_i}.
\end{aligned} \tag{5.6}$$

It follows from $I'_\lambda(\nu_n) \to 0$ that ν_∞ is a critical point of I_λ in $H_0^1(\Omega)$. By Sobolev inequalities, we get

$$S_0 \tilde{\nu}_{x_j}^{\frac{2}{2^*}} \leq \tilde{\mu}_{x_j} \text{ for } j \in \mathcal{J} \text{ and } S_{\mu_i} \tilde{\nu}_{a_i}^{\frac{2}{2^*}} \leq \tilde{\mu}_{a_i} - \mu_i \tilde{\gamma}_{a_i},\ 1 \leq i \leq k.$$

Similar to Lemma 3.1 of [12] we can prove that for any $j \in \mathcal{J}$, either $\tilde{\nu}_{x_j} = 0$ or $Q(x_j)\tilde{\nu}_{x_j} \geq \dfrac{S_0^{N/2}}{Q_M^{\frac{N-2}{2}}}$.
Thus $\mathcal{J}$ is finite.
From (5.6) and using the arguments in Lemma 3.1 of [12], one concludes that

$$\begin{aligned} c &= I_\lambda(\nu_n) - \frac{1}{2}\left\langle I'_\lambda(\nu_n),\ \nu_n \right\rangle + o(1) \\ &= \frac{1}{N}\int_\Omega Q(x)\left(\nu_n^+\right)^{2^*} dx + o(1) \\ &= \frac{1}{N}\left(\int_\Omega Q(x)\left(\nu_n^+\right)^{2^*} dx + \sum_{j\in\mathcal{J}} Q(x_j)\tilde{\nu}_{x_j} + \sum_{i=1}^{k} Q(a_i)\tilde{\nu}_{a_i}\right). \end{aligned} \tag{5.7}$$

If there exists a $j \in \mathcal{J}$ such that $\tilde{\nu}_{x_j} \neq 0$, or there is an $i \in \{1,\ 2,\ ...,\ k\}$, such that $\tilde{\nu}_{a_i} \neq 0$, from $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ we deduce that

$$c \geq \frac{1}{N}\min\left\{\frac{S_{\mu_1}^{N/2}}{Q(a_1)^{\frac{N-2}{2}}},\ \frac{S_{\mu_2}^{N/2}}{Q(a_2)^{\frac{N-2}{2}}},\ ...,\ \frac{S_{\mu_k}^{N/2}}{Q(a_k)^{\frac{N-2}{2}}},\ \frac{S_0^{N/2}}{Q_M^{\frac{N-2}{2}}}\right\} = c^*,$$

which contradicts the assumption $c < c^*$. So $\tilde{\nu}_{a_i} = 0$ for every $1 \leq i \leq k$ and we derive that $\nu_n \longrightarrow \nu_\infty$ strongly in $H_0^1(\Omega)$.

Now we show that there is a nonnegative function $\nu_l \in H_0^1(\Omega)$ such that $\sup_{t\geq 0} I_\lambda(t\nu_l) \leq c^*$ which is given in Lemma 5.2. Set

$$u_{\mu,\epsilon}^{a_i}(x) = \varphi(x)V_{\mu,\epsilon}^{a_i}(x) = \epsilon^{-\frac{N-2}{2}}\varphi(x)U_{\mu,a_i}(\frac{x}{\epsilon}),$$

where $a_i \in \Omega,\ 0 < \mu < \bar{\mu}$ and $\varphi \in C_0^\infty(B_{r_0}(a_i))$ satisfying:

$$0 \leq \varphi \leq 1,\ \varphi \equiv 1,\ \forall x \in B_{r_0/2}(a_i).$$

Completion of the proof of Theorem 2.2.

Proposition 5.3. *Suppose that $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ hold. Then I_λ has at least one nonzero critical point.*

Proof: Let

$$\nu = 0 \ \text{ be the uniqe critical point of } \ I_\lambda. \tag{5.8}$$

Define

$$c_\lambda^* = \inf_{h\in\Gamma}\max_{t\in[0,1]} I_\lambda\big(h(t)\big),$$

where

$$\Gamma = \{h \in C([0,1], H_0^1(\Omega));\ h(\nu = 0) = 0, h(1) = tu_{\mu,\epsilon}^{a_i}\}.$$

By Lemma 5.1, $\nu = 0$ is a local minimizer of I_λ moreover $I_\lambda\big(tu_{\mu,\epsilon}^{a_i}\big) \to -\infty$ as $t \to \infty$.
To use the Mountain Pass theorem whenever $c_\lambda^* > 0$ and the Ghoussoub-Preiss version [22] whenever $c_\lambda^* = 0$, that is enough to estimate the Mountain Pass level c_λ^* such that $c_\lambda^* < c^*$, where c^* is given in

Lemma 5.2.
Thanks to $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\frac{S_{\mu_l}^{\frac{N}{2}}}{Q\left(a_l\right)^{\frac{N-2}{2}}}<\frac{S_0^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}},$$

and

$$c^*=\frac{1}{N}\frac{S_{\mu_l}^{\frac{N}{2}}}{Q\left(a_l\right)^{\frac{N-2}{2}}}.$$

Then we get

$$I_\lambda\big(tu_{\mu,\epsilon}^{a_i}\big)=\frac{t^2}{2}\int_\Omega\Big(|\nabla u_{\mu,\epsilon}^{a_i}|^2-\sum_{i=1}^k\frac{\mu_i}{|x-a_i|^2}\big(u_{\mu,\epsilon}^{a_i}\big)^2\Big)dx-\int_\Omega G\big(tu_{\mu,\epsilon}^{a_i}\big)dx.$$

From the elementary inequality

$$(a+b)^p\geq a^p+b^p+pa^{p-1}b,\quad p>1,\ a,b\geq 0,$$

one has

$$g(x,t)\geq Q\left(x\right)(2^*-1)u_\lambda^{2^*-2}t+Q\left(x\right)t^{2^*-1},$$

and so

$$\begin{aligned}G\big(tu_{\mu,\epsilon}^{a_i}\big)&=\int_0^{tu_{\mu,\epsilon}^{a_i}}g(x,t)dt\geq\int_0^{tu_{\mu,\epsilon}^{a_i}}Q\left(x\right)\Big((2^*-1)u_\lambda^{2^*-2}t+t^{2^*-1}\Big)dt\\&=Q\left(x\right)\Big(\tfrac{2^*-1}{2}t^2u_\lambda^{2^*-2}\big(u_{\mu,\epsilon}^{a_i}\big)^2+\tfrac{1}{2^*}t^{2^*}\big(u_{\mu,\epsilon}^{a_i}\big)^{2^*}\Big),\end{aligned}\tag{5.9}$$

Choose the support of $\varphi(x)$ so small such that $\operatorname{supp}\varphi\subset B_{r_0}$. By Theorem 2.1,

$$u_\lambda\geq N_0>0\ \text{ on }\ B_r(a_i)\backslash\{a_i\},$$

then we have

$$\begin{aligned}I_\lambda\big(u_{\mu,\epsilon}^{a_i}\big)&\leq\frac{t^2}{2}\int_\Omega\Big(|\nabla u_{\mu,\epsilon}^{a_i}|^2-\sum_{i=1}^k\frac{\mu_i}{|x-a_i|^2}\big(u_{\mu,\epsilon}^{a_i}\big)^2\Big)dx\\&\quad-\frac{2^*-1}{2}t^2N_0^{2^*-2}\int_\Omega\big(Q\left(x\right)\big(u_{\mu,\epsilon}^{a_i}\big)^2dx-\frac{1}{2^*}t^{2^*}\int_\Omega Q\left(x\right)\big(u_{\mu,\epsilon}^{a_i}\big)^{2^*}dx\\&\leq\frac{t^2}{2}\int_\Omega\Big(|\nabla u_{\mu,\epsilon}^{a_i}|^2-\sum_{i=1}^k\frac{\mu_i}{|x-a_i|^2}\big(u_{\mu,\epsilon}^{a_i}\big)^2\Big)dx\\&\quad-\frac{t^{2^*}}{2^*}\int_\Omega Q\left(x\right)\big(u_{\mu,\epsilon}^{a_i}\big)^{2^*}dx.\end{aligned}$$

Due to the $\mathcal{H}_2$, set $i=l$ and define $\nu_l=u_{\mu_l,\epsilon}^l$. By Lemma (3.3) of [12] we have

$$\sup_{t\geq 0}I_\lambda\left(t\nu_l\right)\leq\frac{S_{\mu_l}^{\frac{N}{2}}}{NQ\left(a_l\right)^{\frac{N-2}{2}}}=c^*\ \text{ for }\ \mu_l<\bar{\mu}-1.$$

Then for $t\geq 0$, we have that $c_\lambda^*\leq\sup_{t\geq 0}I_\lambda\left(t\nu_l\right)\leq c^*$. So for $t>0$ there exists at least one nonzero critical point of Mountain Pass type and this is contradiction by the assumption (5.8).

Remark 5.4. *In Proposition* (4.1) *of* [12] *by taking* $Q(x) \equiv 1$, $\delta = 0$ *and using Ljusternik-Schnirelman theory, we have the Dirichlet problem*

$$\begin{cases} -\Delta u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u = |\, u \,|^{2^*-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a nontrivial solution.

Remark 5.5. *Using Nehari manifold has been proved that the following semilinear elliptic equation:*

$$\begin{cases} -\Delta u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u = |\, u \,|^{2^*-2}\, u + \lambda |\, u \,|^{q-2}\, u & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

has at lest two positive solutions, there are still some interesting problems that we have not answered, that: Are the solutions obtained by Nehari manifold method different from the solutions that we find in this paper?

References

1. A. Ambrosetti, M. Badiale, The dual variational principle and elliptic problems with discontinuous nonlinearities, *J. Math. Anal. Appl.* 140 (1989) 363-373.
2. A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* 122 (1994) 519-543.
3. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and aplications,*J. Funct. Anal.* 14 (1973) 349-381.
4. J.P. Garcia Azorero , I. Peral Alonso, inequalities and some critical elliptic and parabolic problems, *J. Differential Equations.* 144 (1998) 441-476.
5. T. Bartsch, A. Pankov, and Z-Q. Wang, Nonlinear Schrödinger equations with steep potential well*Commun. Contemp. Math*, 3(2001) 549-569.
6. T. Bartsch and Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations*, 20(1995)1725-1741.
7. H. Brezis, T. Kato, Remarks on the Schrödinger operator with singular complex potentials, *J. Math. Pures Appl.* 58 (1979) 137-151.
8. H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc.Amer. Math. Soc.* 88 (1983)486-490.
9. H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, *Comm. Pure Apple. Math.* 36 (1983) 437-478.
10. J. Byeon, Z.Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, *Arch. Ration. Mech. Anal.* 165(4)(2002) 295-316.
11. D.Cao, P. Han, Solutions for semilinear elliptic equations with critical exponents and Hardy potential, *J. Differential Equations.* 205 (2004) 521-537.
12. D. Cao, P. Han, Solutions to critical elliptic equations with multi-singular inverse square potentials, *Differential Equations.* 224 (2006) 332-372.
13. D. Cao, E. S. Noussair, and S. Yan, Multiscale-bump standing waves with a critical frequency for nonlinear Schrödinger equations, *Trans. Amer. Math. Soc*, 360(2008) 3813-3837.
14. D. Cao, S. Peng, A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms, *J. Differential Equations* 193 (2003) 424-434.
15. F. Catrina, Z.Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of external functions, *Comm. Pure Appl. Math.* 54 (2001) 229-258.
16. J. Chen, Multiple positive solutions for a class of nonlinear elliptic equations,*J. Math. Anal.* Appl. 295 (2004) 341-354.
17. E. Egnell, Elliptic boundary value problems with singular coefficients and critical nonlinearities, *Indiana Univ. Math.* J. 38 (1989) 235-251.

18. I. Ekeland, N. Ghoussoub, Selected new aspects of the calculus of variations in the large, *Bull. Amer. Math.*Soc. 39 (2002) 207-265.

19. V. Felli, E. M. Marchini, S. Terracini, On Schrödinger Operators with Multi-singular Inverse-square Anisotropic Potentials *Indiana University Mathematics Journal*.58. 2 (2009) 617-676.

20. A. Ferrero, F. Gazzola, Existence of solutions for singular critical growth semi linear elliptic equations,*J.Differential Equations* . 177 (2001) 494-522.

21. V. Benci and D. Fortunato, Discreteness conditions of the spectrum of Schrödinger operators, *J. Math. Anal. Appl*, 64(3)(1978) 695-700.

22. N. Ghoussoub, D. Preiss, A general Mountain pass principle for locating and classifying critical points, *Ann. Inst. H. Poincaré Anal. Non Linéaire*. 6 (1989) 321-330.

23. N. Ghoussoub, C. Yuan, Multiple solutions for quasilinear PDEs involving critical Sobolev and Hardy exponents, *Trans. Amer. Math. Soc*. 352 (2000) 5703-5743.

24. T.S. Hsu, Multiple positive solutions for semilinear elliptic equations involving multi-singular inverse square potentials and concave-convex nonlinearities, *Nonlin. Anal*. 74 (2011) 3703-3715.

25. E. Jannelli, The role played by space dimension in elliptic critical problems, *J. Differential Equations*. 156 (1999) 407-426.

26. A. Kristály. Multiple solutions of a sublinear Schrödinger equation, *NoDEA Nonlinear Differential Equations Appl*.14(3-4)(2007)291-301.

27. P.L. Lions, The concentration-compactness principle in the calculus of variations: the limit case, *Rev. Mat. Iberoamericana*. 1 (1985) 145-201 45-121.

28. L. Wei, Exact behavior of positive solutions to elliptic equations with multi-singular inverse square potentials, *Discrete and continuous dynamical systems*. 36. 12(2016) 7169-7189.

29. P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys*. 43(1992)270-291.

30. D. Ruiz, M. Willem, Elliptic problems with critical exponents and Hardy potentials, *J. Differential Equations*. 190 (2003) 524-538.

31. W. A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys*. 55(2)(1977)149-162.

32. P. N. Srikanth, Uniqueness of solutions of ninlinear Dirichlet problems, *Differential Integral Equations* .6 (1993) 663-670.

33. M. Willem, Minimax theorems, *Progress in Nonlinear Differential Equations and their Applications, 24. Birkhauser Boston, Inc., Boston, MA*(1996).

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