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Truncation and Convergence Dynamics: KdV Burgers Model in the Sense of Caputo Derivative

Asif Yokus

ABSTRACT: This study examines the time fractional KdV Burgers equation with the initial conditions by using finite difference method (FDM) via the extended result on Caputo formula. For this reason, various fractional differential operators are defined and analyzed. In order to check the stability of the numerical scheme, the Fourier-von Neumann technique is used. By presenting an example of KdV Burgers equation above mentioned issues are discussed and numerical solutions of the error estimates are found for the FDM. For the errors in L_2 and L_∞ the method accuracy is controlled. Moreover, the obtained results are compared with the exact solution for different cases of non-integer order and the behavior of the potentials u is presented as a graph. The numerical results are shown in tables.

Key Words: Nonlinear time Fractional KdV Burgers Equation, Finite Difference Method, Caputo Formula, Linear Stability.

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1. Introduction

Recent studies reveals the method of expansion by means of inspiration of (G'/G) expansion method which is presented by Wang et al. [1]. Some similar methods have been presented [2,3]. The reason of selecting this method is to find different exact solutions in the form of rational, trigonometric function solutions to the time fractional KdV Burgers equation. When the literature is examined, it is seen that different classes of nonlinear partial differential equations have been presented the solutions by using different methods [4,5,6,7,8,9,10]. It can be stated that finding the exact solutions for some nonlinear partial differential equations becomes complex and this is the case mostly for nonlinear fractional partial differential equations (nfPDE). In this study, we aimed to utilize one of the received exact solutions to the fractional KdV Burgers equation. Moreover we use the extended FDM to approximate numerical solutions.

In order to reach the approximate solutions for nfPDE, various trials have been performed in the last decades [11,12,13,14]. To some extent, it could be expressed that non-linear fractional differential equations could be implemented to explain different problems today. Due to the fact that it could be a useful tool for finding solutions to our daily life problems, obtaining the exact solutions for such equations play an almost importance. For comprehending the physical aspects of fractional nonlinear, studies presenting exact solutions for nfPDE play a significant role. The fact is that solving nfPDE is difficult and a variable transformation on fractional equation is needed to ease the process. Afterwards,

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nfPDE will get the shape of integer-order differential equations. It can be expressed that various types of fractional derivatives and favorable formulas can be found in the literature for such transformations [15,16,17,18,19,20,21,22,23,24,25].

In this study, the non-linear time fractional KdV Burgers equation of initial conditions was examined and extend FDM based on Caputo formula. Our aim is to define various fractional differential operators and reveal the analysis of them. Von Neumann's Stability analysis technique is utilized.

Both numerical and exact methods efficiency of the time fractional KdV Burgers equation are investigated as [26]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+6u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{\partial^3 u(x,t)}{\partial x^3}=0, \tag{1}$$

where $0<\alpha\leq 1$ is a real constant, in the sense of Caputo is defined by

$$\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(m-\alpha)}\int\limits_0^t \frac{\frac{\partial^m}{\partial \xi^m}f(x,\xi)}{(t-\xi)^{\alpha-m+1}}d\xi,$$

where m a constant and integer that $m-1<\alpha\leq m$. $\Gamma(.)$ also represents the gamma function.

2. Methodology of FDM

Some important notations are needed to define the forward FDM, these are:
a) Δx, which is the spatial step
b) Δt, which is the time step
c) $x_i=a+i\Delta x$, $i=0,1,2,...,N$ points, which are the coordinates of mesh and $N=\frac{b-a}{\Delta x}$, $t_j=j\Delta x$, $j=0,1,2,...,M$ and $M=\frac{T}{\Delta t}$.
d) The function $u\left(x,t\right)$ is the values of the solution at these grid points which are $u\left(x_i,t_j\right)\cong u_{ij}$, where $u_{i,j}$ will is the numerical approximations of $u\left(x,t\right)$ at the point $\left(x_i,t_j\right)$.
The difference operators as

$$\begin{aligned}
H_t u_{i,j} &= u_{i,j}-u_{i,j-1}, && (2.1)\\
H_x u_{i,j} &= u_{i+1,j}-u_{i-1,j}, && (2.2)\\
H_{xx} u_{i,j} &= u_{i+1,j}-2u_{i,j}+u_{i-1,j}, && (2.3)\\
H_{xxx} u_{i,j} &= u_{i+2,j}-2u_{i+1,j}+2u_{i-1,j}-u_{i-2,j}, && (2.4)
\end{aligned}$$

the formulas giving approximate values of partial derivatives according to FDM are as follows

$$\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{i,j} &= \frac{H_x u_{i,j}}{2\Delta x}+O(x^2), && (2.5)\\
\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} &= \frac{H_{xx} u_{i,j}}{(\Delta x)^2}+O(x^3), && (2.6)\\
\left.\frac{\partial^3 u}{\partial x^3}\right|_{i,j} &= \frac{H_{xxx} u_{i,j}}{2(\Delta x)^3}+O(x^4), && (2.7)
\end{aligned}$$

the shifted Caputo definition ([27]) is given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\cong\begin{cases}\frac{h^{-\alpha}}{\Gamma(2-\alpha)}H_t u_{i,j+1}+\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum\limits_{k=1}^{j}f(k)H_t u_{i,j+1-k}, & j\geq 1,\\ \frac{h^{-\alpha}}{\Gamma(2-\alpha)}H_t u_{i,0}, & j=0,\end{cases} \tag{2.8}$$

In the FDM, inserting Eq. (2.5), (2.6), (2.7) and (2.8) into Eq. (1), we have the indexed

$$u_{i+1,j} = \frac{\begin{array}{c}[u_{i-2,j} - 2(1+\Delta x)u_{i-1,j} + 4(\Delta x)u_{i,j}\left(1+3(\Delta x)u_{i,j}\right) - u_{i-2,j}] - \\ \vartheta\left[H_t u_{i,j} - \sum\limits_{k=1}^{j} f(k) H_t u_{i,j-k}\right]\end{array}}{2(-1+\Delta x + 6(\Delta x)^2 u_{i,j})}, \tag{2.9}$$

where $\vartheta = \frac{2(\Delta x)^3}{(\Delta t)^\alpha \Gamma(2-\alpha)}$, $f(k) = (k+1)^{1-\alpha} - k^{1-\alpha}$ and the initial values $u_{i,0} = u_0(x_i)$.

3. Consistency of FDM

This section presents the consistency of Eq. (1) with FDM. At first, one can give the Taylor series expansions as follows:

$$u_{i+1,j} = u_{i,j} + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} + O((\Delta x)^3), \tag{3.1}$$

$$u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2} + O((\Delta t)^3), \tag{3.2}$$

$$u_{i-1,j} = u_{i,j} - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} - O((\Delta x)^3), \tag{3.3}$$

$$u_{i+2,j} = u_{i,j} + 2\Delta x \frac{\partial u}{\partial x} + 2(\Delta x)^2\frac{\partial^2 u}{\partial x^2} + O((\Delta x)^3), \tag{3.4}$$

$$u_{i-2,j} = u_{i,j} - 2\Delta x \frac{\partial u}{\partial x} + 2(\Delta x)^2\frac{\partial^2 u}{\partial x^2} - O((\Delta x)^3), \tag{3.5}$$

we define an L operator as,

$$L = \frac{\partial}{\partial t} + 6u\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} + \frac{\partial^3}{\partial x^3},$$

and we write of operator L as

$$L_{i,j}u_{i,j} = \frac{H_t u_{i,j}}{\Delta t} + 6u\frac{H_x u_{i,j}}{\Delta x} - \frac{H_{xx}u_{i,j}}{(\Delta x)^2} + \frac{H_{xxx}u_{i,j}}{2(\Delta x)^3}. \tag{3.6}$$

Putting the indexed form; (3.1),(3.2), (3.3), (3.4), and (3.5) into the Eq. (3.6) and doing some necessary operations, and as $\Delta x \to 0$ and $\Delta t \to 0$, then the Eq. (3.6) will be the same as left hand side of the Eq. (1). It shows that Eq. (1) is consistent with FDM.

4. The Analysis of Truncation Error and Convergence

In this part we study the error and the stability analysis of the forward FDM. In order to investigate the stability, we analyzed whether there was a deterioration in the first case. If there is a minor change, this will not cause a major error in the numerical solution. The stability can be basically defined as the scheme that does not amplify any error.

A theorem is presented below regarding the truncation error of FDM for KdV Burgers equation.

Theorem 4.1. *The truncation error of the FDM for KdV Burgers equation is as follows;*

$$\frac{\left(\Delta x^3 O\left(\Delta t^{2\alpha}\right)\Delta x^2 - \Delta x^3 O\left(\Delta x^3\right)\Delta x^2 + \Delta x^3 O\left(\Delta x^4\right)\Delta x^2 + 6\Delta x^3 O\left(\Delta x^2\right)\Delta x^2 u_{i,j}\right)}{\left(h^3 + \Delta x^2 - 6h^3 \Delta x u_{i,j}\right)}.$$

Proof. When we insert Eq. (2.5), (2.6), (2.7) and (2.8) into Eq. (1) we reach

$$6u_{i,j}\left(\frac{H_x u_{i,j}}{\Delta x} + (\Delta x)^2\right) - \frac{H_{xx}u_{i,j}}{(\Delta x)^2} - (\Delta x)^3 + \frac{H_{xxx}u_{i,j}}{2(\Delta x)^3} + (\Delta x)^4 = 0, \tag{4.1}$$

Eq. (4.1) can be stated as in the form below after various algebraic computations:

$$\frac{h^{-\alpha}}{\Gamma(2-\alpha)}H_t u_{i,j+1} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{i} H_t u_{i,j-k} f(k) + 6u_{i,j}\frac{H_x u_{i,j}}{\Delta x} - \frac{H_{xx}u_{i,j}}{(\Delta x)^2} + \frac{H_{xxx}u_{i,j}}{2(\Delta x)^3} +$$

$$\frac{\left(\Delta x^3 O\left(\Delta t^{2\alpha}\right)\Delta x^2 - \Delta x^3 O\left(\Delta x^3\right)\Delta x^2 + \Delta x^3 O\left(\Delta x^4\right)\Delta x^2 + 6\Delta x^3 O\left(\Delta x^2\right)\Delta x^2 u_{i,j}\right)}{\left(h^3 + \Delta x^2 - 6h^3\Delta x u_{i,j}\right)}, \tag{4.2}$$

when we remove truncation error $\breve{U}$ term from the Eq. (4.2), we get the indexed form of Eq. (1) as follows:

$$\frac{h^{-\alpha}}{\Gamma(2-\alpha)}H_t u_{i,j+1} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{i} H_t u_{i,j-k} f(k) + 6u_{i,j}\frac{H_x u_{i,j}}{\Delta x} - \frac{H_{xx}u_{i,j}}{(\Delta x)^2} + \frac{H_{xxx}u_{i,j}}{2(\Delta x)^3} = 0.$$

and the transaction error $\breve{U}$ has the following form

$$\breve{U} = \frac{\left(\Delta x^3 O\left(\Delta t^{2\alpha}\right)\Delta x^2 - \Delta x^3 O\left(\Delta x^3\right)\Delta x^2 + \Delta x^3 O\left(\Delta x^4\right)\Delta x^2 + 6\Delta x^3 O\left(\Delta x^2\right)\Delta x^2 u_{i,j}\right)}{\left(h^3 + \Delta x^2 - 6h^3\Delta x u_{i,j}\right)}. \tag{4.3}$$

Therefore, we may state the absolute error E of transaction as

$$E = \left|U - \breve{U}\right|,$$

U is the exact solution. Using Theorem 1, if Δx and Δt are considered very small, the transaction error is negligible as well. Thus, the limit of E can be stated as

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} (E) = 0,$$

In the event of Δx and Δt are set up for a value which is close to zero $\varepsilon > 0$, we find the following inequality

$$|E| < \varepsilon.$$

In addition it shows FDM is convergence. ☐

5. Stability with Linearization Technique

The linearized version of Eq. (1) takes the form,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \lambda\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \tag{5.1}$$

where $\alpha = 1$ and $\lambda = 6u(x,t)$ are constant. Eq. (5.1) can be written in indexed form as

$$H_t u_{i,j} + a_1 H_x u_{i,j} + a_2 H_{xx} u_{i,j} + a_3 H_{xxx} u_{i,j} = 0, \tag{5.2}$$

where $a_1 = \frac{\lambda\Delta t}{\Delta x},\ a_2 = -\frac{\Delta t}{(\Delta x)^2},\ a_3 = \frac{\Delta t}{(\Delta x)^3}$.

Theorem 5.1. *The FDM for Eq.* (1) *is conditionally linear stable.*

Proof. Using the von Neumann stability for the FDM of the Eq. (1). For that, let

$$u_{i,j} = u(i\Delta x, j\Delta t) = u(p,q) = e^{kq}e^{\mathbf{i}\xi p} = \varepsilon^q e^{\mathbf{i}\xi p}, \quad \xi \in [-\pi, \pi], \tag{5.3}$$

where $e^k = \varepsilon,\ p = i\Delta x,\ q = j\Delta t, (i = 0, 1, 2, ..., N$ and $j = 0, 1, 2, ..., M)$ and $\mathbf{i} = \sqrt{-1}$. If we substitute the equality (5.3) into the equality (5.2)

$$\varepsilon = \frac{1}{1 - 4a_2 \sin^2 \frac{\xi}{2} - 2\mathbf{i} \sin \xi \left[a_1 - 4a_3 \sin^2 \frac{\xi}{2} \right]}, \tag{5.4}$$

we can written $|\varepsilon| \leq 1$. Therefore, by the Von Neumann's Stability analysis if $|\varepsilon| \leq 1$, FDMs for the Eq. (1) is stable.

□

6. Numerical Appication

We consider time fractional KdV Burgers equation of the form (1) (see [26]) with the initial conditions in the following :

$$u_0(x) = -\frac{1}{50} \left[1 - \tanh\left(\frac{x}{10}\right) \right]^2, \quad 0 \leq x \leq 1. \tag{6.1}$$

When the following numerical results are obtained, α is chosen as $\alpha = 0.8$. The travelling wave solution of the time fractional KdV Burgers equation is given as follows

$$u(x,t) = -\frac{1}{50} \left[1 - \tanh\left(\frac{1}{10}(x + \frac{6t}{25}) \right) \right]^2. \tag{6.2}$$

We use L_2 and L_∞ error norms to present how the analytical results and numerical results are close to each other. The L_2 error norm can be defined as [28]

$$L_2 = \left\| u^{exact} - u^{numeric} \right\|_2 = \sqrt{h \sum_{j=0}^{N} \left| u_j^{exact} - u_j^{numeric} \right|^2},$$

and then L_∞ error norm

$$L_\infty = ||u^{exact} - u^{numeric}||_\infty = \underset{j}{Max} |u_j^{exact} - u_j^{numeric}|.$$

The numerical solutions are secured from the finite difference technique discussed above considering Eq.(2.9). The numerical solutions in $0 \leq x \leq 1$ are as follows L_2 and L_∞ error norm table

Table 1: L_2 and L_∞ error norm when $0 \leq \Delta x = \Delta t \leq 1$.

$\Delta x = \Delta t$	L_2	L_∞
0.05	7.32185×10^{-3}	3.274×10^{-2}
0.02	2.49458×10^{-3}	5.012×10^{-2}

Numerical solutions, exact solutions and absolute errors table

Table 2: Numerical solutions, exact solutions and absolute errors of Eq. (1) with respect to the Caputo derivative for $\Delta x = \Delta t = 0.02$.

x_i	t_j	**Numerical solution**	**Exact Solution**	**Absolute Errors**
0	0.02	-0.0199936	-0.0199808	1.27898×10^{-5}
0.02	0.02	-0.0199139	-0.0199009	1.29565×10^{-5}
0.04	0.02	-0.0198343	-0.0198212	1.31224×10^{-5}
0.06	0.02	-0.0197549	-0.0197416	1.32873×10^{-5}
0.08	0.02	-0.0196757	-0.0196622	1.34514×10^{-5}
0.10	0.02	-0.0195966	-0.0195830	1.36146×10^{-5}
0.12	0.02	-0.0195177	-0.0195039	1.37769×10^{-5}

As we know, that the truncation error depends on the choice of the Δx and Δt. When they approaches zero it means that the truncation error will be very small. By using the values of $\Delta x = 0.02$, this resulting behavior of the numerical and exact solutions is presented in the following graph.

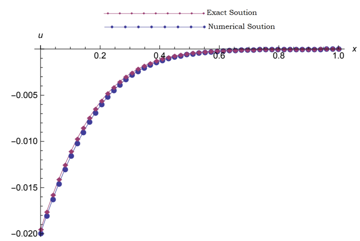

Figure 1: 2D Numerical and exact travelling wave solution of the Eq. (1).

In addition, numerical solutions using Eq. (2.9) and the exact solution using Eq. (6.2) are shown in the figure above:

Figure 2: 3D Numerical and exact travelling wave solution of the Eq. (1).

7. Conclusions

n summary, we have successfully applied FDM to Eq. (1) to search for numerical solutions. Error results obtained in this paper have been introduced that numerical results are very close to the exact solution of Eq.(1). In order to show the solution representing the stationary wave, we have plotted two and three dimensional surfaces of Eq.(1) by giving the appropriate values to the parameters with the aid of computer package program. The stability of FDM is analyzed by using the Fourier-von Neumann technique. We have discussed the equation of KdV Burgers by giving an example and we have found some error estimates via the FDM. By considering these results and figures found in this paper, FDM is easier and more reliable than most numerical methods.

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Asif Yokus,
Department of Actuary,
Firat University,
Turkey.
E-mail address: `asfyokus@yahoo.com`