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An Eigenvalue Problem for a Fractional Differential Equation with an Iterated Fractional Derivative

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ABSTRACT: This paper concerns the investigation of an eigenvalue problem for a nonlinear fractional differential equation. Using the properties of the Green function, Banach contraction principle, Leray-Schauder nonlinear alternative and Guo-Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation are considered. Some sufficient conditions for the existence of at least one positive solution is established. Some examples are presented to illustrate the main results.

Key Words: Fractional Caputo derivative, Banach contraction principle, Leray-Schauder nonlinear alternative, Guo–Krasnosel'skii fixed point theorem on cones.

Contents

1. Introduction

Nonlinear fractional order differential equations have received grat interest in the recent years. Many results ranging from the existence and uniqueness of solutions to the analytic and numerical methods have appeared in the literature, we refer the reader to a series of papers [1, 8, 10] and the references cited therein.

The search for the existence of positive solutions and multiple positive solutions to nonlinear fractional boundary value problems has expanded greatly over the past decade; for some recent examples see [2, 4, 5, 7, 14, 15]. In all of these works and the references cited therein, different techniques and methods have been employed to deal with the solvability of such boundary value problems; for example, the use of fixed point index theory, the classic cone-compressions and expansions fixed point theorems, the method of upper and lower solutions, and Leggett-Williams theorem and its extensions.

On the other hand, eigenvalue problems of nonlinear fractional differential equations have been concerned by some authors; see [3, 13, 17].

In this work, we consider the nonlinear boundary value problem (P)

$$^{c}D_{0^+}^{p}\left({}^{c}D_{0^+}^{q}u(t)\right)=\lambda f\left(t,u\left(t\right)\right),0<t<1, \tag{1.1}$$

$$\begin{gathered} u(0)=0, u'(0)=au'(1),\\ D_{0^+}^{q}u(1)=D_{0^+}^{q}u(\xi),\\ D_{0^+}^{q}u(0)=D_{0^+}^{q+2}u(0)=0 \end{gathered} \tag{1.2}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is given function, denotes $^{c}D_{0^+}^{p}$ the Caputo fractional derivative, $1<q<2, 2<p<3$, $0<a,\xi<1$.

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Using the Banach fixed point theorem, the nonlinear alternative of Leray Schauder type and Guo-krasnosel'skii fixed point theorem on cone, we investigate the eigenvalue interval for the existence and uniqueness of positive solutions. No contributions exist, as far as we know, concerning the existence of solutions of higher fractional differential equation (1.1) jointly with boundary condition in (1.2).

The rest of this paper is organized as follows. First, we recall some notations, definitions and lemmas to be used later. In section 3, we present and prove our main results wich consist of uniqueness and existence theorems. Finally, we give some examples to illustrate our work.

2. Basic result

First of all, we recall some necessary definitions.

Definition 2.1. *[12] If $g \in L_1([a,b])$ and $\alpha > 0$, then the Riemann -Liouville fractional integral is defined by*

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2.2. *[12] Let $\alpha \geq 0,\ n = [\alpha]+1$. If $g \in AC^n([a,b])$ then the Caputo fractional derivative of order α of g is defined by*

$$^{c}D_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds.$$

Lemma 2.3. *[12] For $\alpha > 0$, $g(t) \in C([a,b])$, the homogenous fractional differential equation $^{c}D_{a^+}^{\alpha} g(t) = 0$ has a solution*

$$g(t) = c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},$$

where $c_i \in R$, $i = 0,1,2,...,n$, and $n = [\alpha]+1$.

The following lemmas gives some properties of Riemann -Liouville fractional integrals and Caputo fractional derivative.

Lemma 2.4. *[12] Let $p,\ q \geq 0,\ f \in L_1([a,b])$. Then*

$$I_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^{q} I_{0^+}^{p} f(t)$$

and

$$^{c}D_{a^+}^{q} I_{0^+}^{q} f(t) = f(t),\ \textit{for all}\ t \in [a,b].$$

Let $\beta > \alpha > 0,\ f \in L_1([a,b])$. Then for all $t \in [a,b]$ we have

$$^{c}D_{a^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\beta-\alpha} f(t).$$

The following lemma is fundamental in the proof of the existence Theorems.

Lemma 2.5. *[12](Leray-Schauder nonlinear alternative). Let F be a Banach space and Ω be a bounded open subset of F, $0 \in \Omega$ and let $T : \overline{\Omega} \longrightarrow F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Theorem 2.6. *[6](Guo-Krasnosel'skii fixed point Theorem on cone). Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let*

$$A : K \cap (\overline{\Omega_2}/\Omega_1) \to K$$

be a completely continuous operator such that

$$\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_1, and\, \|Au\| \geq \|u\|, u \in K \cap \partial\Omega_2,\ \ or$$

$$\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1, and\, \|Au\| \leq \|u\|, u \in K \cap \partial\Omega_2.$$

Then A has a fixed point in $K \cap (\overline{\Omega_2}/\Omega_1)$.

We start by solving an auxiliary problem and we give its Green function.

Lemma 2.7. *For* $1<q<2$ *and* $y\in C\left([0,1]\right)$, *the boundary value problem* (P_0)

$$\left\{\begin{array}{c} {}^cD_{0+}^{q}u(t)=y(t),\ 0<t<1, \\ u(0)=0, u'(0)=au'(1); \end{array}\right.$$

has the solution of the form

$$u(t)=\int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s)=\left\{\begin{array}{c} \frac{1}{\Gamma(q)}(t-s)^{q-1}+\frac{at}{(1-a)\Gamma(q-1)}(1-s)^{q-1}, 0\leq s\leq t, \\ \frac{at}{(1-a)\Gamma(q-1)}(1-s)^{q-1}, t\leq s\leq 1. \end{array}\right. \tag{2.1}$$

Proof. We have

$$u(t)=I_{0+}^{q}y(t)+C+Bt,$$

from the conditions $u(0)=0$, we obtain $C=0$, and the condition $u'(0)=au'(1)$ leads to

$$B=\frac{a}{(1-a)\,\Gamma(q-1)}\int_0^1(1-s)^{q-1}y(s)ds,$$

so $u(t)$ can be written as

$$\begin{aligned} u(t) &= I_{0+}^{q}y(t)+\frac{at}{(1-a)\,\Gamma(q-1)}\int_0^1(1-s)^{q-1}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds \end{aligned}$$

where G is defined by (2.1). □

Lemma 2.8. *For* $2<p<3$ *and* $y\in C\left([0,1]\right)$, *the solution of problem* (P_1)

$$\left\{\begin{array}{c} {}^cD_{0+}^{p}\ {}^cD_{0+}^{q}u(t)=\lambda y\left(t\right),0<t<1, \\ u(0)=0, u'(0)=au'(1), \\ D_{0+}^{q}u(1)=D_{0+}^{q}u(\xi), \\ D_{0+}^{q}u(0)=D_{0+}^{q+2}u(0)=0, \end{array}\right. \tag{2.2}$$

is given by

$$u(t)=\frac{\lambda}{\Gamma(p)}\int_0^1 G(t,s)\int_0^1 H(s,r)y\left(r\right)drds,$$

where

$$H(t,s)=\left\{\begin{array}{ll} -\frac{t(1-s)^{p-1}}{1-\xi}+\frac{t(\xi-s)^{p-1}}{1-\xi}+(t-s)^{p-1}, & 0\leq s\leq \min(t,\xi), \\ -\frac{t(1-s)^{p-1}}{1-\xi}+(t-s)^{p-1}, & \xi\leq s\leq t, \\ -\frac{t(1-s)^{p-1}}{1-\xi}+\frac{t(\xi-s)^{p-1}}{1-\xi}, & t\leq s\leq \xi, \\ -\frac{t(1-s)^{p-1}}{1-\xi}, & \max(t,\xi)\leq s\leq 1. \end{array}\right.$$

Proof. Applying the operator I^p on both sides of the differential equation in (2.2), we obtain

$${}^cD_{0+}^{q}u(t)=\lambda I_{0+}^{p}y(t)+a_1+a_2t+a_3t^2. \tag{2.3}$$

The boundary conditions ${}^cD_{0+}^{q}u(0)=\ {}^cD_{0+}^{q+2}u(0)=0$ lead to $a_1=a_3=0$ and the boundary condition ${}^cD_{0+}^{q}u(1)={}^cD_{0+}^{q}u(\xi)$ yields

$$a_2 = \frac{\lambda}{(1-\xi)} I_{0^+}^p y(\xi) - \frac{\lambda}{(1-\xi)} I_{0^+}^p y(1).$$

Consequently, (2.3) takes the form

$$ {}^cD_{0^+}^q u(t) = \lambda \left(I_{0^+}^p y(t) + \frac{t}{(1-\xi)} I_{0^+}^p y(\xi) - \frac{t}{(1-\xi)} I_{0^+}^p y(1) \right),$$

which can be written as

$$ {}^cD_{0^+}^q u(t) = \frac{\lambda}{\Gamma(p)} \int_0^1 H(t,s) y(s)\, ds.$$

The boundary value problem (P_1) reduces to the following problem

$$\begin{cases} {}^cD_{0^+}^q u(t) = \frac{\lambda}{\Gamma(p)} \int_0^1 H(t,s) y(s)\, ds, \ 0 < t < 1, \\ u(0) = 0, u'(0) = au'(1), \end{cases}$$

which in view of lemma 7 yields the required result

$$u(t) = \frac{\lambda}{\Gamma(p)} \int_0^1 G(t,s) \int_0^1 H(s,r) y(r)\, drds.$$

Finally, the integral solution of probem (P) is:

$$u(t) = \frac{\lambda}{\Gamma(p)} \int_0^1 G(t,s) \int_0^1 H(s,r) f(r, u(r))\, drds.$$

□

Lemma 2.9. *For all $s, t \in [0,1]$, the Green fonction $H(t,s)$ and $G(t,s)$ are continuous and satisfy*

i) $H(t,s) \le 0$,

ii) $t(\zeta - \zeta^{p-1}) \frac{(1-s)^{p-1}}{(1-\xi)} \le -H(t,s) \le \frac{(1-s)^{p-1}}{(1-\xi)}$,

iii) $0 \le \frac{at}{(1-a)\Gamma(q-1)} (1-s)^{q-1} \le G(t,s) \le \frac{1}{\Gamma(q)} + \frac{a}{(1-a)\Gamma(q-1)} = A.$

3. Existence and uniqueness results

Let the Banach space $E = C([0,1], \mathbb{R})$ be endowed with the Chebyshev norm

$$\|u\| = \max_{t \in [0,1]} |u(t)|.$$

Define the integral operator $T : E \to E$ by

$$Tu(t) = \frac{\lambda}{\Gamma(p)} \int_0^1 G(t,s) \int_0^1 H(s,r) f(r, u(r))\, drds.$$

Theorem 3.1. *Let $f \in C([a,b] \times \mathbb{R})$, then $u \in E$ is a solution of the fractional boundary value problem (P) if and only if $Tu(t) = u(t)$, for any $t \in [0,1]$.*

To show that the integral solution $u(t)$ is effectively a solution of problem (P), we can follow the same steps as in the paper of Kilbas and Marzan [11].

In this section, we prove the existence and uniqueness of solutions in the Banach space E.

Theorem 3.2. *Assume that there exist a nonnegative function $g \in L^1([0,1], \mathbb{R}_+)$ such that for all x*

$$|f(t,x)| - f(t,y) \le g(t) |x - y|.$$

Then, there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$ the $FBVP$ (P) has a unique solution u in E.

Proof. We transform the fractional boundary value Problem (P) to a fixed point problem. By Lemma 10 the problem (P) has a solution if and only if the operator T has a fixed point in E. Now we prove that T is a contraction. Let $u, v \in E$, we have

$$Tu(t) - Tv(t) = \frac{\lambda}{\Gamma(p)} \int_0^1 G(t,s) \int_0^1 H(s,r) \left(f\left(r, u\left(r\right)\right) - f(r, v(r)\right) drds;$$

with the help of (3.1) we obtain

$$\begin{aligned} |Tu(t) - Tv(t)\ | &\leq \frac{\lambda}{\Gamma(p)} \int_0^1 G\left(t,s\right) \int_0^1 |H(s,r)|\, g(r)\, |u(r) - v(r)|\, drds \\ &\leq \frac{\lambda}{\Gamma(p)} A \left\| u - v \right\| \int_0^1 \int_0^1 \frac{(1-r)^{p-1}}{1-\xi} g(r) drds \\ &\leq \frac{\lambda A}{\Gamma(p)(1-\xi)} \left\| u - v \right\| \int_0^1 (1-r)^{p-1} g(r) dr \\ &\leq \frac{\lambda A \left\| g \right\|_{L^1}}{\Gamma(p)(1-\xi)} \left\| u - v \right\|, \text{ as } (1-r)^{p-1} \leq 1. \end{aligned}$$

Choose $\lambda^* = \frac{\Gamma(p)(1-\xi)}{2A\|g\|_{L^1}}$; then, when $0 < \lambda \leq \lambda^*$, we have

$$\|Tu - Tv\| \leq \frac{1}{2} \left\| u - v \right\|.$$

So, T is a contraction, hence it has a unique fixed point which is the unique solution of the $FBVP$ (P). □

Now, we give an other existence result for the fractional boundary value problem (P).

Theorem 3.3. *Assume that $f(t,0) \neq 0$ and there exist nonnegative functions $h \in L^1([0,1], \mathbb{R}_+)$, $\Psi \in C(\mathbb{R}_+, \mathbb{R}_+^*)$ nondecreasing on $\mathbb{R}_+$ and $\delta > 0$, such that*

$$|f(t,x)| \leq h(t)\Psi\left(|x|\right). \tag{3.1}$$

Then, there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \leq \lambda^*$, the problem (P) has at least one nontrivial solution $u^* \in E$.

Here, we are going to use the Theorem of Leray-Schauder.

Proof. First, let us prove that T is completely continuous. It is easy to see that T is continuous since f, G and H are continuous.

(i)Let $\{u \in E, \|u\| \leq \delta\}$,for $u \in B_\delta$, we get

$$|Tu(t)| \leq \frac{\lambda}{\Gamma(p)} \int_0^1 |G(t,s)| \int_0^1 |H(s,r) f\left(r, u\left(r\right)\right)| \, drds. \tag{3.2}$$

Since Ψ is nondecreasing then

$$\begin{aligned} |Tu(t)| &\leq \frac{\lambda A \Psi\left(\delta\right)}{\Gamma(p)(1-\xi)} \int_0^1 (1-r)^{p-1} h\left(r\right) dr \\ &\leq \frac{\lambda A \Psi\left(\delta\right)}{\Gamma(p)(1-\xi)} \int_0^1 h\left(r\right) dr, \end{aligned}$$

thus

$$\|Tu\| \leq \frac{\lambda A \Psi\left(\delta\right)}{\Gamma(p)(1-\xi)} \left\| h \right\|_{L^1} < +\infty,$$

hence $T(B_\delta)$ is uniformly bounded.

(ii) $T(B_\delta)$ is equicontinuous.

Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for fixed $s\in[0,1]$ and for any $\varepsilon>0$, there exists a constant $\rho>0$, such that for any $t_1,\, t_2\in[0,1]$, $|t_1-t_2|<\rho$,we have

$$|G(t_1,s)-G(t_2,s)|\le\frac{\Gamma(p)(1-\xi)}{\lambda\Psi(\delta)\|h\|_{L^1}}\varepsilon;$$

since

$$\begin{aligned}|Tu(t_1)-Tu(t_2)| &\le \frac{\lambda}{\Gamma(p)}\int_0^1|G(t_1,s)-G(t_2,s)|\int_0^1|H(s,r)|\,|f(r,u(r))|\,drds\\ &\le \frac{\epsilon(1-\xi)}{\Psi(\delta)\|h\|_{L^1}}\int_0^1|H(s,r)|\,h(r)\,\Psi(|u(r)|)\,dr,\end{aligned}$$

we obtain

$$|Tu(t_1)-Tu(t_2)|\le\varepsilon.$$

Consequently, $T(B_r)$ is equicontinuous; by means of the Arzela-Ascoli Theorem, we conclude that T is completely continuous.

Now we apply the Leray-Schauder nonlinear alternative to prove that T has at least a nontrivial solution in E.

Setting $\Omega=\{u\in E:\|u\|<\delta\}$, then for $u\in\partial\Omega$, such that $u=\mu Tu$,

$0<\mu<1$, we have with the help of (4.1)

$$|u(t)|=\mu\,|Tu(t)|\le|Tu(t)|\le\frac{\lambda A\Psi(\delta)}{\Gamma(p)(1-\xi)}\|h\|_{L^1}.$$

Choose $\lambda^*=\frac{\delta\Gamma(p)(1-\xi)}{2A\Psi(\delta)\|h\|_{L^1}}$. Then, for $0<\lambda\le\lambda^*$, we have

$$\|u\|\le\frac{\lambda A\Psi(\delta)}{\Gamma(p)(1-\xi)}\|h\|_{L^1}<\delta.$$

We conclude that T has a fixed point $u^*\in\overline{\Omega}$ and then the FBVP (P) has a nontrivial solution u^* in E. □

4. Existence of a positive solution of problem 2

In this section, we apply theorem 6 to establish an eigenvalue interval for the existence of a positive solutions for the problem (P_2)

$$\begin{gathered}{}^cD^p_{0^+}\ {}^cD^q_{0^+}u(t)+\lambda f(t,u(t))=0,\ 0<t<1,\\ u(0)=0,u'(0)=au'(1),\\ {}^cD^q_{0^+}u(1)={}^cD^q_{0^+}u(\xi),\\ {}^cD^q_{0^+}u(0)={}^cD^{q+2}_{0^+}u(0)=0.\end{gathered}$$

For convenience, we set:

$$A_0=\lim_{u\to0^+}\frac{f(u)}{u},\qquad A_\infty=\lim_{u\to\infty}\frac{f(u)}{u}.$$

Let us define the cone $P\subset E$ by

$$P=\left\{u\in E,u(t)\ge0,0\le t\le1,\min_{\tau\le t\le1}u(t)\ge\frac{a\tau(\xi-\xi^{p-1})}{Aq\,(q+1)\,(1-a)\,\Gamma(q-1)}\|u\|\right\}$$

Lemma 4.1. *If $f \in C([0,1],\mathbb{R}_+)$, then, the solution of problem (P_2) satisfies*

$$\min_{t\in[\tau,1]} u(t) \geq \frac{a\tau(\xi-\xi^{p-1})}{Aq\,(q+1)\,(1-a)\,\Gamma(q-1)}\,\|u\|\,.$$

Proof. By Lemma 7, u can be expressed by

$$\begin{aligned} u(t) &= \frac{\lambda}{\Gamma(p)}\int_0^1 G(t,s)\int_0^1 -H(s,r)f\left(r,u\left(r\right)\right)drds \\ &\leq \frac{\lambda}{\Gamma(p)}\int_0^1 A\int_0^1 (1-r)^{p-1} f\left(r,u\left(r\right)\right)drds \end{aligned}$$

then

$$\|u\| \leq \frac{\lambda A}{\Gamma(p)(1-\xi)}\int_0^1 (1-r)^{p-1} f\left(r,u\left(r\right)\right)dr.$$

Also, we have

$$\begin{aligned} u(t) &\geq \frac{\lambda a\left(\xi-\xi^{p-1}\right)}{q\,(q+1)\,\Gamma(p)\Gamma(q-1)(1-\xi)\,(1-a)}t\int_0^1 (1-r)^{p-1} f\left(r,u(r)\right)dr \\ &\geq \frac{at(\xi-\xi^{p-1})}{Aq\,(q+1)\,(1-a)\,\Gamma(q-1)}\,\|u\|\,, \end{aligned}$$

therefore

$$\min_{\tau\leq t\leq 1} u(t) \geq \frac{a\tau(\xi-\xi^{p-1})}{Aq\,(q+1)\,(1-a)\,\Gamma(q-1)}\,\|u\|\,.$$

□

Theorem 4.2. *Let $f\left(t,u\left(t\right)\right) = \varphi\left(t\right)f\left(u\left(t\right)\right)$ with $f \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\varphi \in C([0,1],\mathbb{R}_+)$, $\int_0^1 s(1-s)^{q-1}\varphi(s)ds \neq 0$ and $\tau\in[0,1]$. Then for each*

$$\lambda \in \left]\frac{\Gamma(p)\left(Aq\left(q+1\right)\Gamma(q-1)\left(1-a\right)\left(1-\xi\right)\right)^2}{a^2\tau A_\infty(\zeta-\zeta^{p-1})^2\int_\tau^1(1-r)^{p-1}\varphi\left(r\right)dr}, \frac{\Gamma(p)(1-\xi)}{AA_0\int_0^1(1-r)^{p-1}\varphi(r)dr}\right[, \tag{4.1}$$

the problem (P_2) has at least one positive solution.

Proof. We apply Guo-Krasnosel'skii fixed point Theorem on cone.

Let u be in P, in view of nonnegativeness and continuity of functions $G(t,s), -H\left(t,s\right)$ and f, we conclude that $Tu\geq 0, t\in[0,1]$, continuous and $T(P)\subset P$.

i)It is clear that $T(B_r)$ is uniformly bounded and equicontinuous, by means of the Arzela-Ascoli Theorem we conclude that T is completely continuous.

From (4.1) there exists $\varepsilon>0$ such that

$$\frac{\Gamma(p)\left(Aq\left(q+1\right)\Gamma(q-1)\left(1-a\right)\left(1-\xi\right)\right)^2}{a^2\tau^2\left(A_\infty-\varepsilon\right)(\zeta-\zeta^{p-1})^2\int_\tau^1(1-r)^{p-1}\varphi\left(r\right)dr} \leq \lambda \leq \frac{\Gamma(p)(1-\xi)}{A\left(A_0+\varepsilon\right)\int_0^1(1-r)^{p-1}\varphi(r)dr}.$$

By the definition of A_0, there exists $\delta_1>0$, such that for any u, $0\leq u\leq\delta_1$, we have

$$f(u)\leq\left(A_0+\varepsilon\right)u.$$

Set $\Omega_1=\left\{u\in E:\|u\|<\delta_1\right\}$. let $u\in P\cap\partial\Omega_1$, then we have

$$\begin{aligned} Tu(t) &= \frac{\lambda}{\Gamma(p)}\int_0^1 G(t,s)\int_0^1 -H(s,r)\varphi\left(r\right)f\left(u\left(r\right)\right)drds \\ &\leq \frac{\lambda A\left(A_0+\varepsilon\right)}{\Gamma(p)(1-\xi)}\int_0^1(1-r)^{p-1}\varphi\left(r\right)u\left(r\right)dr \\ &\leq \frac{\lambda A\left(A_0+\varepsilon\right)}{\Gamma(p)(1-\xi)}\|u\|\int_0^1(1-r)^{p-1}\varphi\left(r\right)dr \\ &\leq \|u\|\,. \end{aligned}$$

On the other hand, by the definition of A_∞, there exists $\delta_2 > 0$, such that

$$f(u) \geq (A_\infty - \varepsilon)\, u, \text{ for any } u \in [\delta_2, +\infty[\,.$$

Setting $R = \max\left\{2\delta_1, \frac{Aq(q+1)\Gamma(q-1)(1-a)(1-\xi)}{a\tau(\zeta-\zeta^{p-1})}\delta_2\right\}$ and $\Omega_2 = \{u \in E : \|u\| < R\}$, then $\overline{\Omega_1} \subset \Omega_2$ and for $u \in P \cap \partial\Omega_2$ we have

$$\begin{aligned}
\|Tu(t)\| &\geq Tu(\tau) \\
&\geq \frac{\lambda}{\Gamma(p)} \int_0^1 \frac{a\tau s(1-s)^{q-1}}{\Gamma(q-1)\,(1-a)} \int_0^1 (\zeta - \zeta^{p-1}) \frac{(1-r)^{p-1}}{(1-\xi)} \varphi\,(r)\, f\,(u\,(r))\, drds \\
&\geq \frac{\lambda a\tau(\zeta - \zeta^{p-1})}{\Gamma(p)\Gamma(q-1)\,(1-a)\,(1-\xi)} \int_0^1 s(1-s)^{q-1} \int_0^1 (1-r)^{p-1} \varphi\,(r)\, f\,(u\,(r))\, drds \\
&\geq \frac{\lambda a^2\tau^2(\zeta - \zeta^{p-1})^2\,(A_\infty - \varepsilon)}{\Gamma(p)\,(Aq\,(q+1)\,\Gamma(q-1)\,(1-a)\,(1-\xi))^2} \|u\| \int_\tau^1 (1-r)^{p-1} \varphi\,(r)\, dr \\
&\geq \|u\|\,.
\end{aligned}$$

According to Theorem 6 T has a fixed point in $P \cap (\overline{\Omega_2}/\Omega_1)$, that means that the problem (P_2) has at least one positive solution in $P \cap (\overline{\Omega_2}/\Omega_1)$. $\square$

5. Examples

We illustrate our work with three examples.

Example 5.1. *For the fractional boundary value problem*

$$\begin{cases}
{}^cD_{0^+}^{\frac{8}{3}}\ {}^cD_{0^+}^{\frac{3}{2}} u(t) = \lambda\left(\frac{t^3}{4}x + \sin t\right),\ 0 < t < 1, \\
u(0) = 0,\ u'(0) = \frac{1}{3}u'(1)), \\
D_{0^+}^{\frac{3}{2}} u(1) = D_{0^+}^{\frac{3}{2}} u(\frac{1}{2}), \\
D_{0^+}^{\frac{3}{2}} u(0) = D_{0^+}^{\frac{7}{2}} u(0) = 0,
\end{cases}$$

we have

$f(t,x) = \frac{t^3}{4}x + \sin t$, $2 < p = \frac{8}{3} < 3$, $1 < q = \frac{3}{2} < 2$, $a = \frac{1}{3} < 1$ *and* $\xi = \frac{1}{2}$.

Then

$$|f(t,x) - f(t,y)| \leq g(t)\,|x-y|\,, \quad \forall x \in \mathbb{R}, t \in [0,1],$$

where

$$g(t) = \frac{t^3}{4}.$$

Simple calculus give:

$\|g\|_{L^1} = 0,0625$, $A = \frac{1}{\Gamma(q)} + \frac{a}{(1-a)\Gamma(q-1)} = 2.0146$, $\lambda^* = \frac{\Gamma(p)(1-\xi)}{2A\|g\|_{L^1}} = 7.4142$.

Hence from Theorem 11 we conclude that for any $0 < \lambda \leq 7.4142$, *the problem (P) has a unique solution in E.*

Example 5.2. *The following fractional boundary value problem*

$$\begin{cases}
{}^cD_{0^+}^{\frac{5}{2}}\ {}^cD_{0^+}^{\frac{4}{3}} u(t) = \lambda\left(\frac{1}{1+t}\right)\frac{\exp\left(\frac{u}{2}\right)}{1+\exp\left(\frac{u}{3}\right)}, \quad 0 < t < 1, \\
u(0) = 0,\ u'(0) = \frac{1}{3}u'(1)), \\
D_{0^+}^{\frac{4}{3}} u(1) = D_{0^+}^{\frac{4}{3}} u(\frac{1}{2}), \\
D_{0^+}^{\frac{4}{3}} u(0) = D_{0^+}^{\frac{4}{3}} u(0) = 0,
\end{cases}$$

has at least one nontrivial solution u^* in E. Indeed, we have

$$\begin{aligned}|f(t,x)| &= \left(\frac{1}{1+t}\right)\frac{\exp\left(\frac{x}{2}\right)}{1+\exp\left(\frac{x}{3}\right)}\\ &\leq \frac{1}{1+t}\exp\left(\frac{|x|}{6}\right)\\ &\leq h(t)\psi\left(|x|\right),\end{aligned}$$

where $h(t)=\frac{1}{1+t}$ and $\psi(x)=\exp\frac{x}{6}$.

For $\delta=0.9$, some computions lead to

$$\|h\|_{L^1}=\ln 2,\ \psi\left(0.9\right)=1.161\,8,$$

hence

$$\lambda^*=\frac{\delta\Gamma(p)(1-\xi)}{2A\Psi\left(\delta\right)\|h\|_{L^1}}=2.715\,3.$$

Theorem 12 implies that for each $0<\lambda\leq 2.715\,3$, the problem (P) has at least one nontrivial solution u^* in E.

Example 5.3. *Consider the fractional boundary value problem*

$$\begin{cases} {}^cD_{0^+}^{\frac{8}{3}}\ {}^cD_{0^+}^{\frac{3}{2}}u(t)+\lambda f\left(t,u\left(t\right)\right)=0,\ 0<t<1,\\ u(0)=0,\ u'(0)=\frac{1}{3}u'(1)),\\ D_{0^+}^{\frac{3}{2}}u(1)=D_{0^+}^{\frac{3}{2}}u(\frac{1}{2}),\\ D_{0^+}^{\frac{3}{2}}u(0)=D_{0^+}^{\frac{7}{2}}u(0)=0.\end{cases}$$

In this example take

$$f\left(t,u\left(t\right)\right)=\left(\frac{\left(1-t\right)u^2+u}{\left(u+7\right)}\right)=\left(1-t\right)\left(\frac{u^2+u}{\left(u+7\right)}\right).$$

Obviously, we have

$$\begin{aligned}A_0 &= \lim_{u\to 0^+}\frac{f(u)}{u}=\frac{1}{7},\\ A_\infty &= \lim_{u\to\infty}\frac{f(u)}{u}=1.\end{aligned}$$

Since $a=\frac{1}{3}$, $\xi=\frac{1}{2}$, $p=\frac{8}{3}$ and $q=\frac{3}{2}$, through a computation, we can get

$$A=2.0146,$$

$$\int_0^1(1-r)^{p-1}\varphi(r)dr=\int_0^1(1-r)^p dr=\frac{3}{11}.$$

Then

$$\frac{\Gamma(p)(1-\xi)}{AA_0\int_0^1(1-r)^{p-1}\varphi(r)dr}=9.5844.$$

Choose $\tau=0.9$; we have

$$\frac{\Gamma(p)\left(Aq\left(q+1\right)\Gamma(q-1)\left(1-a\right)\left(1-\xi\right)\right)^2}{a^2\tau^2A_\infty(\zeta-\zeta^{p-1})^2\int_\tau^1(1-r)^{p-1}\varphi\left(r\right)dr}=1643.8$$

Theorem 13 implies that, for $\lambda\in\,]9.5844,1643.8[$, problem (P_2) has at least one positive solution.

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References

1. B. Ahmad and J. J. Nieto, Existence results for nonlinear boundary value problems of fractional integro differential equations with integral boundary conditions, Bound. Value Porblem. (2009), Art. ID 708576, 11 pages.
2. Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, Nonlinear Analysis **72**, 916–924 (2010).
3. Z. Bai, Eigenvalue intervals for a class of fractional boundary value problem, Computers & Mathematics with Applications, **64** (10) 3253–3257 (2012).
4. L. Bing, Positive solutions of a nonlinear three point boundary value problem, Comput. Math. Appl. **44,** 201-211 (2002).
5. A. Guezane-Lakoud and R. Khaldi, Positive solution to a higher order fractional boundary value problem with fractional integral condition, Romanian Journal of Mathematics and Computer Sciences, **2** 28–40 (2012).
6. D. Guo, V. Lakshmikantham, Nonlinear problems in abstract cones, Academic Press, San Diego, 1988.
7. D. Jiang and C. Yuan, The positive properties of the Green function for Dirichlet type boundary value problems of nonlinear fractional differential equations and its application, Nonlinear Analysis **72**, 710–719 (2010).
8. E. R. Kaufmann and N. Kosmatov, A Second-Order Singular Boundary Value Problem, Comput. Math. Appl. **47**, 1317–1326 (2004).
9. R.A. Khan, Existence and approximation of solutions of nonlinear problems with integral boundary conditions, Dynam. Systems Appl. **14** (2005) 281-296.
10. R.A. Khan, M.R Rehman and J. Henderson, Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions, Fractional Diff. eq. **1** (1) 29–43 (2001) .
11. A. Kilbas and A. Marzan, Cauchy problem for differential equation with Caputo derivative, Fractional calculus and applied analysis **7** (3) (2004).
12. A. Kilbas, M. Srivastava and J. J. Trujillo, Theory and applicaions of fractional differential equations, in: North-Holland Mathematics Studies, 204, Elsevier Science, B.V. Amsterdam, (2006).
13. S. Sun, Y. Zhao, Z. Han, and J. Liu, Eigenvalue problem for a class of nonlinear fractional differential equations, Annals of Functional Analysis, **4 (1)** 25–39 (2013).
14. X. Xu, Positive Solutions for Singular Semi-positone Boundary Value Problems, J. Math.Anal. Appl. **237**, 480–491 (2007).
15. X. Xu, D. Jiang and C. Yuan, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation, Nonlinear Analysis **71**, 4676–4688 (2009).
16. S. Q. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, Electronic Journal of Differential Equations **36**, 1–12 (2006).
17. X. Zhang, L. Liu, and Y. Wu, The eigenvalue problem for a singular higher order fractional differential equation involving fractional derivatives, Applied Mathematics and Computation, **218** (17) 8526–8536 (2012).

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