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On the Sequences of Polynomials and Their Generating Functions

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ABSTRACT: We give first of all, an identity having interesting applications on polynomials and some combinatorial sequences. Secondly, we refer two interesting formulas on generating functions of polynomials. Our results are illustrated by some comprehensive examples.

Key Words: Sequences of polynomials, Appell polynomials, Generating functions of polynomials.

Contents

1. Introduction

Generating functions of specified sequences of polynomials, such as Hermite, Laguerre, Bell, Chebychev, Jacobi polynomials and others, have been studied by a large number of authors, see for example [3,4,12,13,15]. Different methods and techniques are used to develop some relations, formulas and identities. Taylor-Maclaurin expansion and Lagrange inversion formula are the principal tools often used for such studies. In this paper, based on an identity on polynomials established below, we give interesting formulas for the generating functions of polynomials. Indeed, let m, n be natural numbers, z be a complex number and let P_m be a polynomial with degree at most m, we prove below that the following identity holds

$$P_m\left(z\right)=\sum_{k=0}^{n+m}\left(-1\right)^{n+k}k^n\binom{n+m+1}{k+1}P_m\left(-kz\right). \tag{1.1}$$

We use this identity to establish a formula on generating function for any sequence of polynomials. Further formula is established from the Melzak formula [6,7] given, for any polynômial f with degree $\leqslant p$, by

$$f\left(x+\alpha\right)=\alpha\binom{\alpha+p}{p}\sum_{j=0}^{p}\left(-1\right)^j\binom{p}{j}\frac{f\left(x-j\right)}{\alpha+j}, \tag{1.2}$$

where x and α are complex numbers.

2. Identities on polynomials

The key of this paper is the following proposition.

Proposition 2.1. *Let m be a non-negative integer and let H, G be two power series such that $H\left(0\right)=1$. Then*

$$D_{t=0}^{n}\left(\frac{G\left(t\right)}{H\left(t\right)}\right)=\sum_{k=0}^{n+m}\left(-1\right)^k\binom{n+m+1}{k+1}D_{t=0}^{n}\left(H^k\left(t\right)G\left(t\right)\right), \tag{2.1}$$

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and if $G(0)=1$ we also have

$$D_{t=0}^{n}\left(\frac{G(t)}{H(t)}\right)=\sum_{k=0}^{n+m}(-1)^{k}\binom{n+m+1}{k+1}D_{t=0}^{n}\left(H^{k}(t)\,G^{-k}(t)\right), \tag{2.2}$$

where $D_{t=0}^{n}(f(t))$ means the coefficient of t^n in the Taylor expansion of $f(t)$.

Proof. Since $H(0)=1$, it follows that

$$\frac{G(t)}{H(t)}\left(H(t)-1\right)^{n+m+1}=t^{n+m+1}M(t)$$

for some power series M. Then

$$\begin{aligned} 0 &= D_{t=0}^{n}\left(\frac{G(t)}{H(t)}\left(H(t)-1\right)^{n+m+1}\right)\\ &= (-1)^{n+m}\sum_{k=0}^{n+m+1}(-1)^{k-1}\binom{n+m+1}{k}D_{t=0}^{n}\left(H^{k-1}(t)\,G(t)\right)\\ &= (-1)^{n+m}\left[-D_{t=0}^{n}\left(\frac{G(t)}{H(t)}\right)+\sum_{k=0}^{n+m}(-1)^{k}\binom{n+m+1}{k+1}D_{t=0}^{n}\left(H^{k}(t)\,G(t)\right)\right]. \end{aligned}$$

So, the first identity follows. This identity becomes when we set $G(t)=1$:

$$D_{t=0}^{n}\left(\frac{1}{H(t)}\right)=\sum_{k=0}^{n+m}(-1)^{k}\binom{n+m+1}{k+1}D_{t=0}^{n}\left(H^{k}(t)\right).$$

Then, by replacing H by H/G for such power series G with $G(0)=1$, the second identity follows. $\square$

Example 2.2. Let $\left(L_n^{(\alpha,\beta)}(x)\,; n\geq 0\right)$ be a sequence of polynomials defined by

$$\sum_{n\geq 0}L_n^{(\alpha,\beta)}(x)\frac{t^n}{n!}=(1-t)^{\alpha}\exp\left(x\left((1-t)^{\beta}-1\right)\right),$$

for more information above these class of polynomials, see [10,11].

For $\alpha=-(c+1)k$ and $\beta=-\frac{1}{k}$, $k=1,2,\ldots$, $c>-1$, these polynomials are named Konhauser's (biorthogonal) polynomials and can also be viewed as a generalization of Laguerre polynomials, see [4].

For $G(t)=(1-t)^{\alpha}$, $H(t)=\exp\left(-z\left((1-t)^{\beta}-1\right)\right)$ in the formulas (2.1) and (2.2) we obtain respectively

$$\begin{aligned} L_m^{(\alpha,\beta)}(z) &= \sum_{k=0}^{n+m}(-1)^{k}\binom{n+m+1}{k+1}L_m^{(\alpha,\beta)}(-kz),\\ L_m^{(\alpha,\beta)}(z) &= \sum_{k=0}^{n+m}(-1)^{k}\binom{n+m+1}{k+1}L_m^{(-k\alpha,\beta)}(-kz). \end{aligned}$$

For $G(t)=\exp\left(z\left((1-t)^{\beta}-1\right)\right)$, $H(t)=(1-t)^{-\alpha}$ in the formula (2.1) we obtain

$$L_m^{(\alpha,\beta)}(z)=\sum_{k=0}^{n+m}(-1)^{k}\binom{n+m+1}{k+1}L_m^{(-k\alpha,\beta)}(z).$$

By application of Proposition 2.1 on Appell polynomials, we derive an identity on polynomials on which is based the rest of this paper. Recall that an Appell sequence is a sequence $(f_n; n \geq 0)$ of polynomials satisfying $\frac{d}{dx} f_n(x) = n f_{n-1}(x)$ and $deg(f_0) = 0$, see [1].

Proposition 2.3. *Let α, β be real numbers and let $\left(f_n^{(\alpha)}(x)\right)$ be the sequence of Appell polynomials having exponential generating function*

$$\sum_{n\geq 0} f_n^{(\alpha)}(x) \frac{t^n}{n!} = (F(t))^{\alpha} e^{xt}, \quad F(0) = 1. \tag{2.3}$$

Then

$$f_n^{(\alpha+\beta)}(x) = \sum_{k=0}^{n+m} (-1)^k \binom{n+m+1}{k+1} f_n^{(\beta-\alpha k)}(x), \tag{2.4}$$

$$f_n^{(\alpha)}(x) = \sum_{k=0}^{n+m} (-1)^k \binom{n+m+1}{k+1} f_n^{(-\alpha k)}(-kx). \tag{2.5}$$

Proof. When we set $G(t) = e^{xt}(F(t))^{\beta}$ and $H(t) = (F(t))^{-\alpha}$ in the formula (2.1) we obtain the identity (2.4), and, when we set $G(t) = e^{xt}$ and $H(t) = (F(t))^{-\alpha}$ in the formula (2.2), we obtain the identity (2.5). □

The identity (2.5) can be generalized as follows.

Proposition 2.4. *Let α be a real number and let $\left(f_n^{(\alpha)}(x)\right)$ be as above and P_m be a polynomial of degree $\leq m$. Then, for any complex number z, we have*

$$f_n^{(\alpha)}(x) P_m(z) = \sum_{k=0}^{n+m} (-1)^k \binom{n+m+1}{k+1} f_n^{(-\alpha k)}(-kx) P_m(-kz). \tag{2.6}$$

Proof. Since $\frac{d}{dx} f_n^{(\alpha)}(x) = n f_{n-1}^{(\alpha)}(x)$, then by derivation h times the two sides of the identity (2.5), we get

$$f_n^{(\alpha)}(x) = \sum_{k=0}^{n+m+h} (-1)^k (-k)^h \binom{n+m+h+1}{k+1} f_n^{(-\alpha k)}(-kx). \tag{2.7}$$

Setting $P_m(z) = \sum_{j=0}^{m} a_j z^j$. Then, by replacing (m, h) by $(m+h-j, j)$ in (2.7) we obtain

$$f_n^{(\alpha)}(x) z^j = \sum_{k=0}^{n+m+h} (-1)^k (-kz)^j \binom{n+m+h+1}{k+1} f_n^{(-\alpha k)}(-kx).$$

Multiply this identity by a_j and sum it over $j = 0, \ldots, m+h$ to get

$$f_n^{(\alpha)}(x) P_{m+h}(z) = \sum_{k=0}^{n+m+h} (-1)^k \binom{n+m+h+1}{k+1} f_n^{(-\alpha k)}(-kx) P_{m+h}(-kz),$$

which is equivalent, when we replace $m+h$ by m, to the desired identity. □

For $P_m(z) = z(z+1)\cdots(z+m-1)$ and $z = 1$ in Proposition 2.4, we obtain

Corollary 2.5.

$$f_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{k+m}{m} \binom{n+m+1}{k+m+1} f_n^{(-\alpha(k+m))}(-(k+m)x). \tag{2.8}$$

In particular, for $f_n^{(\alpha)}(x) = x^n$ in Proposition 2.4, we get

Corollary 2.6.

$$P_m(z) = \sum_{k=0}^{n+m} (-1)^{n+k} k^n \binom{n+m+1}{k+1} P_m(-kz). \tag{2.9}$$

In particular, for $n = 0$, we get

$$P_m(z) = \sum_{k=0}^{m} (-1)^k \binom{m+1}{k+1} P_m(-kz). \tag{2.10}$$

Remark 2.7. We note that the identity (2.9) can be derived from more elementary computations. Indeed, it can be written as

$$\sum_{k=0}^{n+m+1} (-1)^{k-1} (1-k)^n \binom{n+m+1}{k} P_m((1-k)z) = 0. \tag{2.11}$$

By linearity it suffices to prove the formula (2.11) for $P_m(z) = z^m$. Hence, the factor z^m is common to all the terms of the sum so can be omitted. By expanding $(1-k)^n$ and rearranging the sum, we find that the formula (2.11) is equivalent to

$$\sum_{i=0}^{n+m} (-1)^i \binom{n+m}{i} \sum_{k=0}^{n+m+1} (-1)^k \binom{n+m+1}{k} k^{n+m-i} = 0. \tag{2.12}$$

To prove this identity, we use the identities

$$\sum_{k=0}^{n+m+1} (-1)^k \binom{n+m+1}{k} (k)_s = \left(\frac{d}{dx}\right)^s (1-x)^{n+m+1}\Big|_{x=1}, \quad s = 0, 1, \ldots,$$

and the formula

$$k^{n+m-i} = \sum_{s=0}^{n+m-i} {n+m-i \brace s} (k)_s$$

to prove that the left hand side of (2.12) is to be

$$\begin{aligned} & \sum_{i=0}^{n+m} (-1)^i \binom{n+m}{i} \sum_{s=0}^{n+m-i} {n+m-i \brace s} \sum_{k=0}^{n+m+1} (-1)^k \binom{n+m+1}{k} (k)_s \\ = & \sum_{i=0}^{n+m} (-1)^i \binom{n+m}{i} \sum_{s=0}^{n+m-i} {n+m-i \brace s} \left(\frac{d}{dx}\right)^s (1-x)^{n+m+1}\Big|_{x=1} \\ = & \ 0, \end{aligned}$$

where ${m \brace k}$ is the (m,k)-th Stirling number of the second kind.

Example 2.8. By the identity (2.9) we get

$$L_m^{(\alpha,\beta)}(z) = \sum_{k=0}^{n+m} (-1)^{n+k} k^n \binom{n+m+1}{k+1} L_m^{(\alpha,\beta)}(-kz).$$

In particular, the Lah polynomials $\mathcal{L}_n(z) = L_n^{(0,-1)}(z)$ satisfy

$$\mathcal{L}_m(z) = \sum_{k=0}^{m} L(m,k) z^k = \sum_{k=0}^{n+m} (-1)^{n+k} k^n \binom{n+m+1}{k+1} \mathcal{L}_m(-kz),$$

where $L(n,k)$ are the unsigned Lah numbers.

Example 2.9. The identity (2.10) implies

$$P_m(z) = \sum_{i,j=0}^{m} (-1)^{i+j} \binom{m+1}{i+1} \binom{m+1}{j+1} P_m(ijz). \tag{2.13}$$

Then, for r, s be non-negative integers and

$$P_m(z) = \left(\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j+z)^m \right)^s,$$

we get $P_m(r) = \left({m+r \brace k+r}_r \right)^s$ and by (2.13) we obtain

$$\left({m+r \brace k+r}_r \right)^s = \sum_{i,j=0}^{ms} (-1)^{i+j} \binom{ms+1}{i+1} \binom{ms+1}{j+1} \left({m+ijr \brace k+ijr}_{ijr} \right)^s,$$

where ${m \brace k}_r$ is the (m,k)-th r-Stirling number of the second kind [2], defined by

$$\sum_{m \geq k} {m+r \brace k+r}_r \frac{t^m}{m!} = \frac{1}{k!}(e^t - 1)^k e^{rt}, \quad k, r = 0, 1, 2, \ldots.$$

Example 2.10. Let $(\mathcal{B}_{n,r}(z))$ be the sequence of the r-Bell polynomials [8] and let

$$B_r(t; z) = e^{z(e^t - 1) + rt} := \sum_{n \geq 0} \mathcal{B}_{n,r}(z) \frac{t^n}{n!}.$$

From (2.9), it follows that the sequence $(\mathcal{B}_{n,r}(z))$ satisfies

$$\mathcal{B}_{m,r}(z) = \sum_{k=0}^{n+m} (-1)^{n+k} k^n \binom{n+m+1}{k+1} \mathcal{B}_{m,r}(-kz).$$

Example 2.11. For $P_m(z) = \binom{-z}{m}^r$, the identity (2.9) implies

$$\binom{-z}{m}^r = \sum_{k=0}^{n+mr} (-1)^{n+k} k^n \binom{n+mr+1}{k+1} \binom{kz}{m}^r, \quad m \geq 0, \; n \geq 0,$$

where $\binom{\alpha}{m}$ is defined for any complex number α by

$$\binom{\alpha}{m} = \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} \; \textit{if } m \geq 1 \textit{ and } \binom{\alpha}{0} = 1.$$

Remark 2.12. By replacing $P_m(z)$ by $P_m(y+z)$, the identity (2.9) becomes

$$P_m(y+z) = \sum_{k=0}^{n+m} (-1)^{n+k} k^n \binom{n+m+1}{k+1} P_m(y - kz), \tag{2.14}$$

which can be written by setting $x = y + z$ as

$$P_m(x) = \sum_{k=0}^{n+m} (-1)^{n+k} k^n \binom{n+m+1}{k+1} P_m(x - (k+1)z). \tag{2.15}$$

3. Two formulas for the generating functions of polynomials

We establish in this section a formula for the generating functions of polynomials based on the identity (2.9) and other formula based on the Melzak formula. Indeed, let (a_n) be a sequence of real numbers, $(P_n(z))$ be a sequence of polynomials and let $E(.)$ and $F(.)$ be their generating functions defined by

$$E(t)=\sum_{n\geq 0}a_n t^n,\quad F(t;z)=\sum_{n\geq 0}P_n(z)\,t^n,\quad |t|<\mu,\quad \mu>0. \tag{3.1}$$

The tool used here is the following theorem:

Theorem 3.1. *[14, th. 7.50] Suppose that $c_{m,n}\in\mathbb{C}$ for each $(m,n)\in\mathbb{N}\times\mathbb{N}$ and that ϕ in any one-to-one mapping of $\mathbb{N}$ onto $\mathbb{N}\times\mathbb{N}$. If any of the three sums*

$$(i)\quad \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}|c_{m,n}|\right),\quad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|c_{m,n}|\right),\quad \sum_{k=1}^{\infty}\left|c_{\phi(k)}\right|$$

is finite, then all of the series

$$(ii)\quad \sum_{n=1}^{\infty}c_{m,n}\quad (m=1,2,\ldots),$$

$$(iii)\quad \sum_{m=1}^{\infty}c_{m,n}\quad (n=1,2,\ldots),$$

$$(iv)\quad \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}c_{m,n}\right),\quad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}c_{m,n}\right),\quad \sum_{k=1}^{\infty}c_{\phi(k)}$$

are absolutely convergent and the three series in (iv) all have the same sum, where $\mathbb{C}$ and $\mathbb{N}$ are, respectively, the sets of complex and natural numbers.

3.1. First formula for the generating functions of polynomials

Based on the identity (2.9) and Theorem 3.1 we may state the following theorem.

Theorem 3.2. *If the series*

$$\sum_{n\geq 0}\left(\sum_{k\geq 1}\left|a_n\frac{(-t)^k}{k!}\left(\frac{d}{dt}\right)^k\left(t\left(-(k-1)t\right)^n F\left(t;-(k-1)z\right)\right)\right|\right)$$

converges on $D\subset\left]-\mu,\mu\right[$, then

$$\sum_{k\geq 1}\frac{(-t)^k}{k!}\left(\frac{d}{dt}\right)^k\left(tE\left(-(k-1)t\right)F\left(t;-(k-1)z\right)\right)=-tE(t)\,F(t;z),\quad t\in D. \tag{3.2}$$

Proof. By the identity (2.9) we can write

$$\begin{aligned}
F(t;z) &= \sum_{m\geq 0}P_m(z)\,t^m\\
&= \sum_{m\geq 0}\left(\sum_{k=0}^{n+m}(-1)^{n+k}k^n\binom{n+m+1}{k+1}P_m(-kz)\right)t^m\\
&= -\sum_{k\geq 0}(-k)^n\frac{(-1)^{k+1}}{(k+1)!}\sum_{m\geq\max(k-n,0)}\frac{(n+m+1)!}{(n+m-k)!}P_m(-kz)\,t^m\\
&= -\sum_{k=0}^{n}(-k)^n\frac{(-1)^{k+1}}{(k+1)!}\sum_{m\geq 0}\frac{(n+m+1)!}{(n+m-k)!}P_m(-kz)\,t^m\\
&\quad -\sum_{k\geq n+1}(-k)^n\frac{(-1)^{k+1}}{(k+1)!}\sum_{m\geq k-n}\frac{(n+m+1)!}{(n+m-k)!}P_m(-kz)\,t^m.
\end{aligned}$$

So, we get

$$\begin{aligned} t^{n+1}F(t;z) &= -\sum_{k=0}^{n}(-k)^n\frac{(-t)^{k+1}}{(k+1)!}\left(\frac{d}{dt}\right)^{k+1}\left(t^{n+1}\sum_{m\geq 0}P_m(-kz)t^m\right) \\ &\quad -\sum_{k\geq n+1}(-k)^n\frac{(-t)^{k+1}}{(k+1)!}\left(\frac{d}{dt}\right)^{k+1}\left(t^{n+1}\sum_{m\geq k-n}P_m(-kz)t^m\right) \\ &= -\sum_{k=0}^{n}(-k)^n\frac{(-t)^{k+1}}{(k+1)!}\left(\frac{d}{dt}\right)^{k+1}\left(t^{n+1}F(t;-kz)\right) \\ &\quad -\sum_{k\geq n+1}(-k)^n\frac{(-t)^{k+1}}{(k+1)!}\left(\frac{d}{dt}\right)^{k+1}\left(t^{n+1}F(t;-kz)-T_k(t)\right), \end{aligned}$$

where $T_k(t)=t^{n+1}\sum\limits_{m=0}^{k-n-1}P_m(-kz)t^m$ is a polynomial with degree at most k. So $T_k(t)$ vanishes under the action of $(d/dt)^{k+1}$. Hence, we can write

$$\begin{aligned} t^{n+1}F(t;z) &= -\sum_{k\geq 0}(-k)^n\frac{(-t)^{k+1}}{(k+1)!}\left(\frac{d}{dt}\right)^{k+1}\left(t^{n+1}F(t;-kz)\right) \\ &= -\sum_{k\geq 1}(-1)^{n+k}(k-1)^n\frac{t^k}{k!}\left(\frac{d}{dt}\right)^{k}\left(t^{n+1}F(t;-(k-1)z)\right) \\ &= -\sum_{k\geq 1}\frac{(-t)^k}{k!}\left(\frac{d}{dt}\right)^{k}\left(t(-(k-1)t)^nF(t;-(k-1)z)\right). \end{aligned}$$

This identity implies

$$\begin{aligned} tE(t)F(t;z) &= \sum_{n\geq 0}a_nt^{n+1}F(t;z) \\ &= -\sum_{n\geq 0}a_n\left(\sum_{k\geq 1}\frac{(-t)^k}{k!}\left(\frac{d}{dt}\right)^{k}\left(t(-(k-1)t)^nF(t;-(k-1)z)\right)\right) \\ &= -\sum_{k\geq 1}\frac{(-t)^k}{k!}\left(\frac{d}{dt}\right)^{k}\left(t\sum_{n\geq 0}a_n(-(k-1)t)^nF(t;-(k-1)z)\right) \\ &= -\sum_{k\geq 1}\frac{(-t)^k}{k!}\left(\frac{d}{dt}\right)^{k}\left(tE(-(k-1)t)F(t;-(k-1)z)\right). \end{aligned}$$

For t be a complex number, by Theorem 3.1, a sufficient condition that the main identity holds is such that the series $\sum\limits_{n\geq 0}\left(\sum\limits_{k\geq 1}|c_{n,k}|\right)$ is finite, where
$c_{n,k}=\frac{(-t)^k}{k!}\left(\frac{d}{dt}\right)^k\left(t(-(k-1)t)^nF(t;-(k-1)z)\right)a_n$. □

Example 3.3. Some applications of Theorem 3.2 are given as follows:
For $E(t)=e^{\alpha t}$ and $F(t;z)=e^{\beta t}$, $\alpha\neq 0$, we get

$$\sum_{k\geq 1}\left((\beta-\alpha(k-1))t+k\right)(\beta-\alpha(k-1))^{k-1}\frac{\left(-te^{-\alpha t}\right)^k}{k!}=-t,\quad t\neq\frac{1}{\alpha},$$

for $E(t) = (1+t)^m$ and $F(t;z) = 1$, $m \in \mathbb{N}$, we get

$$\sum_{k=0}^{m} \binom{m}{k} \left(1 - \binom{m+1}{k+1} kt\right) (kt)^k (1-kt)^{m-k-1} = (1+t)^m, \quad t \in \mathbb{R},$$

for $E(t) = (1+t)^\alpha$ and $F(t;z) = 1$ we get

$$\sum_{k\geq 0} \binom{\alpha}{k} \left(1 - \binom{\alpha+1}{k+1} kt\right) (kt)^k (1-kt)^{\alpha-k-1} = (1+t)^\alpha, \quad t \in \left]-1,1\right[,$$

for $E(t) = \frac{\sin t}{t}$ and $F(t;z) = 1$ we get

$$\sum_{k\geq 1} (k-1)^{k-1} \sin\left((k-1)t - k\frac{\pi}{2}\right) \frac{t^k}{k!} = -\sin t, \quad t \in \left]-e^{-1}, e^{-1}\right[.$$

and for $E(t) = 1$, $F(t;z) = e^{z(e^t-1)+rt}$, since $\left(\frac{d}{dt}\right)^k F(t;z) = \mathcal{B}_{k,r}(ze^t)\, e^{z(e^t-1)+rt}$ [5, Th. 9], then the formula (3.2) gives

$$\sum_{k\geq 0} \left(\mathcal{B}_{k,r}(-kze^t) + \frac{t}{k+1}\mathcal{B}_{k+1,r}(-kze^t)\right) \frac{\left(-te^{-z(e^t-1)}\right)^k}{k!} = e^{z(e^t-1)}.$$

3.2. Second formula for the generating functions of polynomials

Based on the Melzak formula and Theorem 3.1 we may state the following theorem.

Theorem 3.4. *Let s be positive integer. If the series*

$$\sum_{n\geq 0} \left(\sum_{k\geq 1} \left| \frac{a_n}{(s-1)!} \frac{(-t)^k}{k!\,(k+s)} \left(\frac{d}{dt}\right)^{k+s} \left(t^{n+s} F(t;z-s-k)\right) \right| \right)$$

converges on $D \subset \left]-\mu, \mu\right[$, then

$$\sum_{k\geq 0} \frac{(-t)^k}{k!\,(k+s)} \left(\frac{d}{dt}\right)^{k+s} \left(t^s E(t) F(t;z-s-k)\right) = E(t) F(t;z), \quad t \in D. \tag{3.3}$$

In particular, for $s = 1$, we get

$$\sum_{k\geq 1} \frac{(-t)^k}{k!} \left(\frac{d}{dt}\right)^k \left(tE(t) F(t;z-k)\right) = -tE(t) F(t;z), \quad t \in D. \tag{3.4}$$

Proof. From the Melzak formula [6,7]

$$P_m(z) = \alpha \binom{\alpha+n+m}{n+m} \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} \frac{P_m(z-\alpha-k)}{\alpha+k}, \quad n \geq 0, \ m \geq 0.$$

We can write for $\alpha = s \geq 1$:

$$P_m(z) = \sum_{k=0}^{n+m} (-1)^k \binom{k+s-1}{s-1} \binom{n+m+s}{k+s} P_m(z-k-s). \tag{3.5}$$

Then, by the identity (3.5) and Theorem 3.1 we can write

$$\begin{aligned}
F(t;z) &= \sum_{m\geq 0} P_m(z)\, t^m \\
&= \sum_{m\geq 0}\left(\sum_{k=0}^{n+m}(-1)^k\binom{k+s-1}{s-1}\binom{n+m+s}{k+s}P_m(z-k-s)\right)t^m \\
&= \frac{1}{(s-1)!}\sum_{k\geq 0}\frac{(-1)^k}{k!\,(k+s)}\sum_{m\geq \max(0,k-n)}\frac{(n+m+s)!}{(n+m-k)!}P_m(z-s-k)\,t^m \\
&= \frac{1}{(s-1)!}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!\,(k+s)}\sum_{m\geq 0}\frac{(n+m+s)!}{(n+m-k)!}P_m(z-s-k)\,t^m \\
&\quad+\frac{1}{(s-1)!}\sum_{k\geq n}\frac{(-1)^k}{k!\,(k+s)}\sum_{m\geq k-n}\frac{(n+m+s)!}{(n+m-k)!}P_m(z-s-k)\,t^m.
\end{aligned}$$

So, we get

$$\begin{aligned}
t^nF(t;z) &= \frac{1}{(s-1)!}\sum_{k=0}^{n-1}\frac{(-t)^k}{k!\,(k+s)}\left(\frac{d}{dt}\right)^{k+s} t^{n+s}\sum_{m\geq 0}P_m(z-s-k)\,t^m \\
&\quad+\frac{1}{(s-1)!}\sum_{k\geq n}\frac{(-t)^k}{k!\,(k+s)}\left(\frac{d}{dt}\right)^{k+s} t^{n+s}\sum_{m\geq 0}P_m(z-s-k)\,t^m \\
&\quad-\frac{1}{(s-1)!}\sum_{k\geq n}\frac{(-t)^k}{k!\,(k+s)}\left(\frac{d}{dt}\right)^{k+s} t^{n+s}\sum_{m=0}^{k-n-1}P_m(z-s-k)\,t^m \\
&= \frac{1}{(s-1)!}\sum_{k\geq 0}\frac{(-t)^k}{k!\,(k+s)}\left(\frac{d}{dt}\right)^{k+s}\left(t^{n+s}F(t;z-s-k)\right).
\end{aligned}$$

This identity implies

$$\begin{aligned}
E(t)\,F(t;z) &= \sum_{n\geq 0}a_n t^{n+1}F(t;z) \\
&= \sum_{n\geq 0}a_n\left(\frac{1}{(s-1)!}\sum_{k\geq 0}\frac{(-t)^k}{k!\,(k+s)}\left(\frac{d}{dt}\right)^{k+s}\left(t^{n+s}F(t;z-s-k)\right)\right) \\
&= \sum_{k\geq 0}\frac{(-t)^k}{k!\,(k+s)}\left(\frac{d}{dt}\right)^{k+s}\left(t^s\sum_{n\geq 0}a_nt^nF(t;z-s-k)\right) \\
&= \sum_{k\geq 0}\frac{(-t)^k}{k!\,(k+s)}\left(\frac{d}{dt}\right)^{k+s}\left(t^sE(t)\,F(t;z-s-k)\right).
\end{aligned}$$

For t be a complex number, by Theorem 3.1, a sufficient condition that the main identity holds is such that the series $\sum_{n\geq 0}\left(\sum_{k\geq 1}|c_{n,k}|\right)$ is finite, where
$c_{n,k}=\frac{a_n}{(s-1)!}\frac{(-t)^k}{k!(k+s)}\left(\frac{d}{dt}\right)^{k+s}\left(t^{n+s}F(t;z-s-k)\right).$ □

Example 3.5. Some applications of Theorem 3.4 are given as follows:
For $s=1,\ E(t)=e^{\alpha t}$ and $F(t;z)=e^{zt}$, $\alpha\neq 0$, we get

$$\sum_{k\geq 0}(k-(\lambda-k)\,t)\,(k-\lambda)^{k-1}\frac{(te^{-t})^k}{k!}=-t,\quad t\neq 1,\quad \lambda=\alpha+z,$$

and for $s=1$, $E(t)=1$ and $F(t;z)=e^{z\left(e^{t}-1\right)+rt}$, we get

$$\sum_{k\geq 0}\left(\mathcal{B}_{k,r}\left(-kze^{t}\right)+\frac{t}{k+1}\mathcal{B}_{k+1,r}\left(-kze^{t}\right)\right)\frac{\left(-te^{-z\left(e^{t}-1\right)}\right)^{k}}{k!}=e^{z\left(e^{t}-1\right)}.$$

4. Conclusion

The knowledge of such properties on polynomials and on their generating functions can help researchers to establish new identities, congruences and generating functions. In this context, our results take applications in combinatorics, algebra and analysis. For example, from the link between the binomial polynomials and the partial Bell polynomials [9, Prop. 1], the identity (1.1) can be exploited to establish new identities on partial Bell polynomials which include many combinatorial numbers. Also, Theorems 3.2 and 3.4 can also be exploited to develop several generating functions as it is shown above.

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