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Nonexistence of Solutions of Higher-order Nonlinear Non-gauge Schrödinger Equation

Ahmed Alsaedi, Bashir Ahmad, Mokhtar Kirane and Abderrazak Nabti

ABSTRACT: A nonexistence result is proved of the space higher-order nonlinear Schrödinger equation

$$i\partial_t u - (-\Delta)^m(|u|^{n-1}u) = \lambda |u|^p,\ x \in \mathbb{R}^N,\ t > 0,$$

where $m > 1,\ n > 1$ and $p > n$. Our method of proof rests on a judicious choice of the test function in the weak formulation of the equation. Then, we obtain an upper bound of the life span of solutions. Furthermore, the necessary conditions for the existence of local or global solutions are provided.

Next, we extend our results to the 2×2 – system

$$\begin{aligned} i\partial_t u - (-\Delta)^m u &= \lambda |v|^p,\ x \in \mathbb{R}^N,\ t > 0, \\ i\partial_t v - (-\Delta)^m v &= \delta |u|^q,\ x \in \mathbb{R}^N,\ t > 0, \end{aligned}$$

where $m > 1$, $p, q > 1$, and $\lambda,\ \delta \in \mathbb{C}\backslash\{0\}$.

Key Words: Schrödinger equation, Nonexistence of solutions, Local and global existence.

Contents

1. Introduction

In this paper, we are concerned not only by the nonexistence of global solutions but also by the life span of solutions of the higher-order space nonlinear non-gauge Schrödinger equation

$$i\partial_t u - (-\Delta)^m(|u|^{n-1}u) = \lambda |u|^p,\ x \in \mathbb{R}^N, t > 0, \tag{1.1}$$

supplemented with the initial condition

$$u(x, 0) = f(x),\ x \in \mathbb{R}^N, \tag{1.2}$$

where $u = u(x, t)$ is the complex-valued unknown function, $i^2 = -1$, $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}\backslash\{0\}$, $\lambda_j \in \mathbb{R}\,(j = 1, 2)$, and $f(x) = f_1(x) + if_2(x),\ f_j = f_j(x) \in L^1_{loc}(\mathbb{R}^N)\,(j = 1, 2)$ are real valued functions.

Before presenting our results, let us dwell a while on some existing literature concerning equations close to (1.1).

In their valuable monograph [7], Galaktionov and al. have discussed the following nonlinear Schrödinger equation with nonlinear diffusion

$$-iu_t = (-1)^{m+1}\Delta^m(|u|^n u) + |u|^n u, \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+. \tag{1.3}$$

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Inspired on nonextensive thermostatistics, Nobre, Rego-Monteiro, and Tsallis [14] introduced the following d–dimensional non-linear generalization of the Schrodinger equation for a particle of mass m

$$i\hbar\frac{\partial}{\partial t}\left(\frac{\phi(x,t)}{\phi_0}\right)=-\frac{1}{2-q}\frac{\hbar^2}{2m}\Delta\left(\frac{\phi(x,t)}{\phi_0}\right)^{2-q}$$

where $1\leq q<3$. See [14] for the reason to introduce such equation.
Let also mention the paper of Majda, McLaughlin and Tabak [12] where they introduced the equation

$$i\frac{\partial}{\partial t}\psi=|\frac{\partial}{\partial x}|^{\alpha}\psi+|\frac{\partial}{\partial x}|^{\beta/4}\left(||\frac{\partial}{\partial x}|^{\beta/4}\psi|^2|\frac{\partial}{\partial x}|^{\beta/4}\psi\right)$$

with parameters $\alpha>0$ and β, where ψ denotes a complex wave field (the operator $|\frac{\partial}{\partial x}|^{\alpha}$ is defined via its Fourier symbol $|k|^{\alpha}$; $|\frac{\partial}{\partial x}|^2=-\frac{\partial^2}{\partial x^2}$); see also [13].
There are many results about nonexistence, blow up and global existence of solutions of nonlinear Schrödinger equations (see [10], [5], [16], [17]).

However, there are few results about upper estimates of the life span of solutions, and necessary conditions of local or global existence for nonlinear Schrödinger equations; we mention that in [9], an upper bound of the life span of solutions for the equation

$$i\partial_t u+\Delta u=\lambda|u|^p,\ \text{in } \mathbb{R}^N\times\mathbb{R}^+,\quad p>1, \tag{1.4}$$

subject to the initial data $u(x,0)=f(x)$, has been obtained. Moreover, in [2], necessary and sufficient conditions for global existence for the equation

$$i\partial_t u+\Delta u+\lambda|u|^{\alpha}u=0,\ \text{in } \mathbb{R}^N\times(-T,T), \tag{1.5}$$

with $u(x,0)=\varphi(x)$, $\lambda\in\mathbb{R}$, $0\leq\alpha<4/(N-2)$, ($0\leq\alpha<\infty$ if $N=1$), have been obtained.

In this paper, using the test function method, we derive nonexistence results for weak solutions to problem (1.1)–(1.2). Then we obtain an upper estimate for the life span of solutions of equation (1.1) with initial data of the form $u(x,0)=\mu f(x)$. Furthermore, we give the necessary conditions of local or global existence of solutions.

Next, we extend our results to a 2×2–system of equations of type (1.1).

2. Preliminaries

Lemma 2.1. *(See Lemma 3.1 in [15]) Let $\phi\in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}\phi(x)dx<0$. Then there exists a test function $0\leq\varphi\leq 1$ such that*

$$\int_{\mathbb{R}^N}\phi(x)\varphi(x)\,dx<0. \tag{2.1}$$

Now, we are in position to announce our results for problem (1.1)–(1.2).

3. Nonexistence and life span of solutions

The definition of a weak solution to (1.1)–(1.2) that we adopt is as follows.

Definition 3.1. *Let $Q_T:=\mathbb{R}^N\times(0,T)$. We say that a function u is a local weak solution of problem (1.1)–(1.2), if $u\in C([0,T];L^p_{loc}(\mathbb{R}^N))$ satisfies*

$$\lambda\int_{Q_T}|u|^p\varphi\,dxdt+i\int_{\mathbb{R}^N}f(x)\varphi(x,0)\,dx=\int_{Q_T}\left(-iu\,\partial_t\varphi-|u|^{n-1}u(-\Delta)^m\varphi\right)\,dxdt \tag{3.1}$$

for any $\varphi\in C_0^{\infty}(Q_T),\varphi\geq 0$ such that $\varphi(\cdot,T)=0$. If $T=+\infty$, we say that u is a global weak solution of problem (1.1)–(1.2).

To study nonexistence of solutions to problem (1.1)–(1.2), we require that the data f satisfies the assumption

$$(H1)\qquad f_1 \in L^1(\mathbb{R}^N),\ \lambda_2 \int_{\mathbb{R}^N} f_1(x)\,dx > 0, \quad \text{or} \quad f_2 \in L^1(\mathbb{R}^N),\ \lambda_1 \int_{\mathbb{R}^N} f_2(x)\,dx < 0.$$

We have the following result.

Theorem 3.2. *Let f satisfy $(H1)$, and let $\lambda \in \mathbb{C}\backslash\{0\}$. If*

$$p \le n + 2m/N, \tag{3.2}$$

then problem (1.1)–(1.2) does not admit a global weak solution.

Proof. Assume that u is a global bounded solution to problem (1.1)–(1.2). First, we construct a test function. For this aim, we shall use a non-negative smooth function Φ which was constucted in the paper [6] and [8]

$$\Phi(x) = \Phi(|x|), \quad \Phi(0) = 1, \quad 0 < \Phi(r) \le 1, \quad \text{for} \quad r \ge 0, \tag{3.3}$$

where $\Phi(r)$ is decreasing and $\Phi(r) \longrightarrow 0$ as $r \longrightarrow \infty$ sufficiently fast. Moreover, there exists a constant k_m such that

$$|\Delta^m \Phi| \le k_m \Phi, \quad x \in \mathbb{R}^N, \tag{3.4}$$

and $\|\Phi\|_{L^1} = 1$. This can be done by setting for example $\Phi(r) = e^{-r^\nu}$ for $r \gg 1$ with $\nu \in (0,1]$ and extending Φ to $[0,\infty)$ by a smooth approximation. Let σ be sufficiently large and

$$\phi(t) = (1-t)_+^\sigma, \quad t \ge 0. \tag{3.5}$$

Now set

$$\varphi(x,t) = \phi(t/R^{2\alpha})\Phi(x/R),\ R > 0,\ \alpha = \frac{m(p-1)}{p-n} > 0.$$

In the sequel of the paper, we will consider the same test functions defined in (3.3)–(3.5).
Here, we consider the case $\int_{\mathbb{R}^N} f_2(x)\,dx < 0$ and $\lambda_1 > 0$ only, since the other cases can be treated similarly (see Remark 3.3). By taking the real parts in (3.1), we have

$$\begin{aligned}
&\lambda_1 \int_{Q_{R^{2\alpha}}} |u|^p \varphi\,dxdt - \int_{\mathbb{R}^N} f_2(x)\varphi(x,0)\,dx \\
&= \int_{Q_{R^{2\alpha}}} (\operatorname{Im} u)\,\partial_t\varphi\,dxdt - \int_{Q_{R^{2\alpha}}} (\operatorname{Re} u)|u|^{n-1}\,(-\Delta)^m\varphi\,dxdt.
\end{aligned}$$

Setting

$$\mathfrak{I} := \int_{Q_{R^{2\alpha}}} |u|^p\varphi\,dxdt,$$

we obtain

$$\begin{aligned}
&\lambda_1 \mathfrak{I} - \int_{\mathbb{R}^N} f_2(x)\varphi(x,0)\,dx \\
&\le \int_{Q_{R^{2\alpha}}} |u||\partial_t\varphi|\,dxdt + \int_{Q_{R^{2\alpha}}} |u|^n|(-\Delta)^m\varphi|\,dxdt =: \mathfrak{I}_1 + \mathfrak{I}_2.
\end{aligned} \tag{3.6}$$

Now, applying ε–Young's inequality

$$XY \le \varepsilon X^p + C(\varepsilon)Y^{\tilde{p}}, \quad p + \tilde{p} = p\tilde{p}, \quad X \ge 0,\ Y \ge 0,\ \text{with } 0 < \varepsilon \ll 1,$$

$$\text{in } \begin{cases} \mathfrak{I}_1 & \text{with} \quad X = |u|\varphi^{\frac{1}{p}}, \quad \text{and} \quad Y = \varphi^{-\frac{1}{p}}|\partial_t\varphi|, \\ \mathfrak{I}_2 & \text{with} \quad X = |u|^n\varphi^{\frac{n}{p}}, \quad \text{and} \quad Y = \varphi^{-\frac{n}{p}}|(-\Delta)^m\varphi|, \end{cases}$$

we have the estimates

$$\mathcal{I}_1 \leq \varepsilon \mathcal{I} + C_1(\varepsilon) \int_{Q_{R^{2\alpha}}} \varphi^{-\frac{1}{p-1}} |\partial_t \varphi|^{\tilde{p}}\, dxdt,$$
$$\mathcal{I}_2 \leq \varepsilon \mathcal{I} + C_2(\varepsilon) \int_{Q_{R^{2\alpha}}} \varphi^{-\frac{n}{p-n}} |(-\Delta)^m \varphi|^{\frac{p}{p-n}}\, dxdt,$$

where $C_1(\varepsilon) = \frac{1}{\tilde{p}}(p\varepsilon)^{-\tilde{p}/p}$ and $C_2(\varepsilon) = \frac{p-n}{p}(\frac{p}{n}\varepsilon)^{-n/(p-n)}$.
So inequality (3.6) can be rewritten

$$\begin{aligned}
&(\lambda_1 - 2\varepsilon)\mathcal{I} - \int_{\mathbb{R}^N} f_2(x)\varphi(x,0)\, dx \\
&\leq C_1(\varepsilon) \int_{Q_{R^{2\alpha}}} \Phi(x/R)\phi(t/R^{2\alpha})^{\frac{-1}{p-1}} |\phi^{'}(t/R^{2\alpha})|^{\tilde{p}}\, dxdt \\
&+ C_2(\varepsilon) \int_{Q_{R^{2\alpha}}} \Phi(x/R)^{-\frac{n}{p-n}} |\Delta^m \Phi(x/R)|^{\frac{p}{p-n}} \phi(t/R^{2\alpha})\, dxdt.
\end{aligned}$$

At this stage, passing to the scaled variables

$$\tau = t/R^{2\alpha}, \quad y = x/R, \tag{3.7}$$

and setting

$$\Omega := \left\{(y,\tau) \in \mathbb{R}^N \times \mathbb{R}^+ \ : \ \tau \leq 1\right\},$$

we have

$$\begin{aligned}
&\int_{Q_{R^{2\alpha}}} \Phi(x/R)\phi(t/R^{2\alpha})^{\frac{-1}{p-1}} |\phi^{'}(t/R^{2\alpha})|^{\tilde{p}}\, dxdt \\
&\leq R^{\frac{2\alpha p}{p-1}+N+2\alpha} \int_\Omega \Phi(y)\phi(\tau)^{\frac{-1}{p-1}} |\phi^{'}(\tau)|^{\tilde{p}}\, dyd\tau,
\end{aligned} \tag{3.8}$$

and by using inequality (3.4), we have

$$\begin{aligned}
&\int_{Q_{R^{2\alpha}}} \Phi(x/R)^{-\frac{n}{p-n}} |\Delta^m \Phi(x/R)|^{\frac{p}{p-n}} \phi(t/R^{2\alpha})\, dxdt \\
&\leq R^{-\frac{2mp}{p-n}+N+2\alpha} \int_\Omega \Phi(y)\phi(\tau)\, dyd\tau.
\end{aligned}$$

Finally, we have the estimate

$$(\lambda_1 - 2\varepsilon)\mathcal{I} - \int_{\mathbb{R}^N} f_2(x)\varphi(x,0)\, dx \leq R^s(\mathcal{A} + \mathcal{B}), \tag{3.9}$$

where

$$\mathcal{A} := C_1(\varepsilon) \int_\Omega \Phi(y)\phi(\tau)^{\frac{-1}{p-1}} |\phi^{'}(\tau)|^{\tilde{p}}\, dyd\tau, \quad \mathcal{B} := C_2(\varepsilon) \int_\Omega \Phi(y)\phi(\tau)\, dyd\tau,$$

and

$$s := \frac{-2m}{p-n} + N.$$

As $p \leq n + 2m/N$, we hace $s \leq 0$. So, we have to consider two cases :

- The case $s_1 < 0$: In this case, letting $R \longrightarrow \infty$ in (3.9), we obtain

$$0 < (\lambda_1 - 2\varepsilon) \int_0^\infty \int_{\mathbb{R}^N} |u|^p\, dxdt - \int_{\mathbb{R}^N} f_2(x)\, dx = 0;$$

a contradiction.

• The case $s_1 = 0$: In this case, we take

$$\varphi(x,t) = \phi(t)\Phi(x/RB),$$

where $\phi(t) = \psi^{\sigma}\left(\frac{t}{R^{2\alpha}}\right)$, $\sigma \gg 1$, $R > 0$, and ψ is a smooth non-increasing function on $[0,\infty)$ such that

$$\psi(r) = \begin{cases} 1 & \text{if} \quad 0 \le r \le 1, \\ 0 & \text{if} \quad\quad\quad r \ge 2, \end{cases}$$

and $1 \le B < R$ is large enough such that when $R \longrightarrow \infty$ we don't have $B \longrightarrow \infty$ in the same time. Now, we estimate the first term on the right hand side of inequality (3.6) using again the ε-Young inequality and the second term by using the Hölder inequality as follows

$$\begin{aligned} &\lambda_1 \mathfrak{I} - \int_{\mathbb{R}^N} f_2(x)\varphi(x,0)\,dx \\ &\le \varepsilon\mathfrak{I} + C_2(\varepsilon)\int_{Q_{R^{2\alpha}}} \Phi(x/R)^{-\frac{n}{p-n}}|\Delta^m\Phi(x/R)|^{\frac{p}{p-n}}\phi(t)\,dxdt \\ &+ \left(\int_{\Omega_1}\int_{\mathbb{R}^N}|u|^p\varphi\,dxdt\right)^{\frac{1}{p}}\left(\int_{\Omega_1}\int_{\mathbb{R}^N}\Phi(x/R)\phi(t)^{\frac{-1}{p-1}}|\phi'(t)|^{\tilde{p}}\,dxdt\right)^{\frac{1}{\tilde{p}}}, \end{aligned} \tag{3.10}$$

where $\Omega_1 = \left\{t \in [0,\infty)\ :\ R^{2\alpha} \le t \le 2R^{2\alpha}\right\}$ is the support of $\phi'(t)$. Note that

$$\int_{\Omega_1}\int_{\mathbb{R}^N}|u|^p\varphi\,dxdt = 0 \quad \text{as} \quad R \longrightarrow \infty,$$

because $u \in L^p(\mathbb{R}^N \times (0,\infty))$. Furthermore, introducing the new variables $y = x/BR$, $\tau = t/R^{2\alpha}$ and using the fact that $p = n + 2m/N$, we rewrite (3.10) as follows

$$\begin{aligned} &(\lambda_1 - \varepsilon)\mathfrak{I} - \int_{\mathbb{R}^N} f_2(x)\varphi(x,0)\,dx \\ &\le C_1 B^{N/\tilde{p}}\left(\int_{\Omega_1}\int_{\mathbb{R}^N}|u|^p\varphi\,dxdt\right)^{\frac{1}{p}} + C_2 B^{-2\alpha}, \end{aligned} \tag{3.11}$$

where C_1, C_2 are independent of R, B. Passing to the limit as $R \longrightarrow \infty$ and then when $B \longrightarrow \infty$, we get

$$0 < (\lambda_1 - \varepsilon)\int_0^\infty\int_{\mathbb{R}^N}|u|^p\,dxdt - \int_{\mathbb{R}^N} f_2(x)\,dx = 0;$$

this is a contradiction. □

Remark 3.3. *For the other cases, setting*

$$\mathfrak{I} := \begin{cases} -\displaystyle\int_{Q_{R^{2\alpha}}} \lambda_1|u|^p\varphi(x,t)\,dxdt \quad if \quad \lambda_1 < 0,\ \lambda_1\int_{\mathbb{R}^N} f_2(x)\,dx < 0, \\ \displaystyle\int_{Q_{R^{2\alpha}}} \lambda_2|u|^p\varphi(x,t)\,dxdt \quad if \quad \lambda_2 > 0,\ \lambda_2\int_{\mathbb{R}^{2N}} f_1(x)\,dx > 0, \\ -\displaystyle\int_{Q_{R^{2\alpha}}} \lambda_2|u|^p\varphi(x,t)\,dxdt \quad if \quad \lambda_2 < 0,\ \lambda_2\int_{\mathbb{R}^N} f_1(x)\,dx > 0, \end{cases}$$

we can prove the same conclusion in the same manner as above.

For an estimate of the life span T_μ from above of a possible weak solution of problem (1.1)–(1.2), we make the following assumption on the data f

$$(H2)\begin{cases} \qquad f_1 \in L^1_{loc}(\mathbb{R}^N),\ \lambda_2 f_1(x) \geq |x|^{-k},\ |x|>1, \\ \text{or} \quad f_2 \in L^1_{loc}(\mathbb{R}^N),\ -\lambda_1 f_2(x) \geq |x|^{-k},\ |x|>1, \end{cases}$$

where $N < k < 2m/(p-n)$. We have the following assertion.

Theorem 3.4. *Suppose that conditions $(H2)$ and (3.2) are satisfied, and let u be the solution of (1.1) with the initial data $u(x,0)=\mu f(x)$, where $\mu>0$. Denote by $[0,T_\mu)$ the life span of u. Then there exists a constant $C>0$ such that*

$$T_\mu \leq C\mu^{1/\rho}, \tag{3.12}$$

where $\rho = k - \frac{2m}{p-n} < 0$.

Remark 3.5. *When $p = n + 2m/N$, we have $\rho = k - N$.*

Proof. Let u be the solution of equation (1.1) with the initial data $u(x,0)=\mu f(x)$. We take

$$\varphi(x,t) = \phi(t/T_\mu^{2\alpha})\Phi(x/T_\mu),\ T_\mu > 0.$$

We only consider the case when $\lambda_1 > 0$; the other cases can be treated in a similar manner. Then, as for the estimate (3.6), we have

$$\begin{aligned} &\lambda_1 \mathfrak{I} - \int_{\mathbb{R}^N} \mu f_2(x)\varphi(x,0)\,dx \\ &\leq \int_{Q_{T_\mu^{2\alpha}}} |u||\partial_t\varphi|\,dxdt + \int_{Q_{T_\mu^{2\alpha}}} |u|^n|(-\Delta)^m\varphi|\,dxdt. \end{aligned} \tag{3.13}$$

Repeating the same calculations as in the proof of Theorem 3.2, we arrive, with $0<\varepsilon \ll 1$, at

$$\begin{aligned} &(\lambda_1 - 2\varepsilon)\mathfrak{I} - \int_{\mathbb{R}^N} \mu f_2(x)\varphi(x,0)\,dx \\ &\leq C_1(\varepsilon)\int_{Q_{T_\mu^{2\alpha}}} \Phi(x/T_\mu)\phi(t/T_\mu^{2\alpha})^{\frac{-1}{p-1}}|\phi^{'}(t/T^{2\alpha})|^{\tilde{p}}\,dxdt \\ &+C_2(\varepsilon)\int_{Q_{T_\mu^{2\alpha}}} \Phi(x/T_\mu)^{-\frac{n}{p-n}}|\Delta^m\Phi(x/T_\mu)|^{\frac{p}{p-n}}\phi(t/T_\mu^{2\alpha})\,dxdt. \end{aligned}$$

If we take the scaled variables $\tau = t/T_\mu^{2\alpha}$, $y = x/T_\mu$, we get

$$(\lambda_1 - 2\varepsilon)\mathfrak{I} - T_\mu^N \int_{\mathbb{R}^N} \mu f_2(T_\mu y)\Phi(y)\,dy \leq T_\mu^s(\mathcal{A}+\mathcal{B}), \tag{3.14}$$

where the domain of integration of $\mathcal{A}$, $\mathcal{B}$ is $Q_{T_\mu^2}$.
By the assumption $(H2)$ on the data f, we have

$$\begin{aligned} \mu T_\mu^N \int_{\mathbb{R}^N} -f_2(T_\mu y)\Phi(y)\,dy \ &\geq\ \mu T_\mu \int_{|y|\geq \frac{1}{T_\mu}} -f_2(T_\mu y)\Phi(y)\,dy \\ &\geq\ \mu\lambda_1^{-1}T_\mu^{N-k}\int_{|y|\geq\frac{1}{T_\mu}} |y|^{-k}\Phi(y)\,dy \\ &\geq\ \mu\lambda_1^{-1}T_\mu^{N-k}\int_{|y|\geq\frac{1}{T_{\mu_0}}} |y|^{-k}\Phi(y)\,dy = C_k\mu T^{N-k}, \end{aligned} \tag{3.15}$$

where T_{μ_0} is a constant independent of T_μ, ε. Now, by the positivity of the first term in the left hand-side of (3.14), we obtain

$$C_k \mu T^{N-k} \leq T^s(\mathcal{A}+\mathcal{B}).$$

Whereupon

$$\mu \leq C_k^{-1}(\mathcal{A}+\mathcal{B}) T_\mu^{k-\frac{2m}{p-n}}.$$

Finally, we have

$$T_\mu \leq C\mu^{1/\rho};$$

this completes the proof of the Theorem. □

4. Necessary conditions for local or global existence

Here, we suppose that the data f satisfy the assumption

$$(H3) \qquad f_1 \in L^\infty(\mathbb{R}^N),\ \lambda_2 f_1(x)\,dx \geq 0, \quad \text{or} \quad f_2 \in L^\infty(\mathbb{R}^N),\ \lambda_1 f_2(x)\,dx \leq 0.$$

In this section, we only consider the case when $\lambda_1 > 0$. Then, the necessary conditions for the existence of local or global solutions to problem (1.1)–(1.2) are presented; these conditions depend on the behavior of the initial condition at infinity.

Theorem 4.1. *(Necessary conditions for global existence)*
Let f satisfy $(H3)$. If u is a global solution to problem (1.1)–(1.2), then there is a positive constant $C > 0$ such that

$$\liminf_{|x|\longrightarrow\infty}\left(-f_2(x)|x|^{\frac{2m}{p-n}}\right) \leq C.$$

Proof. Let u be a global weak solution to problem (1.1)–(1.2). We set

$$\varphi(x,t) = \phi(t/R^{2m})\Phi(x/R),$$

with $supp\Phi(x) := \{R \leq |x| \leq 2R\}$, with ($supp$ stands for support of $\Phi(x)$). Then, we have

$$\begin{aligned}\lambda \mathfrak{I} + \int_{\mathbb{R}^N} i f(x)\varphi(x,0)\,dx &= -\int_{Q_{R^{2m}}} iu\,\Phi(x/R)\phi^{'}(t/R^{2m})\,dxdt \\ &\quad -\int_{Q_{R^{2m}}} |u|^{n-1}u\,(-\Delta)^m\Phi(x/R)\phi(t/R^{2m})\,dxdt. \qquad (4.1)\end{aligned}$$

Now, by taking the real parts in (4.1), we get

$$\begin{aligned}\lambda_1 \mathfrak{I} - \int_{\mathbb{R}^N} f_2(x)\Phi(x/R)\,dx &\leq \int_{Q_{R^{2m}}} |u|\,\Phi(x/R)|\phi^{'}(t/R^{2m})|\,dxdt \\ &\quad + \int_{Q_{R^{2m}}} |u|^n|\Delta^m\Phi(x/R)|\phi(t/R^{2m})\,dxdt. \qquad (4.2)\end{aligned}$$

Applying ε-Young's inequality to the right hand side of (4.2), we obtain, with $0\varepsilon \ll 1$,

$$\begin{aligned}&(\lambda_1 - 2\varepsilon)\mathfrak{I} - \int_{\mathbb{R}^N} f_2(x)\Phi(x/R)\,dx \\ &\leq C_1(\varepsilon)\int_{Q_{R^{2m}}} \Phi(x/R)\phi(t/R^{2m})^{-\frac{1}{p-1}}|\phi^{'}(t/R^{2m})|^{\tilde{p}}\,dxdt \\ &+ C_2(\varepsilon)\int_{Q_{R^{2m}}} \Phi(x/R)^{-\frac{n}{p-n}}|\Delta^m\Phi(x/R)|^{\frac{p}{p-n}}\phi(t/R^{2m})\,dxdt. \qquad (4.3)\end{aligned}$$

If we take $\tau = t/R^{2m}$, $y = x/R$, and use inequality (3.4) in the right-hand side of (4.3), and take account of the positivity of the first term in the left-hand side, we get

$$\begin{aligned}
\int_{supp\Phi} -f_2(Ry)\Phi(y)\,dy &\leq \int_{supp\Phi} -f_2(Ry)\Phi(y)\,dy \\
&\leq CR^{\gamma}\int_{supp\Phi} \Phi(y)\,dy \\
&= CR^{\gamma}\int_{supp\Phi} |Ry|^{-\gamma}|Ry|^{\gamma}\Phi(y)\,dy \\
&\leq \tilde{C}(R)\int_{supp\Phi} |Ry|^{\gamma}\Phi(y)\,dy,
\end{aligned}$$

where $\gamma = \dfrac{-2m}{p-n}$, and $\tilde{C}(R) := CR^{\gamma}(2R)^{-\gamma} := C$. So

$$\int_{supp\Phi} -f_2(Ry)\Phi(y)\,dy \leq C\int_{|y|\leq 2} |Ry|^{\gamma}\Phi(y)\,dy. \tag{4.4}$$

Using the estimate

$$\begin{aligned}
\inf_{|y|\geq 1}\left(-f_2(Ry)|Ry|^{-\gamma}\right)\int_{|y|\leq 2} |Ry|^{\gamma}\Phi(y)\,dy &\leq \int_{supp\Phi} -f_2(Ry)\Phi(y)\,dy \\
&\leq \int_{|y|\leq 2} |Ry|^{\gamma}\Phi(y)\,dy
\end{aligned}$$

in the left-hand side of (4.4), we obtain, after dividing by $\displaystyle\int_{|y|\,\int_{|y|\leq 2}|Ry|^{\gamma}\Phi(y)\,dy2} |Ry|^{\gamma}\Phi(y)\,dy$, that

$$\inf_{|y|>1}\left(-f_2(Ry)|Ry|^{-\gamma}\right) \leq C. \tag{4.5}$$

Passing to the limit in (4.5), as $R \longrightarrow \infty$, we obtain

$$\liminf_{|x|\longrightarrow\infty}\left(-f_2(x)|x|^{\frac{2m}{p-n}}\right) \leq C.$$

$\square$

Corollary 4.2. *(Sufficient conditions for the nonexistence of global solutions)*
Let f satisfy $(H3)$. If

$$\liminf_{|x|\longrightarrow\infty}\left(-f_2(x)|x|^{\frac{2m}{p-n}}\right) = +\infty,$$

then problem (1.1)–(1.2) cannot admit a global solution.

Finally, we give a necessary condition for the local existence.

Theorem 4.3. *(Necessary conditions for the local existence)*
Let f satisfy $(H3)$. If u is a weak local solution of problem (1.1)–(1.2) on $[0,T]$ where $0 < T \leq \infty$, then we have

$$\liminf_{|x|\longrightarrow\infty}\left(-f_2(x)\right) \leq CT^{-\frac{1}{p-1}}, \tag{4.6}$$

for some positive constant $C > 0$.

Proof. We set, for $R > 0$ sufficiently large, $\varphi(x,t) = \phi(t/T)\Phi(x/R)$. Then as for the estimate (4.3), we have for $0 < \varepsilon \ll 1$,

$$\begin{aligned}
&(\lambda_1 - 2\varepsilon)\int_{Q_T} |u|\varphi\,dxdt - \int_{\mathbb{R}^N} f_2(x)\Phi(x/R)\,dx \\
&\leq C_1(\varepsilon)\int_{Q_T} \Phi(x/R)\,\phi(t/T)^{-\frac{1}{p-1}}|\phi^{'}(t/T)|^{\tilde{p}}\,dxdt \\
&+ C_2(\varepsilon)\int_{Q_T} \Phi(x/R)^{-\frac{n}{p-n}}|\Delta^m\Phi(x/R)|^{\frac{p}{p-n}}\phi(t/T)\,dxdt.
\end{aligned} \tag{4.7}$$

Now, we pass to the variables $\tau = t/T$, $y = x/R$, and use (3.4) in the right-hand side of (4.7), while in the left-hand side we use the positivity of the first term, to obtain

$$\int_{supp\Phi} -f_2(Ry)\Phi(y)\,dy \leq \left(C_1 T^{1-\tilde{p}} + C_2 T R^{-\frac{2m}{p-n}}\right)\int_{|y|\leq 2} \Phi(y)\,dy. \tag{4.8}$$

Using the estimate

$$\begin{aligned}
\inf_{|y|>1}(-f_2(Ry))\int_{|y|\leq 2}\Phi(y)\,dy \quad &\leq \quad \int_{supp\Phi} -f_2(Ry)\Phi(y)\,dy \\
&\leq \quad \left(C_1 T^{1-\tilde{p}} + C_2 T R^{-\frac{2m}{p-n}}\right)\int_{|y|\leq 2}\Phi(y)\,dy
\end{aligned}$$

in the left-hand side of (4.8), we obtain, after dividing by $\displaystyle\int_{|y|\leq 2}\Phi(y)\,dy$, that

$$\inf_{|y|\geq 1}(-f_2(Ry)) \leq C_1 T^{1-\tilde{p}} + C_2 T R^{-\frac{2m}{p-n}}. \tag{4.9}$$

Passing to the limit in (4.9), as $R \longrightarrow \infty$, we obtain

$$\liminf_{|x|\longrightarrow\infty}(-f_2(x)) \leq C T^{1-\tilde{p}} = C T^{\frac{-1}{p-1}},$$

which completes the proof of the Theorem. □

5. A system of two equations

Now, we consider the system of space nonlocal nonlinear non-gauge Schrödinger equations

$$\left\{\begin{array}{rcl} i\partial_t u - (-\Delta)^m u &=& \lambda|v|^p,\ x\in\mathbb{R}^N,\ t>0, \\ i\partial_t v - (-\Delta)^m v &=& \delta|u|^q,\ x\in\mathbb{R}^N,\ t>0, \end{array}\right. \tag{5.1}$$

supplemented with the initial conditions

$$u(x,0) = f(x),\ v(x,0) = g(x),\ x\in\mathbb{R}^N, \tag{5.2}$$

where $u = u(x,t)$, $v = v(x,t)$ are the complex-valued unknown functions, $\lambda,\,\delta \in \mathbb{C}\backslash\{0\}$, $\lambda = \lambda_1 + i\lambda_2$, $\delta = \delta_1 + i\delta_2$, $\lambda_j,\,\delta_j \in \mathbb{R}\,(j=1,2)$, and $f = f(x) = f_1(x) + if_2(x)$, $g = g(x) = g_1(x) + ig_2(x)$, $f_j(x),\ g_j(x) \in L^1_{loc}(\mathbb{R}^N)\,(j=1,2)$.

Definition 5.1. *We say that*

$$(u,v) \in C([0,T]; L^q_{loc}(\mathbb{R}^N)) \times C([0,T]; L^p_{loc}(\mathbb{R}^N))$$

is a local weak solution of problem (5.1)–(5.2) *if*

$$\lambda \int_{Q_T} |v|^p \varphi \, dxdt + \int_{\mathbb{R}^N} f(x)\varphi(x,0)\, dx$$
$$= \int_{Q_T} -iu\,\partial_t\varphi - u\,(-\Delta)^m \varphi \, dxdt, \tag{5.3}$$

and

$$\delta \int_{Q_T} |u|^q \varphi \, dxdt + \int_{\mathbb{R}^N} g(x)\varphi(x,0)\, dx$$
$$= \int_{Q_T} -iv\,\partial_t\varphi - v\,(-\Delta)^m \varphi \, dxdt, \tag{5.4}$$

for all $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$ *and satisfying* $\varphi(\cdot, T) = 0$. *If* $T = +\infty$, *we say that* (u, v) *is a global weak solution of* (5.1)–(5.2).

Now, we derive similar results for problem (5.1)–(5.2) as for problem (1.1)–(1.2).

6. Nonexistence and life span of solutions

Suppose that the initial conditions f and g satisfy, respectively, the assumptions

$$(A1) \qquad f_1 \in L^1(\mathbb{R}^N),\ \lambda_2 \int_{\mathbb{R}^N} f_1(x)\, dx > 0, \quad \text{or} \quad f_2 \in L^1(\mathbb{R}^N),\ \lambda_1 \int_{\mathbb{R}^N} f_2(x)\, dx < 0;$$

$$(A2) \qquad g_1 \in L^1(\mathbb{R}^N),\ \delta_2 \int_{\mathbb{R}^N} g_1(x)\, dx > 0, \quad \text{or} \quad g_2 \in L^1(\mathbb{R}^N),\ \delta_1 \int_{\mathbb{R}^N} g_2(x)\, dx < 0.$$

Theorem 6.1. *Let* f *and* g *satisfy, respectively,* (A1), (A2), *and let* $\lambda, \delta \in \mathbb{C}\backslash\{0\}$. *If*

$$N \leq \max\left\{\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right\}, \tag{6.1}$$

then problem (5.1)–(5.2) *has no global nontrivial solution.*

Proof. Let (u, v) be a nontrivial global solution of problem (5.1)–(5.2). Using (5.3), (5.4), and taking the real parts, we have

$$\lambda_1 \int_{Q_T} |v|^p \varphi \, dxdt - \int_{\mathbb{R}^N} f_2(x)\varphi(x,0)\, dx$$
$$= \int_{Q_T} (\mathrm{Im}\ u)\partial_t\varphi \, dxdt - \int_{Q_T} (\mathrm{Re}\ u)(-\Delta)^m \varphi \, dxdt,$$

and

$$\delta_1 \int_{Q_T} |u|^q \varphi \, dxdt - \int_{\mathbb{R}^N} g_2(x)\varphi(x,0)\, dx$$
$$= \int_{Q_T} (\mathrm{Im}\ v)\partial_t\varphi \, dxdt - \int_{Q_T} (\mathrm{Re}\ v)(-\Delta)^m \varphi \, dxdt.$$

We assume that

$$\varphi(x,t) := \phi(t/R^{2m})\Phi(x/R), \quad R > 0.$$

We only consider the case when $\lambda_1 > 0$, $\int_{\mathbb{R}^N} f_2(x)(x)\,dx < 0$, and $\delta_1 > 0$, $\int_{\mathbb{R}^N} g_2(x)\,dx < 0$. Then by the Lemma 2.1, we have the estimates

$$\lambda_1 \int_{Q_{R^{2m}}} |v|^p \varphi\,dxdt \quad \leq \quad \int_{Q_{R^{2m}}} |u||\partial_t \varphi|\,dxdt + \int_{Q_{R^{2m}}} |u||\Delta^m \varphi|\,dxdt, \tag{6.2}$$

and

$$\delta_1 \int_{Q_{R^{2m}}} |u|^q \varphi\,dxdt \quad \leq \quad \int_{Q_{R^{2m}}} |v||\partial_t \varphi|\,dxdt + \int_{Q_{R^{2m}}} |v||\Delta^m \varphi|\,dxdt. \tag{6.3}$$

We set

$$\mathcal{I} := \int_{Q_{R^{2m}}} |v|^p \varphi\,dxdt, \qquad \mathcal{J} := \int_{Q_{R^{2m}}} |u|^q \varphi\,dxdt.$$

By Hölder's inequality applied in the right hand sides of (6.11) and (6.12), respectively, we obtain

$$\lambda_1 \mathcal{I} \quad \leq \quad \mathcal{J}^{\frac{1}{q}} (\mathcal{A}_1 + \mathcal{B}_1), \tag{6.4}$$

$$\delta_1 \mathcal{J} \quad \leq \quad \mathcal{I}^{\frac{1}{p}} (\mathcal{A}_2 + \mathcal{B}_2), \tag{6.5}$$

where

$$\mathcal{A}_1 = \left(\int_{Q_{R^{2m}}} \Phi(x/R) \phi(t/R^{2m})^{\frac{-1}{q-1}} |\phi^{'}(t/R^{2m})|^{\tilde{q}}\,dxdt \right)^{\frac{1}{\tilde{q}}},$$

$$\mathcal{B}_1 = \left(\int_{Q_{R^{2m}}} \Phi(x/R)^{\frac{-1}{q-1}} |\Delta^m \Phi(x/R)|^{\frac{q}{q-1}} \phi(t/R^{2m})\,dxdt \right)^{\frac{1}{\tilde{q}}},$$

$$\mathcal{A}_2 = \left(\int_{Q_{R^{2m}}} \Phi(x/R) \phi(t/R^{2m})^{\frac{-1}{p-1}} |\phi^{'}(t/R^{2m})|^{\tilde{p}}\,dxdt \right)^{\frac{1}{\tilde{p}}},$$

$$\mathcal{B}_2 = \left(\int_{Q_{R^{2m}}} \Phi(x/R)^{\frac{-1}{p-1}} |\Delta^m \Phi(x/R)|^{\frac{p}{p-1}} \phi(t/R^{2m})\,dxdt \right)^{\frac{1}{\tilde{p}}}.$$

Combining (6.4) and (6.5), we obtain

$$\lambda_1 \mathcal{I} \quad \leq \quad \delta_1^{-\frac{1}{q}} \mathcal{I}^{\frac{1}{pq}} \left(\mathcal{A}_2 + \mathcal{B}_2\right)^{\frac{1}{q}} \left(\mathcal{A}_1 + \mathcal{B}_1\right),$$

$$\delta_1 \mathcal{J} \quad \leq \quad \lambda_1^{-\frac{1}{q}} \mathcal{J}^{\frac{1}{pq}} \left(\mathcal{A}_1 + \mathcal{B}_1\right)^{\frac{1}{p}} \left(\mathcal{A}_2 + \mathcal{B}_2\right).$$

Then, we have

$$\mathcal{I}^{1-\frac{1}{pq}} \quad \leq \quad C \left(\mathcal{A}_2 + \mathcal{B}_2\right)^{\frac{1}{q}} \left(\mathcal{A}_1 + \mathcal{B}_1\right), \tag{6.6}$$

$$\mathcal{J}^{1-\frac{1}{pq}} \quad \leq \quad C \left(\mathcal{A}_1 + \mathcal{B}_1\right)^{\frac{1}{p}} \left(\mathcal{A}_2 + \mathcal{B}\right). \tag{6.7}$$

At this stage, we introduce the scaled variables: $\tau = t/R^{2m}$, $y = x/R$; we obtain the estimates

$$\mathcal{I}^{1-\frac{1}{pq}} \quad \leq \quad CR^{\theta_1}, \tag{6.8}$$

$$\mathcal{J}^{1-\frac{1}{pq}} \quad \leq \quad CR^{\theta_2}, \tag{6.9}$$

where
$$\theta_1 := (N+2m)\left(\frac{pq-1}{pq}\right) - 2m\left(\frac{1+q}{q}\right),$$
and
$$\theta_2 := (N+2m)\left(\frac{pq-1}{pq}\right) - 2m\left(\frac{1+p}{p}\right).$$
Note that inequality (6.1) is equivalent to $\theta_1 \leq 0$ or $\theta_2 \leq 0$. So, we have to distinguish two cases :

• The case $\theta_1 < 0$ (resp. $\theta_2 < 0$) : we pass to the limit as $R \longrightarrow \infty$ in (6.8) and (6.9), respectively, we get
$$\int_0^\infty \int_{\mathbb{R}^N} |u|^q \, dxdt = 0 \quad \text{and} \quad \int_0^\infty \int_{\mathbb{R}^N} |v|^p \, dxdt = 0.$$
which implies that $u \equiv v \equiv 0$; contradiction.

• The case $\theta_1 = 0$ (resp. $\theta_2 = 0$) : in this case, we choose the test function as follows
$$\varphi(x,t) = \phi(t/R^{2m})\Phi(x),$$
where $\Phi(x) = \psi^\ell\left(\frac{|x|}{BR}\right)$, $\ell \gg 1$, and $1 \leq B \leq R$ is large enough such that when $R \longrightarrow \infty$ we don't have $B \longrightarrow \infty$ in the same time. We repeat the same calculations as above, we arrive at
$$u \equiv v \equiv 0,$$
which is a contradiction. □

Now, we give an estimate of the life span of solutions to problem (5.1)–(5.2). We introduce the following assumptions on f and g
$$(B1)\begin{cases} \qquad f_1 \in L^1_{loc}(\mathbb{R}^N),\ \lambda_2 f_1(x) \geq |x|^{-k},\ |x|>1, \\ \text{or} \quad f_2 \in L^1_{loc}(\mathbb{R}^N),\ -\lambda_1 f_2(x) \geq |x|^{-k},\ |x|>1, \end{cases}$$
and
$$(B2)\begin{cases} \qquad g_1 \in L^1_{loc}(\mathbb{R}^N),\ \delta_2 g_1(x) \geq |x|^{-k},\ |x|>1, \\ \text{or} \quad g_2 \in L^1_{loc}(\mathbb{R}^N),\ -\delta_1 g_2(x) \geq |x|^{-k},\ |x|>1, \end{cases}$$
where $N < k < \min\{\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\}$.

Theorem 6.2. *Suppose that conditions $(B1)$, $(B2)$ and (6.1) are satisfied, and let (u,v) be the solution of (5.1) with the initial condition $(u(x,0),v(x,0)) = \mu(f(x),g(x))$, where $\mu > 0$. Denote by $[0,T_\mu)$ the life span of (u,v). Then there exists a constant $C > 0$ such that*
$$T_\mu \leq C \min\left\{\mu^{1/\rho_1}, \mu^{1/\rho_2}\right\}, \tag{6.10}$$
where $\rho_1 = k - \frac{2m(p+1)}{pq-1}$, and $\rho_2 = k - \frac{2m(q+1)}{pq-1}$.

Proof. Let (u,v) be the solution of system (5.1) with the initial conditions $(u(x,0),v(x,0)) = \mu(f(x),g(x))$. We assume
$$\varphi(x,t) := \phi(t/T_\mu^{2m})\Phi(x/T_\mu).$$
Then, as for the estimates (6.11) and (6.12), by taking the case when $(\lambda_1, \delta_1 > 0)$ in the assumptions $(B1)$, $B(2)$, we have
$$\begin{aligned} &\lambda_1 \int_{Q_{T_\mu^{2m}}} |v|^p \varphi \, dxdt - \int_{\mathbb{R}^N} \mu f_2(x)\varphi(x,0)\, dx \\ &\leq \int_{Q_{T_\mu^{2m}}} |u||\partial_t \varphi| \, dxdt + \int_{Q_{T_\mu^{2m}}} |u||(-\Delta)^m \varphi| \, dxdt, \end{aligned} \tag{6.11}$$

and

$$\begin{aligned}
&\delta_1 \int_{Q_{T_\mu^{2m}}} |u|^q \varphi \, dxdt - \int_{\mathbb{R}^N} \mu g_2(x)\varphi(x,0)\, dx \\
&\leq \int_{Q_{T_\mu^{2m}}} |v||\partial_t \varphi| \, dxdt + \int_{Q_{T_\mu^{2m}}} |v||(-\Delta)^m \varphi| \, dxdt.
\end{aligned}$$

Now, by the ε-Young's inequality, we have

$$C\mathcal{I} - \int_{\mathbb{R}^N} \mu f_2(x)\varphi(x,0)\, dx \leq C \left(\mathcal{A}_2 + \mathcal{B}_2\right)^{\frac{p}{pq-1}} \left(\mathcal{A}_1 + \mathcal{B}_1\right)^{\frac{pq}{pq-1}}, \tag{6.12}$$

and

$$C\mathcal{J} - \int_{\mathbb{R}^N} \mu g_2(x)\varphi(x,0)\, dx \leq C \left(\mathcal{A}_1 + \mathcal{B}_1\right)^{\frac{q}{pq-1}} \left(\mathcal{A}_2 + \mathcal{B}_2\right)^{\frac{pq}{pq-1}}, \tag{6.13}$$

where the domain of integration of $\mathcal{A}_i$, $\mathcal{B}_i$ $(i = 1, 2)$ is $Q_{T_\mu^2}$.
Making the following change of variables $\tau = t/T_\mu^{2m}, y = x/T_\mu$, we obtain

$$C\mathcal{I} - T_\mu^N \int_{\mathbb{R}^N} \mu f_2(T_\mu y)\Phi(y)\, dy \leq C T_\mu^{\gamma_1}, \tag{6.14}$$

and

$$C\mathcal{J} - T_\mu^N \int_{\mathbb{R}^N} \mu g_2(T_\mu y)\Phi(y)\, dy \leq C T_\mu^{\gamma_2}, \tag{6.15}$$

where

$$\gamma_1 := N - \frac{2m(p+1)}{pq-1}, \quad \gamma_2 := N - \frac{2m(q+1)}{pq-1}.$$

By the assumptions on the data f and g, we have as for the estimate (3.15)

$$\mu T_\mu^N \int_{\mathbb{R}^N} -f_2(T_\mu y)\Phi(y)\, dy \quad \geq \quad \mu \lambda_1^{-1} T_\mu^{N-k} \int_{|y| \geq \frac{1}{T_{\mu_0}}} |y|^{-k} \Phi(y)\, dy = C_k \mu T_\mu^{N-k},$$

and

$$\mu T_\mu^N \int_{\mathbb{R}^N} -g_2(T_\mu y)\Phi(y)\, dy \quad \geq \quad \mu \delta_1^{-1} T_\mu^{N-k} \int_{|y| \geq \frac{1}{T_{\mu_0}}} |y|^{-k} \Phi(y)\, dy = C_k \mu T_\mu^{N-k},$$

where T_{μ_0} is a constant independent of T_μ, ε.
Now, by the positivity of the first term in the left-hand sides of (6.14) and (6.15), we obtain

$$C_k \mu T_\mu^{N-k} \leq C T_\mu^{\gamma_1},$$

and

$$C_k \mu T_\mu^{N-k} \leq C T_\mu^{\gamma_2}.$$

Whereupon

$$\mu \leq C T_\mu^{k - \frac{2m(p+1)}{pq-1}}, \qquad \mu \leq C T_\mu^{k - \frac{2m(q+1)}{pq-1}}.$$

Finally, we can get the following estimates

$$T_\mu \leq C \mu^{1/\rho_1}, \quad T_\mu \leq C \mu^{1/\rho_2},$$

where $\rho_1 = k - \frac{2m(p-1)}{pq-1}$, and $\rho_2 = k - \frac{2m(q+1)}{pq-1}$; which completes the proof of the Theorem. $\square$

7. Necessary conditions for local or global existence

We suppose the following assumptions on the initial conditions f and g,

$$(C1) \qquad f_1 \in L^\infty(\mathbb{R}^N),\ \lambda_2 f_1(x) \geq 0, \quad \text{or} \quad f_1 \in L^\infty(\mathbb{R}^N),\ \lambda_1 f_2(x) \leq 0,$$

and

$$(C2) \qquad g_1 \in L^\infty(\mathbb{R}^N),\ \delta_2 g_1(x) \geq 0, \quad \text{or} \quad g_1 \in L^\infty(\mathbb{R}^N),\ \delta_1 g_2(x) \leq 0.$$

We also consider the case when λ_1, $\delta_1 > 0$ only; the other cases can be treated similarly.

Theorem 7.1. *(Necessary condition for global existence)*
Suppose that assumptions $(C1)$ and $(C2)$ are satisfied. If (u, v) is a weak global solution to problem (5.1)–(5.2), then there is a constant $C > 0$ such that

$$\liminf_{|x|\longrightarrow\infty} \left(-f_2(x)|x|^{\frac{2mp+1)}{pq-1}}\right) \leq C \quad \text{and} \quad \liminf_{|x|\longrightarrow\infty} \left(-g_2(x)|x|^{\frac{2m(q+1)}{pq-1}}\right) \leq C. \tag{7.1}$$

Proof. Let (u, v) be a global weak solution to problem (5.1)–(5.2), Now, we set

$$\varphi(x,t) := \phi(t/R^{2m})\Phi(x/R),\ R > 0,$$

with $supp\Phi(x) := \{R \leq |x| \leq 2R\}$. Then, we have

$$\lambda\mathfrak{I} + \int_{\mathbb{R}^N} if(x)\varphi(x,0)\,dx \;=\; -\int_{Q_{R^{2m}}} iu\,\Phi(x/R)\phi^{'}(t/R^{2m})\,dxdt \tag{7.2}$$

$$\qquad - \int_{Q_{R^{2m}}} u\,(-\Delta)^m\Phi(x/R)\phi(t/R^{2m})\,dxdt, \tag{7.3}$$

and

$$\delta\mathfrak{J} + \int_{\mathbb{R}^N} ig(x)\varphi(x,0)\,dx \;=\; -\int_{Q_{R^{2m}}} iv\,\Phi(x/R)\phi^{'}(t/R^{2m})\,dxdt$$

$$\qquad - \int_{Q_{R^{2m}}} v\,(-\Delta)^m\Phi(x/R)\phi(t/R^{2m})\,dxdt. \tag{7.4}$$

By taking the real parts in (7.3), (7.4), we arrive at

$$\lambda_1\mathfrak{I} - \int_{\mathbb{R}^N} f_2(x)\Phi(x/R)\,dx \;\leq\; \int_{Q_{R^{2m}}} |u|\,\Phi(x/R)|\phi^{'}(t/R^{2m})|\,dxdt$$

$$\qquad + \int_{Q_{R^{2m}}} |u|\,|\Delta^m\Phi(x/R)|\phi(t/R^{2m})\,dxdt, \tag{7.5}$$

and

$$\delta_1\mathfrak{J} - \int_{\mathbb{R}^N} g_2(x)\Phi(x/R)\,dx \;\leq\; \int_{Q_{R^{2m}}} |v|\,\Phi(x/R)|\phi^{'}(t/R^{2m})|\,dxdt$$

$$\qquad + \int_{Q_{R^{2m}}} |v||\Delta^m\Phi(x/R)|\phi(t/R^{2m})\,dxdt. \tag{7.6}$$

Using Hölder and ε-Young inequalities, we obtain

$$C\mathfrak{I} - \int_{\mathbb{R}^N} \mu f_2(x)\varphi(x,0)\,dx \leq C\,(\mathcal{A}_2 + \mathcal{B}_2)^{\frac{p}{pq-1}}\,(\mathcal{A}_1 + \mathcal{B}_1)^{\frac{pq}{pq-1}}, \tag{7.7}$$

and

$$C\mathcal{J}-\int_{\mathbb{R}^N}\mu g_2(x)\varphi(x,0)\,dx \le C\left(\mathcal{A}_1+\mathcal{B}_1\right)^{\frac{q}{pq-1}}\left(\mathcal{A}_2+\mathcal{B}_2\right)^{\frac{pq}{pq-1}}. \tag{7.8}$$

Now, we pass to the new variables $\tau=t/R^{2m}$, $y=x/R$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}-f_2(Ry)\Phi(y)\,dy &\le CR^{\alpha_1}\int_{|y|\le 2}\Phi(y)\,dy,\\ \int_{\mathbb{R}^N}-g_2(Ry)\Phi(y)\,dy &\le CR^{\alpha_2}\int_{|y|\le 2}\Phi(y)\,dy,\end{aligned}$$

where $\alpha_1:=-\frac{2m(p+1)}{pq-1}$, and $\alpha_2:=-\frac{2m(q+1)}{pq-1}$. Furthermore, we have

$$\int_{\mathbb{R}^N}-f_2(Ry)\Phi(y)\,dy \le C(R)\int_{|y|\le 2}|Ry|^{-\alpha_1}|Ry|^{\alpha_1}\Phi(y)\,dy,$$

and

$$\int_{\mathbb{R}^N}-g_2(Ry)\Phi(\xi)\,d\xi \le \tilde{C}(R)\int_{|y|\le 2}|Ry|^{-\alpha_2}|Ry|^{\alpha_2}\Phi(y)\,dy.$$

Thus

$$\int_{\mathbb{R}^N}-f_2(Ry)\Phi(y)\,dy \le C\int_{|y|\le 2}|Ry|^{\alpha_1}\Phi(y)\,dy,$$

and

$$\int_{\mathbb{R}^N}-g_2(Ry)\Phi(\xi)\,d\xi \le \tilde{C}\int_{|y|\le 2}|Ry|^{\alpha_2}\Phi(y)\,dy.$$

Using the estimates

$$\inf_{|y|\ge 1}\left(-f_2(Ry)|Ry|^{-\gamma}\right)\int_{|y|\le 2}|Ry|^{\alpha_1}\Phi(y)\,dy \le \int_{supp\Phi}-f_2(Ry)\Phi(y)\,dy, \tag{7.9}$$

and

$$\inf_{|y|\ge 1}\left(-g_2(Ry)|Ry|^{-\gamma_2}\right)\int_{|y|\le 2}|Ry|^{\alpha_2}\Phi(y)\,dy \le \int_{supp\Phi}-g_2(Ry)\Phi(y)\,d\xi, \tag{7.10}$$

in the right-hand sides of (7.9) and (7.10), we conclude, after dividing by $\int_{|y|\le 2}|Ry|^{\alpha_1}\Phi(y)\,dy$ and $\int_{|y|\le 2}|Ry|^{\alpha_2}\Phi(y)\,dy$, respectively, that

$$\inf_{|y|\ge 1}\left(-f_2(Ry)|Ry|^{-\alpha_1}\right)\le C, \tag{7.11}$$

and

$$\inf_{|y|\ge 1}\left(-g_2(Ry)|Ry|^{-\alpha_2}\right)\le C. \tag{7.12}$$

Passing to the limit in (7.11), (7.12), as $R\longrightarrow\infty$, we obtain

$$\liminf_{|x|\longrightarrow\infty}\left(-f_2(x)|x|^{\frac{2m(p+1)}{pq-1}}\right)\le C \quad\text{and}\quad \liminf_{|x|\longrightarrow\infty}\left(-g_2(x)|x|^{\frac{2m(q+1)}{pq-1}}\right)\le C;$$

this completes the proof. $\square$

Corollary 7.2. *(Sufficient conditions for the nonexistence of global solutions)*
Let f, g satisfy the assumptions $(C1)$, $(C2)$. If

$$\liminf_{|x|\longrightarrow\infty}\left(-f_2(x)|x|^{\frac{2m(p+1)}{pq-1}}\right)=+\infty,\quad \textit{or}\quad \liminf_{|x|\longrightarrow\infty}\left(-g_2(x)|x|^{\frac{2m(q+1)}{pq-1}}\right)=+\infty,$$

then problem (5.1)–(5.2) cannot admit a global solution.

Theorem 7.3. *(Necessary condition for local existence)*
Suppose that assumptions $(C1)$ and $(C2)$ are satisfied. If (u,v) is a local solution to problem (5.1)–(5.2) on $[0,T]$ where $0<T<\infty$, then we have the estimates

$$\liminf_{|x|\longrightarrow\infty}(-f_2(x))\le CT^{-\frac{p+1}{pq-1}}\quad \textit{and}\quad \liminf_{|x|\longrightarrow\infty}(-g_2(x))\le CT^{-\frac{q+1}{pq-1}},$$

for some positive constant $C>0$.

Proof. Let (u,v) be a local solution to problem (5.1)–(5.2). Setting $\varphi(x,t):=\phi(t/T)\Phi(x/R)$, and repeating the same calculations as above, we get

$$\int_{supp\Phi}-f_2(Ry)\Phi(y)\,dy \quad\le\quad C_1(T,R)\int_{supp\Phi}\Phi(y)\,dy, \tag{7.13}$$

and

$$\int_{supp\Phi}-g_2(Ry)\Phi(y)\,dy \quad\le\quad C_2(T,R)\int_{supp\Phi}\Phi(y)\,dy, \tag{7.14}$$

where

$$C_1(T,R):=\left(T^{-\frac{p}{pq-1}}+T^{\frac{pq-p}{pq-1}}R^{-\frac{2mpq}{pq-1}}\right)\left(T^{-\frac{1}{pq-1}}+T^{\frac{p-1}{pq-1}}R^{-\frac{2mp}{pq-1}}\right),$$

and

$$C_2(T,R):=\left(T^{-\frac{q}{pq-1}}+T^{\frac{pq-q}{pq-1}}R^{-\frac{2mpq}{pq-1}}\right)\left(T^{-\frac{1}{pq-1}}+T^{\frac{q-1}{pq-1}}R^{-\frac{2mq}{pq-1}}\right).$$

Using the estimates

$$\begin{aligned}\inf_{|y|\ge 1}(-f_2(Ry))\int_{|y|\le 2}\Phi(y)\,dy &\le \int_{supp\Phi}-f_2(Ry)\Phi(y)\,dy\\ &\le C_1(T,R)\int_{|y|\le 2}\Phi(y)\,dy,\end{aligned}\tag{7.15}$$

and

$$\begin{aligned}\inf_{|y|\ge 1}(-g_2(Ry))\int_{|y|\le 2}\Phi(y)\,dy &\le \int_{supp\Phi}-g_2(Ry)\Phi(y)\,dy\\ &\le C_2(T,R)\int_{|y|\le 2}\Phi(y)\,dy\end{aligned}\tag{7.16}$$

in the left-hand side of (7.13), (7.14), respectively, we conclude, after dividing by the term $\int_{|y|\le 2}\Phi(y)\,dy$, that

$$\inf_{|y|\ge 1}(-f_2(Ry))\le C_1(T,R), \tag{7.17}$$

and

$$\inf_{|y|\ge 1}(-g_2(Ry))\le C_2(T,R). \tag{7.18}$$

Passing to the limit in (7.17) and (7.18), as $R\longrightarrow\infty$, we arrive at

$$\liminf_{|x|\longrightarrow\infty}(-f_2(x))\le T^{-\frac{p+1}{pq-1}}\quad\text{and}\quad \liminf_{|x|\longrightarrow\infty}(-g_2(x))\le T^{-\frac{q+1}{pq-1}};$$

which completes the proof of the Theorem. ∎

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Ahmed Alsaedi,
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University, P.O. Box 80203, Jeddah 21589,
Saudi Arabia.
E-mail address: aalsaedi@hotmail.com

and

Bashir Ahmad,
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University, P.O. Box 80203, Jeddah 21589,
Saudi Arabia.
E-mail address: bashirahmad_qau@yahoo.com

and

Mokhtar Kirane,
LaSIE, Université de La Rochelle,
Avenue M. Crépeau, 17000 La Rochelle, France
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University, P.O. Box 80203, Jeddah 21589,
Saudi Arabia.
E-mail address: mokhtar.kirane@univ-lr.fr

and

Abderrazak Nabti,
Département de Mathématiques et Informatique, Université Cheikh El Arbi Tébessi,
12002, Tébessa, Algeria.
E-mail address: abderrazaknabti@gmail.com