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Stability Analysis of the High-order Multistep Collocation Method for the Functional Integral Equations with Constant Delays

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ABSTRACT: The results on the stability of recurrences play an important role in the theory of dynamical systems and computer science in connection to the notions of shadowing and controlled chaos. In this paper, stability properties of high-order multistep collocation method for functional integral equations of Volterra integral equations with constant delays type with respect to significant test equations are investigated.

Key Words: Functional integral equations, Delay integral equations, Convergence and superconvergence analysis, Stability analysis.

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1. Introduction

For many years, the subject of functional equations has held a prominent place in the attention of mathematicians. In more recent years this attention has been directed to a particular kind of functional equation, an integral equation, wherein the unknown function occurs under the integral sign. The study of this kind of equation is sometimes referred to as the inversion of a definite integral.

Probably, one of the first question in classical dynamical systems can be stability which motivated the introduction of new mathematical tools in engineering, particularly in control engineering. Stability theory has been of interest to mathematicians and astronomers for a long time and has had a stimulating impact on these fields. The specific problem of attempting to prove that the solar system is stable accounted for the introduction of many new methods (see e.g. [2]- [4] and reference therein).

Functional delay integral equations model physical systems where the evolution does not only depend on the present state of the system but also on the past history. Such models are found, for example, in population dynamics and epidemiology, where the delay is due to a gestation or maturation period, or in numerical control, where the delay arises from the processing in the controller feedback loop.

In [5], H. Brunner applied collocation type methods for numerical solution of functional integral equations and discussed about their connection with iterated collocation methods. V. Horvat [6], had investigated the collocation methods for Volterra integral equations with delay arguments. P. Darania [7], had considered the nonlinear Volterra integral equations with constant delays $\theta(t) = t - \tau,\ \tau > 0$, of the form

$$y(t) = \begin{cases} g(t) + (Vy)(t) + (V_\tau y)(t), & t \in I = [0, T], \\ \phi(t), & t \in [-\tau, 0), \end{cases} \tag{1.1}$$

where

$$(Vy)(t) = \int_0^t k_1(t, s, y(s))ds, \tag{1.2}$$

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$$(V_\tau y)(t) = \int_0^{t-\tau} k_2(t,s,y(s))ds, \tag{1.3}$$

and the given functions, $\phi : [-\tau, 0] \to \mathbb{R}$, $g : I \to \mathbb{R}$, $k_1 : D \times \mathbb{R} \to \mathbb{R}, D = \{(t,s) : 0 \le s \le t \le T\}$ and $k_2 : D_\tau \times \mathbb{R} \to \mathbb{R}, D_\tau = I \times [-\tau, T-\tau]$ are at least continuous on their domains.

In the first part of this note, we will study the basic materials of the multistep collocation method. Next, we will study the convergence and superconvergence of this method, which has been presented in [7]- [9]. The linear stability is analyzed in section 3 and the paper is closed in section 4, by showing the stability regions for the multistep collocation method on some numerical examples.

2. Preliminaries

Let $t_n = nh$, $(n = 0, \dots, N, \quad t_N = T, \quad h = \frac{\tau}{\tilde{r}}$ for som $\tilde{r} \in \mathbb{N})$ define a uniform partition for $I = [0, T]$, and let $\Omega_N := \{0 = t_0 < t_1 < \cdots < t_N = T\}, \sigma_0 := [t_0, t_1], \sigma_n := (t_n, t_{n+1}]$ $(1 \le n \le N-1)$. With a given mesh Ω_N, we associate the set of its interior points, $Z_N := \{t_n : n = 1, \dots, N-1\}$. For a fixed $N \ge 1$ and, for given integer $m \ge 1$, the piecewise polynomial space $S_{m-1}^{(-1)}(Z_N)$ is defined by

$$S_{m-1}^{(-1)}(Z_N) := \{u : u|_{\sigma_n} = u_h \in \Pi_{m-1}, \quad 0 \le n \le N-1\},$$

where Π_{m-1} denotes the set of (real) polynomials of a degree not exceeding $m-1$.

Let $u_h = u|_{\sigma_n}, u \in S_{m-1}^{(-1)}(Z_N)$, for all $t \in \sigma_n$, we have

$$u_h(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j}, \ s \in [0,1], \ n = r, r+1, \dots, N-1, \tag{2.1}$$

where $U_{n,j} = u_h(t_{n,j}), y_{n-k} = u_h(t_{n-k})$ and

$$\varphi_k(s) = \prod_{i=1}^{m} \frac{s - c_i}{-k - c_i} \cdot \prod_{\substack{i=0 \\ i \ne k}}^{r-1} \frac{s+i}{-k+i}, \qquad \psi_j(s) = \prod_{i=0}^{r-1} \frac{s+i}{c_j + i} \cdot \prod_{\substack{i=1 \\ i \ne j}}^{m} \frac{s - c_i}{c_j - c_i}. \tag{2.2}$$

The collocation solution u_h will be determined by imposing the condition that u_h satisfies the integral equation (1.1) on the finite set $X_N = \{t_{n,j} = t_n + c_j h, \ j = 1, 2, ..., m\}$

$$u_h(t) = \begin{cases} g(t) + (Vu_h)(t) + (V_\tau u_h)(t), & t \in X_N, \\ \phi(t), & t \in [-\tau, 0), \end{cases} \tag{2.3}$$

where $\{c_j\}_{j=1}^m$, $0 \le c_1 < \cdots < c_m \le 1$, the set of collocation parameters. After some computations, the exact multistep collocation method is obtained by collocating both sides of (2.3) at the points $t = t_{n,j}$ for $j = 1, 2, \dots, m$ and computing $y_{n+1} = u_h(t_{n+1})$:

$$\begin{cases} U_{n,j} = D_{n,j}, & j = 1, 2, \dots, m, \\ y_{n+1} = \displaystyle\sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) U_{n,j}, & n = r, r+1, \dots, N-1, \end{cases} \tag{2.4}$$

where $D_{n,j} = D(t_{n,j})$ and

$$D(t_{n,j}) = g(t_{n,j}) + \begin{cases} (Vu_h)(t_{n,j}) + \Phi(t_{n,j}), & t_{n,j} - \tau < 0, \\ (Vu_h)(t_{n,j}) + (V_\tau u_h)(t_{n,j}), & t_{n,j} - \tau \ge 0, \end{cases} \tag{2.5}$$

$$\Phi(t_{n,j}) = \int_0^{t_{n,j}-\tau} k_2(t_{n,j}, s, \phi(s))ds, \qquad j = 1, 2, \dots, m, \quad n = 0, 1, \dots, \tilde{r} - 1, \tag{2.6}$$

$$
(V_\tau u_h)(t_{n,j}) = \begin{cases} -h\left[\int_{c_j}^{1} k_2(t_{n,j}, t_{n-\tilde{r}}+sh, \phi(t_{n-\tilde{r}}+sh))ds \right.\\[2mm] \left. +\sum_{i=n-\tilde{r}+1}^{-1}\int_0^1 k_2(t_{n,j}, t_i+sh, \phi(t_i+sh))ds\right], \qquad t_{n,j}-\tau<0,\\[2mm] h\left[\sum_{i=0}^{n-\tilde{r}-1}\int_0^1 k_2(t_{n,j}, t_i+sh, u_h(t_i+sh))ds \right.\\[2mm] \left. +\int_0^{c_j} k_2(t_{n,j}, t_{n-\tilde{r}}+sh, u_{n-\tilde{r}}(t_{n-\tilde{r}}+sh))ds\right], \quad t_{n,j}-\tau\geq 0, \end{cases} \tag{2.7}
$$

$$
\begin{aligned} (Vu_h)(t_{n,j}) = \quad & h\sum_{i=0}^{n-1}\int_0^1 k_1(t_{n,j}, t_i+sh, u_h(t_i+sh))ds \\ & +h\int_0^{c_j} k_1(t_{n,j}, t_n+sh, u_h(t_n+sh))ds. \end{aligned} \tag{2.8}
$$

By using quadrature formulas with the weights w_l and nodes $d_l, l=1,\ldots,\mu_1$, for integrating on $[0,1]$, and the weights $w_{j,l}$ and nodes $d_{j,l}, l=1,\ldots,\mu_0$ for integrating on $[0,c_i]$, with positive integers μ_0 and μ_1, one can write

$$
\begin{cases} Y_{n,j} = \bar{D}_{n,j}, & j=1,2,\ldots,m,\\ y_{n+1} = \sum_{k=0}^{r-1}\varphi_k(1)y_{n-k} + \sum_{j=1}^{m}\psi_j(1)Y_{n,j}, & n=r,r+1,\ldots,N-1, \end{cases} \tag{2.9}
$$

where

$$
\bar{D}(t_{n,j}) = g(t_{n,j}) + (\bar{V}u_h)(t_{n,j}) + (\bar{V}_\tau u_h)(t_{n,j}), \tag{2.10}
$$

$$
\begin{aligned} (\bar{V}u_h)(t_{n,j}) = \quad & h\sum_{i=0}^{n-1}\sum_{l=1}^{\mu_1} w_l k_1(t_{n,j}, t_i+d_l h, P_i(t_i+d_l h)) \\ & +h\sum_{l=1}^{\mu_0} w_{j,l}k_1(t_{n,j}, t_n+d_{j,l}h, P_n(t_n+d_{j,l}h)), \end{aligned} \tag{2.11}
$$

$$
(\bar{V}_\tau u_h)(t_{n,j}) = \begin{cases} -h\left(\sum_{i=n-\tilde{r}+1}^{-1}\sum_{l=1}^{\mu_1} w_l k_2(t_{n,j}, t_i+d_l h, \phi(t_i+d_l h)) \right.\\ \left. +\sum_{l=1}^{\mu_1}\bar{w}_{j,l}k_2(t_{n,j}, t_{n-\tilde{r}}+\xi_{j,l}h, \phi(t_{n-\tilde{r}}+\xi_{j,l}h))\right), \qquad t_{n,j}-\tau<0,\\[2mm] h\left(\sum_{i=0}^{n-\tilde{r}-1}\sum_{l=1}^{\mu_1} w_l k_2(t_{n,j}, t_i+d_l h, P_i(t_i+d_l h)) \right.\\ \left. +\sum_{l=1}^{\mu_0} w_{j,l}k_2(t_{n,j}, t_{n-\tilde{r}}+d_{j,l}h, P_{n-\tilde{r}}(t_{n-\tilde{r}}+d_{j,l}h))\right), \quad t_{n,j}-\tau\geq 0, \end{cases} \tag{2.12}
$$

and $\xi_{j,l} := c_j + (1-c_j)d_l$, $\quad \bar{w}_{j,l} := (1-c_j)w_l$, $\quad j=1,\ldots,m$, $\quad l=1,\ldots,\mu_1$. Also, the discretized multistep collocation polynomial, denoted by

$$
P_n(t_n+sh) = \sum_{k=0}^{r-1}\varphi_k(s)y_{n-k} + \sum_{j=1}^{m}\psi_j(s)Y_{n,j}, \quad s\in[0,1], \quad n=r,\ldots,N-1. \tag{2.13}
$$

For more detail see [7].

Let $u_h \in S_{m-1}^{(-1)}(Z_N)$ denote the (exact) collocation solution to (1.1) defined by (2.4). In convergence analysis, we consider the linear test equation

$$y(t) = \begin{cases} g(t) + \int_0^t k_1(t,s)y(s)ds + \int_0^{t-\tau} k_2(t,s)y(s)ds, & t \in I, \\ \phi(t), & t \in [-\tau, 0), \end{cases} \tag{2.14}$$

where $k_1 \in C(D)$ and $k_2 \in C(D_\tau)$.

Theorem 2.1. *Let the given functions in (2.14) satisfy $g \in C^p(I)$, $k_1 \in C^p(D)$, $k_2 \in C^p(D_\tau)$, $\phi \in C^p([-\tau, 0])$, and for $t \in [0, \tau]$ the integral*

$$\Phi(t) := \int_0^{t-\tau} k_2(t,s)\phi(s)ds, \tag{2.15}$$

is known exactly. Also, suppose that the starting error is

$$\| y - u_h \|_{\infty,[0,t_r]} = O(h^p), \tag{2.16}$$

and

$$\rho(\mathbf{A}) < 1, \tag{2.17}$$

where $p = m + r$ and ρ denotes the spectral radius and

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{r-1} \\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1), ..., \varphi_0(1) \end{array} \right]. \tag{2.18}$$

Then for all sufficiently small $h = \frac{\tau}{\tilde{r}}$, $(\tilde{r} \in \mathbb{N})$ the constrained mesh collocation solution $u_h \in S_{m-1}^{(-1)}(Z_N)$ to (2.14), satisfies

$$\| \mathcal{E} \|_\infty \leq Ch^p, \tag{2.19}$$

where $\mathcal{E}(t) = y(t) - u_h(t)$ be the error of the exact collocation method (2.10) and C is positive constant not depending on h. This estimate holds for all collocation parameters $\{c_j\}$ with $0 \leq c_1 < \cdots < c_m \leq 1$.

Proof. For proof see [7]. □

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold, except that the integrals*

$$\Phi(t) = \int_0^{t-\tau} k_2(t,s)\phi(s)ds, \;\; t = t_{n,j}, \;\; n = 0, 1, ..., \tilde{r} - 1,$$

are now approximated by quadrature formulas $\bar{\Phi}(t)$, with corresponding quadrature errors $E_0(t) := \Phi(t) - \bar{\Phi}(t)$, such that

$$\|E_0(t)\| \leq h^q \tag{2.20}$$

for some $q > 0$. Then the collocation solution $u_h \in S_{m-1}^{(-1)}(Z_N)$ satisfies, for all sufficiently small $h > 0$,

$$\| \mathcal{E} \|_\infty \leq Ch^p, \tag{2.21}$$

with $p := \min\{m + r, q\}$, where C are finite constants not depending on h.

Proof. For proof see [7]. □

3. Stability analysis

The solution of the given Volterra integral equations with constant delays is found by solving associated collocation solutions. If no rounding errors were introduced into this process then their exact solution $U_{n,j}$ would be obtained at each mesh points $t_{n,j}$. The essential idea defining stability is that the numerical process should not cause any small perturbations introduced through rounding at any stage to grow and ultimately dominate the solution.

To make the analysis of stability amenable to mathematical analysis, a definition based on the growth of the exact solution is used. Then, if rounding errors or perturbations are introduced at any stage in time, then these errors will also be bounded if the exact solution is bounded. The first approach to stability analysis is called matrix stability analysis. To establish the criterion for stability properties of exact and discretized multistep method, consider the basic test equation

$$y(t) = \begin{cases} 1 + \lambda \int_0^t y(s)ds + \lambda \int_0^{t-\tau} y(s)ds, & t \in [0,T], \\ \phi(t), & t \in [-\tau, 0). \end{cases} \tag{3.1}$$

To state the main results of stability properties of the method, we define

$$\begin{aligned} &\int_0^{c_j} \varphi_k(s)ds = \Omega_{jk}, \qquad \int_0^{c_j} \psi_l(s)ds = \rho_{jl}, \\ &\int_0^1 \varphi_k(s)ds = \beta_k, \qquad \int_0^1 \psi_l(s)ds = \gamma_l, \\ &\int_0^1 \phi(t_i + sh)ds = \tilde{\phi}_i, \end{aligned} \tag{3.2}$$

and introduce the vectors and matrices

$$\begin{aligned} &\mathbf{U}_n = [U_{n,1}, ..., U_{n,m}]^T, && \mathbf{y}_n^{(r)} = [y_n, ..., y_{n-r+1}]^T, \\ &\mathbf{u} = [1, ..., 1]^T \in \mathbb{R}^m, && \boldsymbol{\beta} = [\beta_0, ..., \beta_{r-1}]^T, \\ &\boldsymbol{\gamma} = [\gamma_1, ..., \gamma_m]^T, && \boldsymbol{\psi}(1) = [\psi_1(1), ..., \psi_m(1)]^T, \\ &\boldsymbol{\varphi}(1) = [\varphi_0(1), ..., \varphi_{r-1}(1)]^T, && \boldsymbol{\varphi}(0) = [\varphi_0(0), ..., \varphi_{r-1}(0)]^T, \\ &\boldsymbol{\Omega} = (\Omega_{ik}) \in \mathbb{R}^{m\times r}, && \boldsymbol{\rho} = (\rho_{ij}) \in \mathbb{R}^{m\times m}, \end{aligned}$$

$$\hat{\boldsymbol{\phi}}_{n-\tilde{r}} = [\int_{c_1}^1 \phi(t_{n-\tilde{r}} + sh)ds, \ldots, \int_{c_m}^1 \phi(t_{n-\tilde{r}} + sh)]^T, \tag{3.3}$$

$$\mathbf{A}_1 = \left[\begin{array}{c} -\boldsymbol{\psi}^T(1) \\ \hline \mathbf{0}_{r\times m} \end{array}\right]_{(r+1)\times m}, \qquad \mathbf{B}_1 = \left[\begin{array}{c|c} \mathbf{0}_{1\times r} & 0 \\ \hline \mathbf{I}_r & \mathbf{0}_{r\times 1} \end{array}\right]_{(r+1)\times(r+1)},$$

$$\mathbf{A}_2 = \left[\begin{array}{c} -\boldsymbol{\psi}^T(0) \\ \hline \mathbf{0}_{r\times m} \end{array}\right]_{(r+1)\times m}, \qquad \mathbf{B}_2 = \left[\begin{array}{c|c} \mathbf{0}_{1\times r} & 0 \\ \hline \mathbf{I}_r & \mathbf{0}_{r\times 1} \end{array}\right]_{(r+1)\times(r+1)}.$$

$$\mathbf{C}_1 = \left[\begin{array}{c|c} 1 & -\boldsymbol{\phi}^T(1) \\ \hline \mathbf{0}_{r\times 1} & \mathbf{I}_r \end{array}\right]_{(r+1)\times(r+1)}, \qquad \mathbf{C}_2 = \left[\begin{array}{c|c} 1 & -\boldsymbol{\phi}^T(0) \\ \hline \mathbf{0}_{r\times 1} & \mathbf{I}_r \end{array}\right]_{(r+1)\times(r+1)},$$

Remark 3.1. *[5] In the following, we assume that, the solution of equation (3.1) is continuous at $t = 0$, in the other words, the initial function $\phi(t)$ is such that*

$$g(0) + \int_0^{-\tau} k_2(0,s)\phi(s)ds = \phi(0).$$

Theorem 3.2. *The exact multistep collocation method (2.1) and (2.4), applied to the test equation (3.1), leads to the following recurrence relation*

$$\mathbf{Y}_n = \mathbf{T}(z)\mathbf{Y}_{n-1} + \mathbf{N}(z), \tag{3.4}$$

where

$$Y_n = \begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \\ \mathbf{U}_n \\ y_{n-\tilde{r}+1} \\ \mathbf{y}_{n-\tilde{r}}^{(r)} \\ \mathbf{U}_{n-\tilde{r}} \end{bmatrix}, \tag{3.5}$$

$\mathbf{T}(z) \in \mathbb{R}^{(2m+2r+2)\times(2m+2r+2)}$ *is the stability matrix and is given by*

$$\mathbf{T}(z) = [\mathbf{P}(z)]^{-1}\mathbf{M}(z),\ \ \mathbf{N}(z) = [\mathbf{P}(z)]^{-1}\mathbf{J}(z), \tag{3.6}$$

with $z = \lambda h$ *and*

$$\mathbf{P}(z) = \left[\begin{array}{c|c|c|c|c|c} \mathbf{0} & -z\boldsymbol{\Omega} & \mathbf{I}_m - z\boldsymbol{\rho} & \mathbf{0} & -z\boldsymbol{\Omega} & -z\boldsymbol{\rho} \\ \hline \mathbf{0} & -z\boldsymbol{\Omega} & \mathbf{I}_m - z\boldsymbol{\rho} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{C}_1} & \mathbf{A}_1 & \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} & \multicolumn{2}{c|}{\mathbf{C}_2} & \mathbf{A}_2 \end{array}\right], \tag{3.7}$$

$$\mathbf{M}(z) = \left[\begin{array}{c|c|c|c|c|c} \mathbf{0} & z(\boldsymbol{u}\boldsymbol{\beta}^T - \boldsymbol{\Omega}) & \mathbf{I}_m + z(\boldsymbol{u}\boldsymbol{\gamma}^T - \boldsymbol{\rho}) & \mathbf{0} & z(\boldsymbol{u}\boldsymbol{\beta}^T - \boldsymbol{\Omega}) & z(\boldsymbol{u}\boldsymbol{\gamma}^T - \boldsymbol{\rho}) \\ \hline \mathbf{0} & z(\boldsymbol{u}\boldsymbol{\beta}^T - \boldsymbol{\Omega}) & \mathbf{I}_m + z(\boldsymbol{u}\boldsymbol{\gamma}^T - \boldsymbol{\rho}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{B}_1} & \mathbf{0} & \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} & \multicolumn{2}{c|}{\mathbf{B}_2} & \mathbf{0} \end{array}\right], \tag{3.8}$$

$$\mathbf{J}(z) = -z \begin{bmatrix} \mathbf{0} \\ \mathbf{u}\tilde{\phi}_{n-\tilde{r}-1} - \hat{\boldsymbol{\phi}}_{n-\tilde{r}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{2m+2r+2\times 1}. \tag{3.9}$$

Proof. By the notations of this section and from equation

$$y(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) Y_{n,j},\ \ s \in [0,1],$$

we have

$$y_{n+1} = \boldsymbol{\varphi}^T(1)\mathbf{y}_n^{(r)} + \boldsymbol{\psi}^T(1)\mathbf{U}_n, \tag{3.10}$$

and

$$y_{n-\tilde{r}} = \boldsymbol{\varphi}^T(0)\mathbf{y}_{n-\tilde{r}}^{(r)} + \boldsymbol{\psi}^T(0)\mathbf{U}_{n-\tilde{r}}, \tag{3.11}$$

which can be written in the following matrices forms

$$\mathbf{C}_1 \begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \end{bmatrix} + \mathbf{A}_1\mathbf{U}_n = \mathbf{B}_1 \begin{bmatrix} y_n \\ \mathbf{y}_{n-1}^{(r)} \end{bmatrix}, \tag{3.12}$$

$$\mathbf{C}_2 \begin{bmatrix} y_{n-\tilde{r}} \\ \mathbf{y}_{n-\tilde{r}}^{(r)} \end{bmatrix} + \mathbf{A}_2\mathbf{U}_{n-\tilde{r}} = \mathbf{B}_2 \begin{bmatrix} y_{n-\tilde{r}} \\ \mathbf{y}_{n-\tilde{r}-1}^{(r)} \end{bmatrix}, \tag{3.13}$$

where $\mathbf{A}_i, \mathbf{C}_i$ and $\mathbf{B}_i,\ i = 1, 2$ are given by (3.3). Now, we apply the equations (2.9)-(2.12) (multistep method) on the test equation (3.1) to get

$$\mathbf{U}_n = \begin{cases} \mathbf{u} + z\Big\{\mathbf{u}\Big[\sum_{i=0}^{n-1}\boldsymbol{\beta}^T\mathbf{y}_i^{(r)} + \sum_{i=0}^{n-1}\boldsymbol{\gamma}^T\mathbf{U}_i + \sum_{i=0}^{n-\tilde{r}-1}\boldsymbol{\beta}^T\mathbf{y}_i^{(r)} + \sum_{i=0}^{n-\tilde{r}-1}\boldsymbol{\gamma}^T\mathbf{U}_i\Big] \\ +\boldsymbol{\Omega}(\mathbf{y}_{n-\tilde{r}}^{(r)} + \mathbf{y}_n^{(r)}) + \boldsymbol{\rho}(\mathbf{U}_{n-\tilde{r}} + \mathbf{U}_n)\Big\}, & t_{n,j} - \tau \geq 0, \\ \mathbf{u} + z\Big\{\mathbf{u}\Big[\sum_{i=0}^{n-1}\boldsymbol{\beta}^T\mathbf{y}_i^{(r)} + \sum_{i=0}^{n-1}\boldsymbol{\gamma}^T\mathbf{U}_i - \sum_{i=n-\tilde{r}}^{-1}\tilde{\phi}_i\Big] \\ +\boldsymbol{\Omega}\mathbf{y}_n^{(r)} + \boldsymbol{\rho}\mathbf{U}_n - \hat{\boldsymbol{\phi}}_{n-\tilde{r}}\Big\}, & t_{n,j} - \tau < 0. \end{cases} \tag{3.14}$$

The computation of the difference $\mathbf{U}_n - \mathbf{U}_{n-1}$ by substituting the (3.14) for both terms $\mathbf{U}_n$ and $\mathbf{U}_{n-1}$, for $t_{n,j} - \tau \geq 0$, leads to

$$\begin{aligned}(\mathbf{I}_m - z\boldsymbol{\rho})\mathbf{U}_n - z\boldsymbol{\rho}\mathbf{U}_{n-\tilde{r}} - z\boldsymbol{\Omega}\mathbf{y}_n^{(r)} - z\boldsymbol{\Omega}\mathbf{y}_{n-\tilde{r}}^{(r)} = \;& (\mathbf{I}_m + z(\mathbf{u}\boldsymbol{\gamma}^T - \boldsymbol{\rho}))\mathbf{U}_{n-1} \\ & +z\left\{(\mathbf{u}\boldsymbol{\gamma}^T - \boldsymbol{\rho})\mathbf{U}_{n-\tilde{r}-1}\right. \\ & +(\mathbf{u}\boldsymbol{\beta}^T - \boldsymbol{\Omega})\mathbf{y}_{n-1}^{(r)} \\ & \left.+(\mathbf{u}\boldsymbol{\beta}^T - \boldsymbol{\Omega})\mathbf{y}_{n-\tilde{r}-1}^{(r)}\right\},\end{aligned} \tag{3.15}$$

and for $t_{n,j} - \tau < 0$, we get

$$(\mathbf{I}_m - z\boldsymbol{\rho})\mathbf{U}_n - z\boldsymbol{\Omega}\mathbf{y}_n^{(r)} = \quad (\mathbf{I}_m + z(\boldsymbol{\gamma}^T - \boldsymbol{\rho}))\mathbf{U}_{n-1} - z\mathbf{u}\tilde{\phi}_{n-\tilde{r}-1} - z\hat{\boldsymbol{\phi}}_{n-\tilde{r}}. \tag{3.16}$$

Now we conclude from (3.12),(3.13), (3.15) and (3.16) that

$$\mathbf{P}(z)\begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \\ \mathbf{U}_n \\ y_{n-r+1} \\ \mathbf{y}_{n-r}^{(r)} \\ \mathbf{U}_{n-r} \end{bmatrix} = \mathbf{M}(z)\begin{bmatrix} y_n \\ \mathbf{y}_{n-1}^{(r)} \\ \mathbf{U}_{n-1} \\ y_{n-r} \\ \mathbf{y}_{n-r-1}^{(r)} \\ \mathbf{U}_{n-r-1} \end{bmatrix} + \mathbf{J}(z), \tag{3.17}$$

or more compactly as

$$\mathbf{Y}_n = \mathbf{T}(z)\mathbf{Y}_{n-1} + \mathbf{N}(z), \tag{3.18}$$

where $\mathbf{T}(z)$ and $\mathbf{N}(z)$ are known and this completes the proof. □

Theorem 3.3. *Suppose that the hypotheses of Theorem 3.2 hold and $\|\mathbf{T}(z)\| \leq 1$. Then the difference method (3.18) stable.*

Proof. In general the difference equations for a equation (3.1) can be written in matrix form

$$\mathbf{Y}_n = \mathbf{T}(z)\mathbf{Y}_{n-1} + \mathbf{N}(z). \tag{3.19}$$

Applied recursively (3.19) gives

$$\mathbf{Y}_n = \mathbf{T}^n(z)\mathbf{Y}_0 + \mathbf{T}^{n-1}(z)\mathbf{N}(z) + \mathbf{T}^{n-2}(z)\mathbf{N}(z) + \cdots + \mathbf{N}(z), \tag{3.20}$$

where $\mathbf{Y}_0$ is the vector of initial values and $\mathbf{N}(z)$ is vector of the known initial condition $\phi(x)$.

The next stage is to consider the propagation of a perturbation, and to do this end consider the vector of initial values $\mathbf{Y}_0$ perturbed to $\mathbf{Y}_0^*$ (if we assume no more rounding errors occur), then we have

$$\mathbf{Y}_n^* = \mathbf{T}^n(z)\mathbf{Y}_0^* + \mathbf{T}^{n-1}(z)\mathbf{N}(z) + \mathbf{T}^{n-2}(z)\mathbf{N}(z) + \cdots + \mathbf{N}(z). \tag{3.21}$$

Define the perturbation or error vector $\mathbf{e}$ to be $\mathbf{e} = \mathbf{Y}^* - \mathbf{Y}$. Then

$$\mathbf{e}_n = \mathbf{Y}_n^* - \mathbf{Y}_n = \mathbf{T}^n(z)\mathbf{e}_0, \ n = r, ..., N-1. \tag{3.22}$$

The multistep collocation method will be stable when $\mathbf{e}_n$ remains bounded at n increases indefinitely. In other words $\mathbf{e}_0$ propagates according to

$$\mathbf{e}_n = \mathbf{T}^n(z)\mathbf{e}_0. \tag{3.23}$$

Hence, for compatible matrix and vector norms,

$$\|\mathbf{e}_n\| \le \|\mathbf{T}^n(z)\|\|\mathbf{e}_0\|. \tag{3.24}$$

Lax defines [17] the difference scheme as stable if there exists $L > 0$, such that $\|\mathbf{T}^n(z)\| \le L$. This condition clearly limits the amplification of any initial perturbation and therefore of any rounding errors since $\|\mathbf{e}_n\| \le L\|\mathbf{e}_0\|$. Since

$$\|\mathbf{T}^n(z)\| \le \|\mathbf{T}(z)\|^n, \tag{3.25}$$

it follows that the Lax definition of stability is satisfied as long as $\rho(\mathbf{T}(z)) \le \|\mathbf{T}(z)\| \le 1$.

□

Now, let us define

$$\tilde{\Omega}_{jk} = \sum_{l=1}^{\mu_0} \omega_{jl}\varphi_k(d_{jl}), \quad \tilde{\rho}_{ik} = \sum_{l=1}^{\mu_0} \omega_{il}\psi_k(d_{il}),$$

$$\tilde{\beta}_k = \sum_{l=1}^{\mu_1} \omega_l\varphi_k(d_l), \qquad \tilde{\gamma}_j = \sum_{l=1}^{\mu_1} \omega_l\psi_j(d_l),$$

and introduce the vectors and matrices

$$\tilde{\boldsymbol{\beta}} = [\tilde{\beta}_0, ..., \tilde{\beta}_{r-1}]^T, \quad \tilde{\boldsymbol{\gamma}} = [\tilde{\gamma}_1, ..., \tilde{\gamma}_m]^T, \quad \tilde{\boldsymbol{\Omega}} = (\tilde{\Omega}_{ik}) \in \mathbb{R}^{m\times r}, \quad \tilde{\boldsymbol{\rho}} = (\tilde{\rho}_{ij}) \in \mathbb{R}^{m\times m}.$$

Then, we have the following Theorem:

Theorem 3.4. *The discretized multistep collocation method (2.9) and (2.13), applied to the test equation (3.1), leads to the following recurrence relation*

$$\mathbf{Y}_n = \tilde{\mathbf{T}}(z)\mathbf{Y}_{n-1} + \tilde{\mathbf{N}}(z),$$

where

$$Y_n = \begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \\ \mathbf{U}_n \\ y_{n-\tilde{r}+1} \\ \mathbf{y}_{n-\tilde{r}}^{(r)} \\ \mathbf{U}_{n-\tilde{r}} \end{bmatrix},$$

$\tilde{\mathbf{T}}(z) \in \mathbb{R}^{(2m+2r+2)\times(2m+2r+2)}$ *is the stability matrix and is given by*

$$\tilde{\mathbf{T}}(z) = \left[\tilde{\mathbf{P}}(z)\right]^{-1}\tilde{\mathbf{M}}(z), \ \tilde{\mathbf{N}}(z) = \left[\tilde{\mathbf{P}}(z)\right]^{-1}\mathbf{J}(z),$$

with $z = \lambda h$ *and*

$$\tilde{\mathbf{P}}(z) = \left[\begin{array}{c|c|c|c|c|c} \mathbf{0} & -z\tilde{\boldsymbol{\Omega}} & \mathbf{I}_m - z\tilde{\boldsymbol{\rho}} & \mathbf{0} & -z\tilde{\boldsymbol{\Omega}} & -z\tilde{\boldsymbol{\rho}} \\ \hline \mathbf{0} & -z\tilde{\boldsymbol{\Omega}} & \mathbf{I}_m - z\tilde{\boldsymbol{\rho}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{C}_1} & \mathbf{A}_1 & \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} & \multicolumn{2}{c|}{\mathbf{C}_2} & \mathbf{A}_2 \end{array}\right],$$

$$\tilde{\mathbf{M}}(z) = \left[\begin{array}{c|c|c|c|c|c} \mathbf{0} & z(\boldsymbol{u}\tilde{\boldsymbol{\beta}}^T - \tilde{\boldsymbol{\Omega}}) & \mathbf{I}_m + z(\boldsymbol{u}\tilde{\boldsymbol{\gamma}}^T - \tilde{\boldsymbol{\rho}}) & \mathbf{0} & z(\boldsymbol{u}\tilde{\boldsymbol{\beta}}^T - \tilde{\boldsymbol{\Omega}}) & z(\boldsymbol{u}\tilde{\boldsymbol{\gamma}}^T - \tilde{\boldsymbol{\rho}}) \\ \hline \mathbf{0} & z(\boldsymbol{u}\tilde{\boldsymbol{\beta}}^T - \tilde{\boldsymbol{\Omega}}) & \mathbf{I}_m + z(\boldsymbol{u}\tilde{\boldsymbol{\gamma}}^T - \tilde{\boldsymbol{\rho}}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{B}_1} & \mathbf{0} & \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} & \multicolumn{2}{c|}{\mathbf{B}_2} & \mathbf{0} \end{array}\right],$$

$\mathbf{A}_i, \mathbf{B}_i,\ \mathbf{C}_i, i = 1, 2$ *and* $\mathbf{J}(z)$ *are matrices defined in* (3.3) *and* (3.9)*.*

Proof. It is similar to the proof of Theorem 3.2. □

The stability function of the method with respect to (3.1) is defined as

$$p(w, z) = \det(w\mathbf{I}_{2m+2r+2} - \mathbf{T}(z)). \tag{3.26}$$

To investigate the stability properties of the exact multistep method, it is more convenient to work with the polynomial obtained by multiplying the stability function (3.26) by its denominator. The resulting polynomial will be denoted by the same symbol $p(w, z)$. Denoting by $w_1, w_2, \ldots, w_{2m+2r+2}$, the roots of the polynomial $p(w, z)$, the region of absolute stability of the methods is defined by

$$\mathcal{S} := \{z \in \mathbb{C} : |w_i(z)| < 1,\ i = 1, 2, \ldots, 2m + 2r + 2\}.$$

For this method, we have

$$p(w, z) = \sum_{i=0}^{2m+2r+2} p_i(z) w^i, \tag{3.27}$$

where $p_i(z),\ i = 0, 1, \ldots, 2m + 2r + 2$, are polynomials of degree less than or equal to $2m$. To obtain the region of absolute stability, we use the boundary locus method [10]. Inserting $w = e^{i\theta}$, the roots of (3.27) describe the stability region [16].

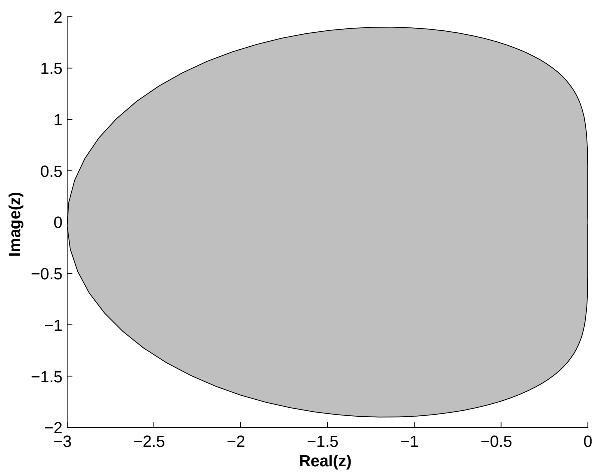

Figure 1: Stability region for multistep collocation method with $m = 1, r = 3$ and $c_1 = 1$ (convergence).

4. Examples for stability regions

In this section, we illustrate the theoretical results obtained in the previous section by the following examples. All computations are performed by MATLAB software.

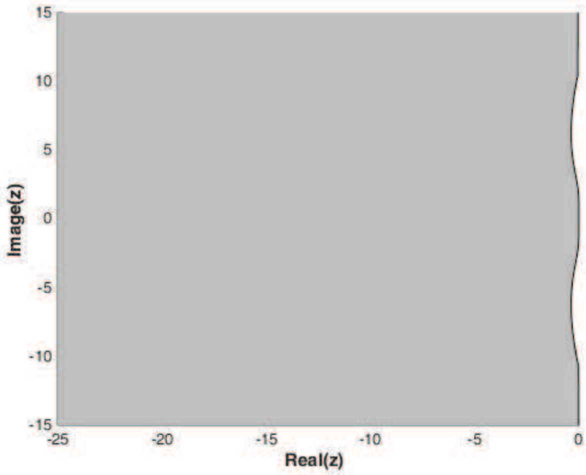

Figure 2: Stability region for multistep collocation method with $m = 2, r = 3$ and $c_1 = 0.7, c_2 = 1$ (convergence).

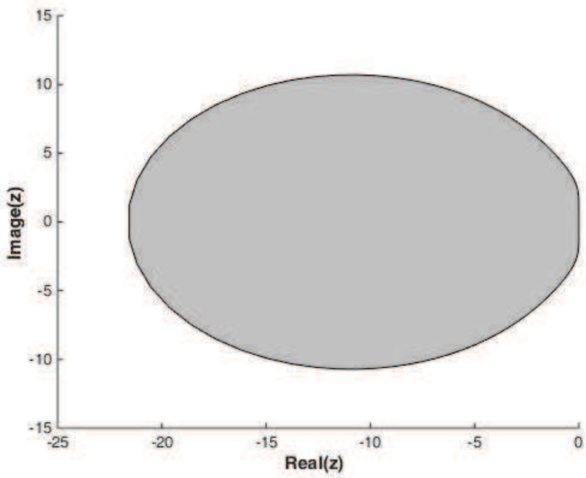

Figure 3: Stability region for multistep collocation method with $m = 2, r = 3$ and $c_1 = \frac{21}{38}, c_2 = 1$ (superconvergence).

Consider multistep collocation method with r steps and m collocation parameters $c_i, i = 1, 2, \ldots, m$. The stability polynomial for this family of methods is of the form

$$p(w, z) = \sum_{i=0}^{2m+2r+2} p_i(z) w^i,$$

where $p_i(z), i = 0, 1, \ldots, 2m + 2r + 2$ are polynomials of degree less than or equal to $2m$.

Note that, Figure 1. shows the stability region for multistep collocation method with $m = 1, r = 3$ and $c_1 = 1$. Figures 2. and 3. show stability region for multistep collocation method with $m = 2, r = 3$, $(c_1, c_2) = (0.7, 1)$ and $m = 2, r = 3$ with superconvergence collocation parameters [8] $(c_1, c_2) = (\frac{21}{38}, 1)$, respectively.

Remark 4.1. *In the discretized multistep collocation method, the order of applied quadrature rules is at least the same proved order for multistep collocation method in section 2. These rules are exact for the*

$\varphi_k(s), k = 0, 1, \ldots, r-1$ and $\psi_j(s), j = 1, 2, \ldots, m$, since these polynomials are of degree $2m + 2r + 2$. Thus for multistep collocation method, we have $T(z) = \tilde{T}(z)$ and so the stability regions plotted in Figures 1-3 don't change for the discretized cases.

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