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The Lack of Polynomial Stability to Mixtures with Memory *

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ABSTRACT: We consider the system modeling a mixture of n materials with memory. We show that the corresponding semigroup is exponentially stable if and only if the imaginary axis is contained in the resolvent set of the infinitesimal generator. In particular this implies the lack of polynomial stability to the corresponding semigroup.

Key Words: Mixture of materials, Materials with memory, Exponential stability, Polynomial stability.

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1. Introduction

Under the theory of non-classical elastic solids we understand certain generalizations of the classical theory of elasticity. The most known non-classical elastic solids are the elastic solids with voids, micropolar elastic solids, nonsimple elastic solids and the mixtures of elastic solids. The theory of mixtures of solids has been widely investigated in the last decades, see for example [5], [6], [8], [9], [13], [14], [15], [26], [27]. In recent years, an increasing interest has been directed to the study of the qualitative properties of solutions related to mixtures composed of two interacting continua. Several results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature [1]- [4], [13], [18]- [25]. In [10] Córdova and Rivera, made a full characterization of the asymptotic behavior of the following mixture model

$$\begin{aligned} R\,U_{tt} - A\,U_{xx} + B\,U_t &= \quad O\,, \quad 0 < x < \ell\,,\ t > 0 \\ U(x,0) &= U_0(x)\,, \ \ 0 \le x \le \ell \\ U_t(x,0) &= U_1(x)\,, \ \ 0 \le x \le \ell \\ U(0,t) = U(\ell,t) &= \quad O\,, \quad t \ge 0\,, \end{aligned}$$

where $R = diag(\rho_1, \cdots, \rho_n)$ and ρ_i denotes the mass density of the $i-$ component of the mixture, $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $B \in \mathbb{R}^{n\times n}$ is a semipositive definite matrix with $rank(B) < n$. They obtained the following result: let us denote by $\mathscr{A} = R^{-1}A$, then the following statements are equivalents

- $\left(e^{\mathcal{A}\,t}\right)_{t\ge 0}$ is exponentially stable.
- $\left(e^{\mathcal{A}\,t}\right)_{t\ge 0}$ is strongly stable.
- $dim\, span\left\{B_j\,,\ B_j\mathscr{A}\,,\ B_j\mathscr{A}^2\,,\ \cdots\,,\ B_j\mathscr{A}^{n-1}\ :\ j = 1,\cdots,n\right\} = n\,,$
  where B_j is $j-$ row vector B.

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In particular this implies the lack of polynomial stability to the corresponding semigroup. Here we study the one dimensional model of a mixture of n of solids with memory interacting continua with reference configuration over $[0,\ell]$. Let us denote by $U^1 := U^1(x_1,t)\,,\, U^2 := U^2(x_2,t)\,, \cdots, U^n := U^n(x_n,t)\,,$ where $x_i \in [0,\ell]$. We assume that the particles under consideration occupy the same position at time $t=0$, so that $x = x_i$, therefore we can assume that

$$U^i : [0,\ell] \times [0,+\infty) \longrightarrow \mathbb{R}\,, \quad \text{for all } \; i = 1,\cdots,n\,.$$

Then, according to the Boltzmann superposition principle, the stress-strain relationship that characterizes viscoelasticity is

$$\sigma_{ij} := a_{ij}\, U_x^j - b_{ij}\int_0^t g(t-\tau)\, U_x^j(\cdot,\tau)\, d\tau\,, \quad \text{for all } \; i\,,\, j = 1,\cdots,n\,,$$

being $g : \mathbb{R} \longrightarrow \mathbb{R}$ the relaxation kernel which accounts for the viscoelastic behavior. The corresponding motion equations are given by

$$\rho_i U_{tt}^i = T_x^i + P^i + F^i\,, \quad \text{for all } \; i = 1,\cdots,n\,, \tag{1.1}$$

where ρ_i denotes the mass density, T^i is the stress contribution of the $i-$ component of the mixture, P^i is the internal body force that depend on the relative displacements $\left(U^1,\cdots,U^n\right)$ and F^i stand for the external forces associated with the constituents $\left(U^i\right)$. The constitutive law we use is

$$\begin{aligned}
T^i &= \sigma_{i1} + \sigma_{i2} + \cdots + \sigma_{in} \\
&= a_{i1}\, U_x^1 - b_{i1}\int_0^t g(t-\tau)\, U_x^1(\cdot,\tau)\, d\tau + a_{i2}\, U_x^2 - b_{i2}\int_0^t g(t-\tau)\, U_x^2(\cdot,\tau)\, d\tau + \cdots + \\
&\qquad a_{in}\, U_x^n - b_{in}\int_0^t g(t-\tau)\, U_x^n(\cdot,\tau)\, d\tau\,, \quad \text{for all } \; i = 1,\cdots,n\,. \qquad (1.2)\\
P^i &= -d_{i1}U^1 - d_{i2}U^2 - \cdots - d_{in}U^n\,, \quad \text{for all } \; i = 1,\cdots,n\,. \qquad (1.3)
\end{aligned}$$

Here we assume that F^i is small such that it can be neglected.
Substitution of relations (1.2)-(1.3) into system (1.1) we get

$$R\, U_{tt} - A\, U_{xx} + N\, U + \int_0^t g(t-\tau)\, B\, U_{xx}(x,\tau)\, d\tau = 0\,, \quad x \in (0,\ell)\,,\; t \in \mathbb{R}^+\,, \tag{1.4}$$

with $U = \left(U^1,\cdots,U^n\right)^T$ and

$$R = \left(\rho_i\delta_{ij}\right)_{n\times n}\,,\; A := \left(a_{ij}\right)_{n\times n}\,,\; N := \left(d_{ij}\right)_{n\times n}\,,\; B := \left(b_{ij}\right)_{n\times n}\,,$$

where δ_{ij} is the Kroneckers delta, A is a positive definite (real) matrix, N a semipositive definite (real) matrix and B a semipositive definite (real) matrix.
The initial conditions are given by

$$U(x,0) = U_0(x)\,, \quad U_t(x,0) = U_1(x)\,, \quad x \in (0,\ell)\,. \tag{1.5}$$

Finally, we consider Dirichlet boundary conditions

$$U(0,t) = U(\ell,t) = 0\,, \quad t \in \mathbb{R}^+\,. \tag{1.6}$$

Therefore the dissipative mechanism is reflected by the rank of the matrix B. If $rank(B) = 0$, then the system (1.4) is conservative. Here the question is, what happen in case of

$$0 < rank(B) \leq n\,.$$

Is it possible that the above system is exponentially stable? or polynomially stable? or there exists oscillating solutions? An important result of this paper is to show, when $rank(B) = n$ (fully dissipative), that the solution semigroup is exponentially stable for all values of the structural parameters. On the other hand, the main result of this paper is to show, when $0 < rank(B) < n$, that the semigroup associated to (1.4)-(1.6) is exponentially stable if and only if

$$dim\, span\left\{B_j^{1/2}\,,\, B_j^{1/2}\mathcal{D}\,,\, B_j^{1/2}\mathcal{D}^2\,,\,\cdots\,,\, B_j^{1/2}\mathcal{D}^{n-1}\,:\, j=1,\cdots,n\right\} = n\,,$$

where $B_j^{1/2}$ is the $j-$ row vector of $B^{1/2}$ and $\mathcal{D} = R^{-1}\left(A - B\displaystyle\int_0^{+\infty} g(s)\,ds\right)$.
Moreover we prove that the above system never is polynomially stable. That is, we show that if the system is not exponentially stable then there exists oscillating solutions. In particular our result implies in the corresponding semigroup is exponential stable if and only if it is strongly stable (as in the finite dimensional case). We believe that this property holds because the system has only second order coupling terms.
To formulate system (1.4) in a history space setting, we follow Dafermos [11] and Fabrizio [12]. For the sake of simplicity, we assume that the past history of U up to 0 satisfies a homogeneous Dirichlet boundary condition, that is

$$U(0,t) = 0 = U(\ell,t)\,,\;\; t\in\mathbb{R}^-\,.$$

Then we introduce the auxiliary variable, $\eta\,,$ which is defined by

$$\eta(x,t,s) = B^{1/2}U(x,t) - B^{1/2}U(x,t-s)\,,\;\; x\in(0,\ell)\,,\; t\in\mathbb{R}^+\,,\; s\in\mathbb{R}^+\,, \tag{1.7}$$

where $\eta = \left(\eta^1,\cdots,\eta^n\right)$. One can easily check that η satisfies the first-order linear evolution equation (this is as a supplementary equation to be added)

$$\eta_t(x,t,s) + \eta_s(x,t,s) = B^{1/2}U_t(x,t)\,,\;\; x\in(0,\ell)\,,\; t\in\mathbb{R}^+\,,\; s\in\mathbb{R}^+\,,$$

along with the boundary conditions

$$\begin{aligned} \eta(x,t,0) = &\;\; 0\,, \quad x\in(0,\ell)\,,\; t\in\mathbb{R}^+ \\ \eta(0,t,s) = &\;\; 0\,, \quad t\in\mathbb{R}^+\,,\; s\in\mathbb{R}^+ \\ \eta(\ell,t,s) = &\;\; 0\,, \quad t\in\mathbb{R}^+\,,\; s\in\mathbb{R}^+\,, \end{aligned}$$

and the initial condition

$$\eta(x,0,s) := \eta_0(x,s) = B^{1/2}U_0(x) - B^{1/2}U(x,-s)\,,\;\; x\in(0,\ell)\,,\; s\in\mathbb{R}^+\,.$$

We assume the following set of hypotheses about relaxation kernel:

- (•) $g\in C^1(0,+\infty)\cap L^1(0,+\infty)\,.$ (1.8)
- (•) $0 < g(0^+) := \lim\limits_{s\to 0^+} g(s) < \infty\,.$ (1.9)
- (•) $g(s) > 0$ and $g'(s) < 0\,,\;\; \forall\, s\in(0,+\infty)\,.$ (1.10)
- (•) $\left(A - B\displaystyle\int_0^{+\infty} g(s)\,ds\right)$ is a positive definite matrix. (1.11)
- (•) There is a constant $\kappa > 0$ such that $g'(s) \le -\kappa\, g(s)\,,\;\; \forall\, s\in(0,+\infty)\,.$ (1.12)

So that the original problem (1.4)-(1.6) turns into the the following (equivalent) initial and boundary value problem:
Problem. Find the solution (U,η) to the system

$$R\,U_{tt} - \mathcal{C}\,U_{xx} + N\,U - \int_0^{+\infty} g(s)\,B^{1/2}\eta_{xx}\,ds = \;\; 0 \;\;\; ,\;\; x\in(0,\ell)\,,\; t\in\mathbb{R}^+ \tag{1.13}$$

$$\eta_t + \eta_s = B^{1/2}U_t\,,\;\; x\in(0,\ell)\,, t\in\mathbb{R}^+\,, s\in\mathbb{R}^+ \tag{1.14}$$

which satisfies the initial conditions

$$\begin{aligned} U(x,0) = & \quad U_0(x) & , \; x \in (0,\ell) \\ U_t(x,0) = & \quad U_1(x) & , \; x \in (0,\ell) \\ \eta(x,0,s) := \eta_0(x,s) = & \quad B^{1/2}U_0(x) - B^{1/2}U(x,-s) & , \; x \in (0,\ell)\,, \; s \in \mathbb{R}^+ \end{aligned} \tag{1.15}$$

and the boundary conditions

$$\begin{aligned} U(0,t) = U(\ell,t) = & \quad 0\,, \quad t \in \mathbb{R}^+ \\ \eta(x,t,0) = & \quad 0\,, \quad x \in (0,\ell)\,, \; t \in \mathbb{R}^+ \\ \eta(0,t,s) = \eta(\ell,t,s) = & \quad 0\,, \quad t \in \mathbb{R}^+\,, \; s \in \mathbb{R}^+\,, \end{aligned} \tag{1.16}$$

being $\mathcal{C} = (c_{ij}) \in \mathbb{R}^{n\times n}$ such that $\mathcal{C} = A - B\displaystyle\int_0^{+\infty} g(s)\,ds$.

This paper is organized as follows. In section 2 we establish the well posedness of the system. Finally, in Section 3 we prove the exponential stability when $rank(B) = n$ and the equivalence between the strong and uniform stability when $rank(B) < n$.

2. Semigroup formulation

Theorem 2.1. *Let $A \in M_n$ be Hermitian and positive semidefinite and let $\kappa \in \mathbb{N}\backslash\{1\}$. Then*

- *There is a unique positive semidefinite matrix $B \in M_n$ such that $B^\kappa = A$.*
- *There is a polynomial $p \in \mathbb{R}[s]$ such that $B = p(A)$. Consequently, B commutes with any matrix that commute with A.*
- $rank(A) = rank(B)$.
- *B is real if A is real.*

Proof. See [16], pag. 439. □

Theorem 2.2. *Let $\mathcal{A}\,,\,\mathcal{B} \in \mathbb{F}^{n\times n}$ be positive semidefinite (definite) matrix, then any eigenvalue of $\mathcal{A}\,\mathcal{B}$ is nonnegative (positive).*

Proof. See [7], pag. 424. □

Other important tool we use is the characterization of the exponential stability of a C_0- semigroup was obtained by Huang [17] and Pruss [23] independently. Here we use the version due to Pruss.

Theorem 2.3. *Let $\mathcal{S}_{\mathcal{A}}(t)$ be a C_0- semigroup of contractions of linear operators on Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Then $\mathcal{S}_{\mathcal{A}}(t)$ is exponentially stable if and only if*

$$i\,\mathbb{R} \subset \rho(\mathcal{A}) \quad \textit{and} \quad \limsup_{|\lambda|\longrightarrow+\infty} \left\|(i\,\lambda\,I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty\,.$$

Proof. See [23]. □

From now on we use the semigroup theory to show the well posedness as well as the asymptotic properties. To do that let us introduce the phase space

$$\mathcal{H} = \left[H_0^1(0,\ell)\right]^n \times \left[L^2(0,\ell)\right]^n \times L_g^2\left(0,+\infty;\left[H_0^1(0,\ell)\right]^n\right),$$

that is a Hilbert space endowed with the inner product

$$\left(\widetilde{\mathcal{U}},\mathcal{U}\right)_{\mathcal{H}} = \int_0^\ell U_x^*\mathcal{C}\widetilde{U}_x dx + \int_0^\ell U^*N\widetilde{U}\,dx + \int_0^\ell V^*R\widetilde{V}\,dx + \int_0^{+\infty} g(s)\int_0^\ell \eta_x^*\widetilde{\eta}_x dx\,ds\,,$$

for all $\mathcal{U}=(U,V,\eta)\,,\,\widetilde{\mathcal{U}}=\left(\widetilde{U},\widetilde{V},\widetilde{\eta}\right)\in\mathcal{H}\,.$
Let us introduce the operator $\mathcal{A}$ given by

$$\mathcal{A}\,\mathcal{U}=\left(\begin{array}{c} V \\ R^{-1}\mathcal{C}\,U_{xx}-R^{-1}N\,U+\displaystyle\int_0^{+\infty}g(s)\,R^{-1}B^{1/2}\eta_{xx}ds \\ B^{1/2}V-\eta_s \end{array}\right),\quad \mathcal{U}=(U,V,\eta)$$

with domain

$$\mathcal{D}(\mathcal{A})=\Big\{\mathcal{U}=(U,V,\eta)\in\mathcal{H}:\left(\mathcal{C}\,U+\int_0^{+\infty}g(s)\,B^{1/2}\eta\,ds\right)\in\left[H^2(0,\ell)\right]^n,$$
$$V\in\left[H_0^1(0,\ell)\right]^n,\ \eta(x,t,0)=0\quad\text{and}\quad\eta_s\in L^2_g\left(0,+\infty;\left[H_0^1(0,\ell)\right]^n\right)\Big\}.$$

Under this conditions the initial-boundary value problem can be rewritten as the linear evolution equation in $\mathcal{H}$

$$\frac{d}{dt}\,\mathcal{U}=\mathcal{A}\,\mathcal{U}\,,\quad \mathcal{U}(0)=\mathcal{U}_0\,, \tag{2.1}$$

where $\mathcal{U}=(U,V,\eta)\in\mathcal{D}(\mathcal{A})$ and $\mathcal{U}_0=(U_0,U_1,\eta_0)\in\mathcal{H}\,.$

Theorem 2.4. *Assume that the relaxation kernel g satisfy conditions (1.8)-(1.12). Then $\mathcal{A}$ is the infinitesimal generator of a C_0- semigroup $\mathcal{S}_{\mathcal{A}}(t)=e^{\mathcal{A}\,t}$ of contractions on $\mathcal{H}$.*

Proof. We first show that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Indeed, note that

$$\left[C_0^\infty(0,\ell)\right]^n\times\left[C_0^\infty(0,\ell)\right]^n\times W_g^1\left(\mathbb{R}_+;\left[C_0^\infty(0,\ell)\right]^n\right)\subset\mathcal{D}(\mathcal{A})\,,$$

where

$$W_g^1\left(\mathbb{R}_+;\left[C_0^\infty(0,\ell)\right]^n\right):=\left\{\eta\in L_g^2\left(0,+\infty;\left[C_0^\infty(0,\ell)\right]^n\right):\eta_s\in L_g^2\left(0,+\infty;\left[C_0^\infty(0,\ell)\right]^n\right)\right\}.$$

Thus we conclude the density. Further, the operator $\mathcal{A}$ is dissipative, that is

$$Re\,(\mathcal{A}\,\mathcal{U},\mathcal{U})_{\mathcal{H}}=\frac{1}{2}\int_0^{+\infty}g'(s)\int_0^\ell\eta_x^*\eta_x\,dx\,ds\le 0\,,\quad\text{for all }\ \mathcal{U}\in\mathcal{D}(\mathcal{A})\,. \tag{2.2}$$

Therefore we only need to show that $0\in\rho(\mathcal{A})$ (See Liu and Zheng [19]). Let $\mathcal{F}=\left(\widetilde{U},\widetilde{V},\widetilde{\eta}\right)\in\mathcal{H}$ and consider the equation $\mathcal{A}\,\mathcal{U}=\mathcal{F}$ which, written in components, reads

$$V=\widetilde{U} \tag{2.3}$$

$$\mathcal{C}\,U_{xx}-N\,U+\int_0^{+\infty}g(s)\,B^{1/2}\eta_{xx}ds=R\,\widetilde{V} \tag{2.4}$$

$$B^{1/2}V-\eta_s=\widetilde{\eta}\,. \tag{2.5}$$

From (2.3) and (2.5) we have that

$$V=\widetilde{U}\in\left[H_0^1(0,\ell)\right]^n\quad\text{and}\quad\eta(x,t,s)=s\,B^{1/2}\widetilde{U}(x,t)-\int_0^s\widetilde{\eta}(x,t,\tau)\,d\tau\,.$$

Also, it follows from hypothesis (1.12) the following inequality

$$\int_0^{+\infty}g(s)\int_0^\ell\eta_x^*\eta_x dx\,ds\le\frac{4}{\kappa^2}\int_0^{+\infty}g(s)\int_0^\ell\eta_{xs}^*\eta_{xs}dx\,. \tag{2.6}$$

Thus we conclude that $\eta\in L_g^2\left(0,+\infty;\left[H_0^1(0,\ell)\right]^n\right)$. Further, by means of Lax-Milgram theorem, the elliptic problem (2.4) admits a unique (weak) solution $U\in\left[H_0^1(0,\ell)\right]^n$. Finally, from (2.2)-(2.5) we have that there exists a constant $C>0$, which does not depend on $\mathcal{U}$ and $\mathcal{F}$, such that $\|\mathcal{U}\|_{\mathcal{H}}\le C\,\|\mathcal{F}\|_{\mathcal{H}}$. Thus $0\in\rho(\mathcal{A})$. □

3. On the Stability of Semigroup

For simplicity, we agree to denote by

$$\|U\|^2_{[H_0^1]^n} = \int_0^\ell U_x^* U_x dx\,, \quad \|V\|^2_{[L^2]^n} = \int_0^\ell V^* V\, dx \quad \text{and}$$
$$\|\eta\|^2_{[L_g^2]^n} = \int_0^{+\infty} g(s) \int_0^\ell \eta_x^* \eta_x dx\, ds\,,$$

for all $U \in \left[H_0^1(0,\ell)\right]^n$, $V \in \left[L^2(0,\ell)\right]^n$ and $\eta \in L_g^2\left(0,+\infty; \left[H_0^1(0,\ell)\right]^n\right)$.
Consider the resolvent equation

$$(i\,\lambda\, I - \mathcal{A})\, \mathcal{U} = \mathcal{F}\,, \tag{3.1}$$

which written in components, reads

$$i\,\lambda U - V = \widetilde{U}\,, \tag{3.2}$$
$$i\,\lambda\, R\, V - \mathcal{C}\, U_{xx} + N\, U - \int_0^{+\infty} g(s)\, B^{1/2} \eta_{xx}\, ds = R\, \widetilde{V}\,, \tag{3.3}$$
$$i\,\lambda\, \eta - B^{1/2} V + \eta_s = \widetilde{\eta}\,, \tag{3.4}$$

where $\mathcal{F} = \left(\widetilde{U}, \widetilde{V}, \widetilde{\eta}\right) \in \mathcal{H}$, $\mathcal{U} = (U, V, \eta) \in \mathcal{D}(\mathcal{A})$ and $\lambda \in \mathbb{R}$ with $|\lambda| > 1$.
Taking the inner product with $\mathcal{U}$ in (3.1) and using the dissipative property (2.2) we get

$$\int_0^{+\infty} g(s) \int_0^\ell \eta_x^*\, \eta_x\, dx\, ds \le \frac{2}{\kappa}\, \|\mathcal{U}\|_{\mathcal{H}}\, \|\mathcal{F}\|_{\mathcal{H}}\,. \tag{3.5}$$

The next Lemma will play an important role in the sequel.

Lemma 3.1. *If* $i\,\mathbb{R} \nsubseteq \rho(\mathcal{A})$ *then there exist* $\lambda_0 > 0$ *and* $\widehat{\mathcal{U}} = \left(\widehat{U}, \widehat{V}, 0\right) \in \mathcal{D}(\mathcal{A})$ *such that*

$$\left\|\widehat{\mathcal{U}}\right\|_{\mathcal{H}} = 1 \quad \textit{and} \quad i\,\lambda_0\, \widehat{\mathcal{U}} - \mathcal{A}\, \widehat{\mathcal{U}} = 0\,.$$

Proof. From the hypothesis follows (see Liu and Zheng [19], pag. 25) that there exist $\lambda_0 > 0$, $(\mathcal{U}_m)_{m\ge1} \subset \mathcal{D}(\mathcal{A})$ and $(\lambda_m)_{m\ge1} \subset \mathbb{R}_+$ such that

$$i\,\lambda_m \in \rho(\mathcal{A})\,,\ \forall\, m \ge 1 \quad \text{and} \quad \lambda_m \longrightarrow \lambda_0\,, \tag{3.6}$$
$$\|\mathcal{U}_m\|_{\mathcal{H}} = 1\,,\ \forall\, m \ge 1\,, \tag{3.7}$$
$$(i\,\lambda_m - \mathcal{A})\, \mathcal{U}_m \longrightarrow 0 \quad \text{in} \quad (\mathcal{H}, \|\cdot\|_{\mathcal{H}})\,. \tag{3.8}$$

From the hypothesis (3.6)-(3.8) follows that $\|\mathcal{U}_m\|_{\mathcal{D}(\mathcal{A})} \le C$, for all $m \ge 1$. Thus, from the reflexivity of $\left(\mathcal{D}(\mathcal{A}), (\cdot,\cdot)_{\mathcal{D}(\mathcal{A})}\right)$, there exist an $\widehat{\mathcal{U}} = \left(\widehat{U}, \widehat{V}, \widehat{\eta}\right) \in \mathcal{D}(\mathcal{A})$ and a subsequence of $(\mathcal{U}_m)_{m\ge1}$ such that $\mathcal{U}_m$ converges weakly to $\widehat{\mathcal{U}}$ in $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$. Using the dissipative property (2.2) we get

$$\int_0^{+\infty} g(s) \int_0^\ell (\eta_m)_x^*\, (\eta_m)_x\, dx\, ds \longrightarrow 0\,,\ m \longrightarrow +\infty\,. \tag{3.9}$$

In addition, from the compactness of $\left[H_0^1(0,\ell)\right]^n$ in $\left[L^2(0,\ell)\right]^n$ and $\left[H^2(0,\ell)\right]^n$ in $\left[H_0^1(0,\ell)\right]^n$, there exist $\chi_2 \in \left[L^2(0,\ell)\right]^n$ and $\chi_1 \in \left[H_0^1(0,\ell)\right]^n$ such that

$$V_m \longrightarrow \chi_2 \ \text{in} \ \left[L^2(0,\ell)\right]^n \quad \text{and} \quad U_m \longrightarrow \chi_1 \ \text{in} \ \left[H_0^1(0,\ell)\right]^n\,. \tag{3.10}$$

From (3.9) and (3.10) we have

$$\widehat{\mathcal{U}} = \left(\widehat{U}, \widehat{V}, 0\right) \in \mathcal{D}(\mathcal{A}) \quad \text{and} \quad \mathcal{U}_m \longrightarrow \widehat{\mathcal{U}} \ \text{ in } \ (\mathcal{H}, \|\cdot\|_{\mathcal{H}}).$$

Thus, since that $\mathcal{A}$ is a closed linear operador we get $i\,\lambda_0\,\widehat{\mathcal{U}} - \mathcal{A}\,\widehat{\mathcal{U}} = O$ and $\left\|\widehat{\mathcal{U}}\right\|_{\mathcal{H}} = 1$. □

Next we prove, when $rank(B) = n$, that the solution semigroup is exponentially stable for all values of the structural parameters.

Theorem 3.2. *If the relaxation kernel g satisfy conditions (1.8)-(1.12) and $rank(B) = n$, then the C_0- semigroup $\mathcal{S}_{\mathcal{A}}(t) = e^{\mathcal{A}\,t}$ is exponentially stable.*

Proof. To demonstrate the strong stability we proceed by contradiction, and assume that the assertion is false. Then applying Lemma 3.1, there exist $\lambda_0 > 0$ and $\widehat{\mathcal{U}} = \left(\widehat{U}, \widehat{V}, 0\right) \in \mathcal{D}(\mathcal{A})$ such that $\left\|\widehat{\mathcal{U}}\right\|_{\mathcal{H}} = 1$ and

$$\begin{aligned} i\,\lambda_0\,\widehat{U} - \widehat{V} &= 0\,,\\ i\,\lambda_0 R\,\widehat{V} - \mathcal{C}\,\widehat{U}_{xx} + N\,\widehat{U} &= 0\,,\\ B^{1/2}\widehat{V} &= 0\,. \end{aligned}$$

From the Theorem (2.1) follows that the matrix $B^{1/2}$ is invertible, then $\widehat{U} = 0 = \widehat{V}$, which is a contradiction. To demonstrate that the operator is uniformly bounded, we integrate (3.4) on $[0, s]$ and we get

$$i\,\lambda \int_0^s \eta(x,t,\tau)\,d\tau - s\,B^{1/2}V(x,t) + \eta(x,t,s) = \int_0^s \widetilde{\eta}(x,t,\tau)\,d\tau\,. \tag{3.11}$$

We denote by $\eta_1(x,t,s) = \int_0^s \eta(x,t,\tau)\,d\tau$ and $\eta_2(x,t,s) = \int_0^s \widetilde{\eta}(x,t,\tau)\,d\tau$. From (2.6) we have

$$\begin{aligned} &\eta_1 \in L^2_g\left(0, +\infty; \left[H^1_0(0,\ell)\right]^n\right) \quad \text{and} \quad \|\eta_1\|_{[L^2_g]^n} \le \frac{2}{\kappa}\,\|\eta\|_{[L^2_g]^n} \le \frac{8}{\kappa^3}\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\,,\\ &\eta_2 \in L^2_g\left(0, +\infty; \left[H^1_0(0,\ell)\right]^n\right) \quad \text{and} \quad \|\eta_2\|_{[L^2_g]^n} \le \frac{2}{\kappa}\,\|\widetilde{\eta}\|_{[L^2_g]^n} \le \frac{4}{\kappa^2}\,\|\mathcal{F}\|^2_{\mathcal{H}}\,. \end{aligned}$$

Using (3.11) and the previous inequality we get

$$\begin{aligned} \frac{1}{|\lambda|^2}\left\|s\,B^{1/2}V\right\|^2_{[L^2_g]^n} &\le 4\,\|\eta_1\|^2_{[L^2_g]^n} + 4\,\|\eta_2\|^2_{[L^2_g]^n} + 2\,\|\eta\|^2_{[L^2_g]^n}\\ &\le \frac{32}{\kappa^3}\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}} + \frac{16}{\kappa^2}\,\|\mathcal{F}\|^2_{\mathcal{H}} + \frac{4}{\kappa}\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\,. \end{aligned}$$

Follows that

$$\frac{1}{|\lambda|^2}\int_0^\ell V_x^*\,B\,V_x\,dx \le C\,\|\mathcal{F}\|^2_{\mathcal{H}} + C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\,, \tag{3.12}$$

where B is a positive semidefinite matrix and the positive number C does not depend on $\mathcal{U}$, $\mathcal{F}$ and λ. Multiplying from left to equation (3.3) by V^*, using the dissipative property (2.2) and the inequality (3.12) we have

$$\begin{aligned} \int_0^{+\infty} g(s)\int_0^\ell \eta_x^*\,\eta_x\,dx\,ds &\le \frac{2}{\kappa}\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\,,\\ \int_0^\ell U_x^*\mathcal{C}\,U_x dx + \int_0^\ell U^*N\,U\,dx &\le C\,\|\mathcal{F}\|^2_{\mathcal{H}} + C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\,,\\ \int_0^\ell V^*R\,V\,dx &\le C\,\|\mathcal{F}\|^2_{\mathcal{H}} + C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\,, \end{aligned}$$

where the positive number C does not depend on $\mathcal{U}$, $\mathcal{F}$ and λ. Therefore the semigroup is exponentially stable. $\square$

Let us denote by $\mathcal{D} = R^{-1}\mathcal{C}$. In addition, we assume that $N = 0$ and B a positive semidefinite matrix with $rank(B) < n$. From the Theorem 2.1 follows that $B^{1/2}$ is a positive semidefinite matrix with $rank(B^{1/2}) = rank(B)$. The next Lemma will play an important role in the sequel.

Lemma 3.3. *If* $dim\,span\left\{B_j^{1/2}, B_j^{1/2}\mathcal{D}, B_j^{1/2}\mathcal{D}^2, \cdots, B_j^{1/2}\mathcal{D}^{n-1} : j = 1, \cdots, n\right\} < n$, *then there exist* $\tau > 0$ *and* $\mathcal{Y} \in \mathbb{R}^n\backslash\{0\}$ *such that*

$$B^{1/2}\mathcal{Y} = 0 \quad and \quad (\mathcal{D} - \tau\, I)\,\mathcal{Y} = 0\,.$$

Proof. From the hypothesis follows that there exists a $\mathcal{X}_0 \in \mathbb{R}^n\backslash\{O\}$ such that

$$B^{1/2}\mathcal{D}^j\mathcal{X}_0 = 0\,, \quad \text{for all } j = 0, 1, \cdots, n-1\,. \tag{3.13}$$

Let us denote by $p(s) \in \mathbb{R}[s]$ the characteristic polynomial of $\mathcal{D}$. Since $\mathcal{C}$ and R^{-1} are positive defined matrices, Theorem 2.2 implies that $\mathcal{D}$ only has positive eigenvalues. Therefore $p(s)$ can be written as

$$p(s) = (s - \mu_1)\,(s - \mu_2)\,\cdots\,(s - \mu_n) \quad \text{and} \quad p(\mathcal{D}) = 0\,, \tag{3.14}$$

where μ_i are the positive eigenvalues of $\mathcal{D}$. We define the subset M de $\mathbb{N}$ as

$$M = \left\{j \in \mathbb{N} : \exists\, \tau_1, \cdots, \tau_j \in \mathbb{R}^+ \text{ such that } (\mathcal{D} - \tau_1 I)\cdots(\mathcal{D} - \tau_j I)\,\mathcal{X}_0 = 0\right\}.$$

From (3.14) we have that $M \neq \emptyset$, so by the Well-Ordering Principle it has a least element m. Then there exist $\tau_1, \cdots, \tau_m \in \mathbb{R}^+$, being $m \leq n$, such that

$$(\mathcal{D} - \tau_1 I)\,\cdots\,(\mathcal{D} - \tau_m I)\,\mathcal{X}_0 = 0\,. \tag{3.15}$$

- If $m = 1$, from (3.13) and (3.15) we have that

  $$B^{1/2}\mathcal{X}_0 = 0 \quad \text{and} \quad (\mathcal{D} - \tau_1 I)\,\mathcal{X}_0 = 0\,.$$

  Taking $\tau := \tau_1$ and $\mathcal{Y} := \mathcal{X}_0$, our conclusion follows.

- If $m \geq 2$, from (3.13) and (3.15) we have that

  $$B^{1/2}\mathcal{D}^j\,\mathcal{X}_0 = 0\,,\ \forall\, j = 0, 1, \cdots, m-1 \text{ and } (\mathcal{D} - \tau_1 I)\,\cdots\,(\mathcal{D} - \tau_m I)\,\mathcal{X}_0 = 0\,. \tag{3.16}$$

  Since m is the least element in M, then $\mathcal{Y}_1 := (\mathcal{D} - \tau_2 I)\,\cdots\,(\mathcal{D} - \tau_m I)\,\mathcal{X}_0 \neq 0$. From (3.16) we have that $B^{1/2}\mathcal{Y}_1 = 0$ and $(\mathcal{D} - \tau_1 I)\,\mathcal{Y}_1 = 0$. Taking $\tau := \tau_1$ and $\mathcal{Y} := \mathcal{Y}_1$, our conclusion follows.

$\square$

As a consequence of the above Lemma, we prove a characterization of strong stability.

Theorem 3.4. $i\,\mathbb{R} \subset \rho(\mathcal{A})$ *if and only if*

$$dim\,span\left\{B_j^{1/2}, B_j^{1/2}\mathcal{D}, B_j^{1/2}\mathcal{D}^2, \cdots, B_j^{1/2}\mathcal{D}^{n-1} : j = 1, \cdots, n\right\} = n\,, \tag{3.17}$$

where $B_j^{1/2}$ *is the* $j-$ *row vector of* $B^{1/2}$.

Proof. We proceed by contradiction, and assume that the assertion (3.17) is false. Then applying Lemma 3.3, there exist $\tau > 0$ and $\mathcal{Y} \in \mathbb{R}^n \backslash \{0\}$ such that

$$B^{1/2}\mathcal{Y} = 0 \quad \text{and} \quad (\mathcal{D} - \tau\, I)\,\mathcal{Y} = 0\,.$$

Then the functions

$$\mathcal{U}_m := \left(\mathcal{Y}\,\sin\left(\frac{m\,\pi}{\ell}\,x\right), i\,\lambda_m \mathcal{Y}\,\sin\left(\frac{m\,\pi}{\ell}\,x\right), 0\right) \in \mathcal{D}(\mathcal{A})\,,\; m \in \mathbb{N}\,,$$

are the eigenvectors of $\mathcal{A}$ with $\lambda_m := \frac{m\,\pi}{\ell}\sqrt{\tau} > 0$ the corresponding imaginary eigenvalues, for $m \in \mathbb{N}$. Therefore $i\,\mathbb{R} \nsubseteq \rho(\mathcal{A})$.
To prove the other implication, let us suppose $i\,\mathbb{R} \nsubseteq \rho(\mathcal{A})$. Then applying Lemma 3.1, there exist $\lambda_0 > 0$ and $\widehat{\mathcal{U}} = \left(\widehat{U}, \widehat{V}, 0\right) \in \mathcal{D}(\mathcal{A})$ such that

$$\left\|\widehat{\mathcal{U}}\right\|_{\mathcal{H}} = 1 \quad \text{and} \quad i\,\lambda_0\,\widehat{\mathcal{U}} - \mathcal{A}\,\widehat{\mathcal{U}} = 0\,,$$

which, written in components, reads

$$\begin{aligned} i\,\lambda_0 \widehat{U} - \widehat{V} &= 0\,,\\ i\,\lambda_0 \widehat{V} - \mathcal{D}\,\widehat{U}_{xx} &= 0\,,\\ B^{1/2}\widehat{V} &= 0\,. \end{aligned}$$

Then we get

$$-\lambda_0^2 \widehat{U} = \mathcal{D}\,\widehat{U}_{xx} \quad \text{and} \quad B^{1/2}\widehat{U} = 0\,. \tag{3.18}$$

So, we have

$$B^{1/2}\mathcal{D}\,\widehat{U}_{xx} = 0 \quad \text{then} \quad B^{1/2}\mathcal{D}\,\widehat{U} = 0\,.$$

Multiplying by $B^{1/2}\mathcal{D}$ the first equation in (3.18) we get $B^{1/2}\mathcal{D}^2\widehat{U} = 0$. Using induction we get that $B^{1/2}\mathcal{D}^m\widehat{U} = 0$, for all $m \in \mathbb{N}$. Therefore, the above is equivalent to

$$B_j^{1/2}\mathcal{D}^m\,\widehat{U} = 0\,, \;\text{ for all }\; j = 1, 2, \cdots, n \;\text{ and }\; m = 0, 1, \cdots, n-1\,,$$

then applying hypothesis (3.17) we get $\widehat{U} = 0$, so we have $\widehat{V} = 0$. Therefore $\widehat{\mathcal{U}} = 0$, which is a contradiction. □

Finally, we prove the equivalence between exponential and strong stability, this in particular implies that the semigroup is never polynomially stable. But, before we show an important lemma.

Lemma 3.5. *Consider the resolvent equation (3.1). Then for all $\epsilon > 0$, there exists a positive number C_ϵ, which depends on only ϵ, such that*

$$\int_0^\ell \left|B^{1/2}\mathcal{D}^m U_x\right|^2 dx + \int_0^\ell \left|B^{1/2}\mathcal{D}^m V\right|^2 dx \le \epsilon\,\|\mathcal{U}\|_{\mathcal{H}}^2 + C_\epsilon\,\|\mathcal{F}\|_{\mathcal{H}}^2\,,$$

for all $m = 0\,, 1\,, \cdots\,, n-1$.

Proof. Multiplying from left to equation (3.2) by $B^{1/2}$ and using (3.12) we get

$$\begin{aligned} \left\|B^{1/2}U\right\|_{[H_0^1]^n}^2 &\le \frac{2}{|\lambda|^2}\left\|B^{1/2}V\right\|_{[H_0^1]^n}^2 + C\left\|\widetilde{U}\right\|_{[H_0^1]^n}^2\\ &\le C\,\|\mathcal{F}\|_{\mathcal{H}}^2 + C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\,, \end{aligned} \tag{3.19}$$

where the positive number C does not depend on $\mathcal{U}$, $\mathcal{F}$ and λ. Then we have

$$\left\|B^{1/2}U\right\|^2_{[H_0^1]^n} \leq \epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}} + C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}}\,, \tag{3.20}$$

where the positive number C_ϵ depends on only ϵ.
Multiplying from left to equation (3.3) by $U^* B\, R^{-1}$ we get

$$i\,\lambda\int_0^\ell U^* B\, V\,dx = \int_0^\ell U^* B\,\widetilde{V}\,dx - \int_0^\ell \left(B^{1/2}U\right)_x^* \left(B^{1/2}R^{-1}\mathcal{C}\,U\right)_x dx - \int_0^{+\infty} g(s)\int_0^\ell \left(B^{1/2}U\right)_x^* \left(B^{1/2}R^{-1}B^{1/2}\,\eta\right)_x dx\,ds\,. \tag{3.21}$$

Multiplying from left to equation (3.2) by $V^* B$ and taking conjugate we get

$$-i\,\lambda\int_0^\ell U^* B\,V\,dx - \int_0^\ell V^* B\,V\,dx = \int_0^\ell \widetilde{U}^* B\,V\,dx\,. \tag{3.22}$$

Adding (3.21) and (3.22) we have

$$\begin{aligned}\left\|B^{1/2}V\right\|^2_{[L^2]^n} &\leq \left|\int_0^\ell \widetilde{U}^* B\,V\,dx\right| + \left|\int_0^\ell \left(B^{1/2}U\right)_x^* \left(B^{1/2}R^{-1}\mathcal{C}\,U\right)_x dx\right| + \\ &\quad \left|\int_0^\ell U^* B\,\widetilde{V}\,dx\right| + \left|\int_0^{+\infty} g(s)\int_0^\ell \left(B^{1/2}U\right)_x^* \left(B^{1/2}R^{-1}B^{1/2}\,\eta\right)_x dx\,ds\right| \\ &\leq C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}} + C\left\|B^{1/2}U\right\|_{[H_0^1]^n}\|U\|_{[H_0^1]^n} + C\left\|B^{1/2}U\right\|_{[H_0^1]^n}\|\eta\|_{[L_g^2]^n} \\ &\leq C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}} + C\,\|\mathcal{U}\|_{\mathcal{H}}\left\|B^{1/2}U\right\|_{[H_0^1]^n}\,,\end{aligned} \tag{3.23}$$

where the positive number C does not depend on $\mathcal{U}$, $\mathcal{F}$ and λ. Using (3.19) we have

$$\left\|B^{1/2}V\right\|^2_{[L^2]^n} \leq \epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}} + C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}}\,, \tag{3.24}$$

where the positive number C_ϵ depends on only ϵ. Now consider $m = 1, 2, \cdots, n-1$. Similarly, multiplying from left to equation (3.3) by $U^*\mathcal{D}^m B\,\mathcal{D}^{m-1}R^{-1}$ we get

$$i\lambda\int_0^\ell U^*\mathcal{D}^m B\mathcal{D}^{m-1}V\,dx + \int_0^\ell \left(B^{1/2}\mathcal{D}^m U\right)_x^* \left(B^{1/2}\mathcal{D}^m U\right)_x dx = \int_0^\ell U^*\mathcal{D}^m B\mathcal{D}^{m-1}\widetilde{V}\,dx - \int_0^{+\infty} g(s)\int_0^\ell \left(B^{1/2}\mathcal{D}^m U\right)_x^* \left(B^{1/2}\mathcal{D}^{m-1}R^{-1}B^{1/2}\eta\right)_x dx\,ds\,. \tag{3.25}$$

Multiplying from left to equation (3.2) by $V^*\mathcal{D}^{m-1}B\,\mathcal{D}^m$ and taking conjugate we get

$$-i\,\lambda\int_0^\ell U^*\mathcal{D}^m B\,\mathcal{D}^{m-1}V\,dx = \int_0^\ell V^*\mathcal{D}^m B\,\mathcal{D}^{m-1}V\,dx + \int_0^\ell \widetilde{U}^*\mathcal{D}^m B\,\mathcal{D}^{m-1}V\,dx\,. \tag{3.26}$$

Adding (3.25) and (3.26) we have

$$\begin{aligned}\left\|B^{1/2}\mathcal{D}^m U\right\|^2_{[H_0^1]^n} &\leq \left|\int_0^\ell U^*\mathcal{D}^m B\,\mathcal{D}^{m-1}\widetilde{V}\,dx\right| + \left|\int_0^\ell \left(B^{1/2}\mathcal{D}^m V\right)^* \left(B^{1/2}\mathcal{D}^{m-1}V\right) dx\right| + \\ &\quad \left|\int_0^{+\infty} g(s)\int_0^\ell \left(B^{1/2}\mathcal{D}^m U\right)_x^* \left(B^{1/2}\mathcal{D}^{m-1}R^{-1}B^{1/2}\eta\right)_x dx\,ds\right| + \left|\int_0^\ell \widetilde{U}^*\mathcal{D}^m B\,\mathcal{D}^{m-1}V\,dx\right|\end{aligned}$$

$$\leq \|U\|_{[L^2]^n}\left\|\mathcal{D}^m B\,\mathcal{D}^{m-1}\widetilde{V}\right\|_{[L^2]^n}+\left\|B^{1/2}\mathcal{D}^m U\right\|_{[L_g^2]^n}\left\|B^{1/2}\mathcal{D}^{m-1}R^{-1}B^{1/2}\eta\right\|_{[L_g^2]^n}+$$
$$\left\|B^{1/2}\mathcal{D}^m V\right\|_{[L^2]^n}\left\|B^{1/2}\mathcal{D}^{m-1}V\right\|_{[L^2]^n}+\left\|\widetilde{U}\right\|_{[L^2]^n}\left\|\mathcal{D}^m B\,\mathcal{D}^{m-1}V\right\|_{[L^2]^n}$$
$$\leq C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}+C\,\|\mathcal{U}\|_{\mathcal{H}}\left\|B^{1/2}\mathcal{D}^{m-1}V\right\|_{[L^2]^n}+\frac{1}{2}\left\|B^{1/2}\mathcal{D}^m U\right\|_{[H_0^1]^n}^2+C\,\|\eta\|_{[L_g^2]^n}^2\,,$$

and using (3.5) we get

$$\left\|B^{1/2}\mathcal{D}^m U\right\|_{[H_0^1]^n}^2\leq C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}+C\,\|\mathcal{U}\|_{\mathcal{H}}\left\|B^{1/2}\mathcal{D}^{m-1}V\right\|_{[L^2]^n}\,, \tag{3.27}$$

where the positive number C does not depend on $\mathcal{U}$, $\mathcal{F}$ and λ. Follow from the above

$$\left\|B^{1/2}\mathcal{D}^m U\right\|_{[H_0^1]^n}^2\leq \epsilon\,\|\mathcal{U}\|_{\mathcal{H}}^2+C_\epsilon\,\|\mathcal{F}\|_{\mathcal{H}}^2+C_\epsilon\left\|B^{1/2}\mathcal{D}^{m-1}V\right\|_{[L^2]^n}\,, \tag{3.28}$$

where the positive number C_ϵ depends on only ϵ.
Multiplying from left to equation (3.2) by $V^*\mathcal{D}^m B\,\mathcal{D}^m$ and taking conjugate we get

$$-i\,\lambda\int_0^\ell U^*\mathcal{D}^m B\,\mathcal{D}^m V\,dx-\int_0^\ell V^*\mathcal{D}^m B\,\mathcal{D}^m V\,dx=\int_0^\ell \widetilde{U}^*\mathcal{D}^m B\,\mathcal{D}^m V\,dx\,. \tag{3.29}$$

Similarly, multiplying from left to equation (3.3) by $U^*\mathcal{D}^m B\,\mathcal{D}^m R^{-1}$ we get

$$i\,\lambda\int_0^\ell U^*\mathcal{D}^m B\mathcal{D}^m V dx=\int_0^\ell U^*\mathcal{D}^m B\mathcal{D}^m\widetilde{V}dx-\int_0^\ell\left(B^{1/2}\mathcal{D}^m U\right)_x^*\left(B^{1/2}\mathcal{D}^{m+1}U\right)_x dx$$
$$-\int_0^{+\infty}g(s)\int_0^\ell\left(B^{1/2}\mathcal{D}^m U\right)_x^*\left(B^{1/2}\mathcal{D}^m R^{-1}B^{1/2}\eta\right)_x dx\,ds\,. \tag{3.30}$$

Adding (3.29) and (3.30) we have

$$\left\|B^{1/2}\mathcal{D}^m V\right\|_{[L^2]^n}^2\leq\left|\int_0^\ell U^*\mathcal{D}^m B\,\mathcal{D}^m\widetilde{V}\,dx\right|+\left|\int_0^\ell\left(B^{1/2}\mathcal{D}^m U\right)_x^*\left(B^{1/2}\mathcal{D}^{m+1}U\right)_x dx\right|+$$
$$\left|\int_0^\ell\widetilde{U}^*\mathcal{D}^m B\,\mathcal{D}^m V\,dx\right|+\left|\int_0^{+\infty}g(s)\int_0^\ell\left(B^{1/2}\mathcal{D}^m U\right)_x^*\left(B^{1/2}\mathcal{D}^m R^{-1}B^{1/2}\eta\right)_x dx\,ds\right|$$
$$\leq C\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}+C\,\|\mathcal{U}\|_{\mathcal{H}}\left\|B^{1/2}\mathcal{D}^m U\right\|_{[H_0^1]^n}\,, \tag{3.31}$$

where the positive number C does not depend on $\mathcal{U}$, $\mathcal{F}$ and λ. In addition, repeating the same procedure above for any positive number D_ϵ, which depends on only ϵ, we have

$$D_\epsilon\left\|B^{1/2}\mathcal{D}^m V\right\|_{[L^2]^n}^2\leq C_\epsilon\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}+C_\epsilon\,\|\mathcal{U}\|_{\mathcal{H}}\left\|B^{1/2}\mathcal{D}^m U\right\|_{[H_0^1]^n}\,, \tag{3.32}$$

where the positive number C_ϵ depends on only ϵ. Using (3.27) in (3.31) and (3.32) we get

$$\left\|B^{1/2}\mathcal{D}^m V\right\|_{[L^2]^n}^2\leq \epsilon\,\|\mathcal{U}\|_{\mathcal{H}}^2+C_\epsilon\,\|\mathcal{F}\|_{\mathcal{H}}^2+C_\epsilon\left\|B^{1/2}\mathcal{D}^{m-1}V\right\|_{[L^2]^n}^2\,, \tag{3.33}$$

$$D_\epsilon\left\|B^{1/2}\mathcal{D}^m V\right\|_{[L^2]^n}^2\leq \epsilon\,\|\mathcal{U}\|_{\mathcal{H}}^2+C_\epsilon\,\|\mathcal{F}\|_{\mathcal{H}}^2+C_\epsilon\left\|B^{1/2}\mathcal{D}^{m-1}V\right\|_{[L^2]^n}^2\,, \tag{3.34}$$

where the positive number C_ϵ depends on only ϵ. Using (3.34) inductively in (3.28) and (3.33) we have

$$\left\|B^{1/2}\mathcal{D}^m V\right\|_{[L^2]^n}^2\leq \epsilon\,\|\mathcal{U}\|_{\mathcal{H}}^2+C_\epsilon\,\|\mathcal{F}\|_{\mathcal{H}}^2+C_\epsilon\left\|B^{1/2}V\right\|_{[L^2]^n}^2\,, \tag{3.35}$$

$$\left\|B^{1/2}\mathcal{D}^m U\right\|_{[H_0^1]^n}^2\leq \epsilon\,\|\mathcal{U}\|_{\mathcal{H}}^2+C_\epsilon\,\|\mathcal{F}\|_{\mathcal{H}}^2+C_\epsilon\left\|B^{1/2}V\right\|_{[L^2]^n}^2\,, \tag{3.36}$$

where the positive number C_ϵ depends on only ϵ. Finally, using (3.19) and (3.23) in (3.35) and (3.36) we get

$$\left\|B^{1/2}\mathcal{D}^m U\right\|^2_{[H_0^1]^n} \leq \epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}} + C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}} \quad \text{and}$$
$$\left\|B^{1/2}\mathcal{D}^m V\right\|^2_{[L^2]^n} \leq \epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}} + C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}},$$

where the positive number C_ϵ depends on only ϵ. □

Theorem 3.6. *$\mathcal{S}_{\mathcal{A}}(t)$ is exponentially stable if and only if $\mathcal{S}_{\mathcal{A}}(t)$ is strongly stable.*

Proof. If $i\,\mathbb{R}\subset\rho(\mathcal{A})$ then by Theorem 3.2 we have that

$$dim\,span\left\{B_j^{1/2},\, B_j^{1/2}\mathcal{D},\, B_j^{1/2}\mathcal{D}^2,\,\cdots,\, B_j^{1/2}\mathcal{D}^{n-1}\,:\, j=1,\cdots,n\right\}=n, \tag{3.37}$$

where $B_j^{1/2}$ is the $j-$ row vector of $B^{1/2}$. Without loss of generality, let us consider an orthonormal base of the vector space in (3.37)

$$\mathcal{V}_i := B_{j_i}^{1/2}\mathcal{D}^{m_i},\ i=1,\cdots,n, \tag{3.38}$$

where $1\leq j_i\leq n$ and $0\leq m_i\leq n-1$, for all $i=1,\cdots,n$. Then we get

$$U_x(x)=\alpha_1(x)\,\mathcal{V}_1^*+\alpha_2(x)\,\mathcal{V}_2^*+\cdots+\alpha_n(x)\,\mathcal{V}_n^*,$$
$$V(x)=\beta_1(x)\,\mathcal{V}_1^*+\beta_2(x)\,\mathcal{V}_2^*+\cdots+\beta_n(x)\,\mathcal{V}_n^*,$$

being $\alpha_i(x)=\mathcal{V}_i\,U_x(x)$ and $\beta_i(x)=\mathcal{V}_i\,V(x)$, for all $i=1,\cdots,n$. Follow from the above

$$\|U\|^2_{[H_0^1]^n}\leq\sum_{i=1}^n C\,\|\alpha_i\|^2_{L^2} \quad\text{and}\quad \|V\|^2_{[L^2]^n}\leq\sum_{i=1}^n C\,\|\beta_i\|^2_{L^2}. \tag{3.39}$$

Using the Lemma 3.5 in the resolvent equation (3.1) we have

$$\left\|B^{1/2}\mathcal{D}^m U\right\|^2_{[H_0^1]^n} \leq \epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}} + C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}} \quad \text{and}$$
$$\left\|B^{1/2}\mathcal{D}^m V\right\|^2_{[L^2]^n} \leq \epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}} + C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}}, \tag{3.40}$$

for all $m=1,\cdots,n-1$, where the positive number C_ϵ depends on only ϵ. Further, using (3.40) we get

$$\|\alpha_i\|^2_{L^2}\leq\epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}}+C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}} \quad\text{and}\quad \|\beta_i\|^2_{L^2}\leq\epsilon\,\|\mathcal{U}\|^2_{\mathcal{H}}+C_\epsilon\,\|\mathcal{F}\|^2_{\mathcal{H}}, \tag{3.41}$$

for all $i=1,\cdots,n$, where the positive number C_ϵ depends on only ϵ.
Finally, using (3.5), (3.39) and (3.41) we have that the semigroup is exponentially stable. □

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