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Some Metrical φ-Fixed Point Results of Wardowski Type with Applications to Integral Equations

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Abstract: In this paper, we introduce the notions of $(\mathcal{F}^*, \varphi)$-contraction as well $(\mathcal{F}^*, \varphi)$-expansion mappings and utilize the same to prove some φ-fixed point results in complete metric spaces. An example is provided to exhibit the utility of our results. As applications, we deduce some fixed point theorems in partial metric spaces besides proving an existence and uniqueness result on the solution of nonlinear integral equations.

Key Words: Partial metric spaces, φ-fixed point, $(\mathcal{F}^*, \varphi)$-contraction, $(\mathcal{F}^*, \varphi)$-expansion, Integral equation.

Contents

1 Introduction and preliminaries 1

2 Main Results 3

3 Results in partial metric spaces 7

4 An application to integral equations 9

1. Introduction and preliminaries

In 1922, S. Banach [1] proved a very famous fixed point result known as Banach contraction principle. It has been originally proved on the set of continuous functions $C[0,1]$ equipped with maximum metric and used the same to establish the existence and uniqueness of solution of an integral equation. Due to its simplicity, usefulness and natural applications, it is perhaps the most widely applied fixed point theorem in many branches of mathematical analysis. In the last several decades, many mathematicians have obtained several generalizations and extensions of Banach contraction principle in different directions (see [2,3,4,5] and references therein).

In 2012, as a new extension of Banach contraction, Wardowski [6] proposed a class of auxiliary functions called $\mathcal{F}$-functions and defined the notion of $\mathcal{F}$-contraction as follows:

Definition 1.1. *[6] Assume that $\mathcal{F} : [0,\infty) \to \mathbb{R}$ is a function satisfying:*

($\mathcal{F}1$) *$\mathcal{F}$ is strictly increasing;*

($\mathcal{F}2$) *for every sequence $\{u_n\} \subset (0,\infty)$,*

$$\lim_{n\to\infty} u_n = 0 \ \text{ if and only if } \ \lim_{n\to\infty} \mathcal{F}(u_n) = -\infty;$$

($\mathcal{F}3$) *there exists $\lambda \in (0,1)$ satisfying $\lim_{\beta\to 0^+} \beta^\lambda \mathcal{F}(\beta) = 0$.*

We denote by $\mathbb{F}$ the family of all functions $\mathcal{F}$ satisfying the conditions ($\mathcal{F}1$), ($\mathcal{F}2$) and ($\mathcal{F}3$).

Definition 1.2. *[6] Let (X,d) be a metric space. A mapping $T : X \to X$ is called an $\mathcal{F}$-contraction if there exist $\mathcal{F} \in \mathbb{F}$ and $\tau > 0$ such that*

$$Tu \neq Tv \ \text{ implies } \ \tau + \mathcal{F}(d(Tu,Tv)) \leq \mathcal{F}(d(u,v)), \ \text{for all } u,v \in X.$$

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Wardowski [6] proved that every $\mathcal{F}$-contraction mapping defined on a complete metric space admits a unique fixed point. Recently, there are many authors who extended the concept of $\mathcal{F}$-contraction in order to obtain some other general classes. For more details see [7,8,9,10,11,12,13,14,15].

On the other hand, the notion of expansion mapping has been introduced by Wang et al. [16] in metric spaces. In recent years, the theory of expansive mappings have made a considerable progress (e.g. [17,18,19,20,21] and references cited therein). In this regard, Górnicki [22] defined the concept of $\mathcal{F}$-expansion as under:

Definition 1.3. *[22] Let (X, d) be a metric space. A mapping $T : X \to X$ is called an $\mathcal{F}$-expansion if there exist $\mathcal{F} \in \mathbb{F}$ and $\tau > 0$ such that*

$$Tu \neq Tv \;\text{ implies }\; \mathcal{F}(d(Tu, Tv)) \geq \mathcal{F}(d(u, v)) + \tau, \text{ for all } u, v \in X.$$

Based on this definition Górnicki [22] presented some results for $\mathcal{F}$-expansion mappings on metric and G-metric spaces.

Now, let us recall the definition of partial metric space and other results given in [23]. For further details, we refer the readers to [24,25,23].

Definition 1.4. *[23] Let X be a non-empty set. A partial metric p on X is a mapping $p : X \times X \to [0, \infty)$ satisfying the following conditions (for all $u, v, w \in X$):*

(P1) $p(u, u) = p(v, v) = p(u, v) \Leftrightarrow u = v$;

(P2) $p(u, u) \leq p(u, v)$;

(P3) $p(u, v) = p(v, u)$;

(P4) $p(u, v) \leq p(u, w) + p(w, v) - p(w, w)$.

The pair (X, p) called as partial metric space. Naturally, if $p(u, v) = 0$, then by $(P2)$ and $(P3)$, $p(u, u) = p(v, v) = p(u, v) = 0$, so by $(P1)$ $u = v$. But if $u = v$, $p(u, v)$ may not be zero. On the other hand, if $p(w, w) = 0$, for each $w \in X$, then the partial metric space reduces to a metric space. Several interesting examples of partial metric spaces which are not metric spaces can be found in [23].

Remark 1.5. *[23] If p is a partial metric on a nonempty set X, then the mapping $d_p : X \times X \to [0, \infty)$ given by*

$$d_p(u, v) = 2p(u, v) - p(u, u) - p(v, v), \;\forall\; u, v \in X \tag{1.1}$$

defines a metric on X.

Lemma 1.6. *[23] Let (X, p), (X, d_p) be the partial metric space and the metric defined as in (1.1) respectively, Then*

(a) a sequence $\{u_n\}$ is Cauchy in (X, p) if and only if it is Cauchy in (X, d_p);

(b) if (X, d_p) is complete, then (X, p) is complete and vise versa.

In 2014, Jleli et al. [26] introduced the notion of φ-fixed point and established φ-fixed point theorems in metric spaces. They also deduced some fixed point results in partial metric spaces.

Definition 1.7. *[26] Let X be a nonempty set and $\varphi : X \to [0, \infty)$ be a function. An element $w \in X$ is said to be φ-fixed point of a mapping $T : X \to X$, if and only if $w \in F_T \cap Z_\varphi$, where F_T and Z_φ defined as $F_T = \{u \in X : Tu = u\}$ and $Z_\varphi = \{u \in X : \varphi(u) = 0\}$.*

The following auxiliary results will be needed in the sequel.

Lemma 1.8. *[27] Let $\{u_n\}$ be a sequence in a metric space (X, d). If $\{u_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and subsequences $\{u_{n_q}\}$ and $\{u_{m_q}\}$ of $\{u_n\}$ such that*

$$m_q > n_q > q, \quad d(u_{m_q - 1}, u_{n_q}) < \epsilon \leq d(u_{m_q}, u_{n_q}), \text{ for all } q \in \mathbb{N}.$$

Lemma 1.9. *[28] If p is a partial metric on a nonempty set X, then the function $\varphi : X \to [0,\infty)$ given by $\varphi(u) = p(u,u)$, for all $u \in X$ is lower semi-continuous with respect to d_p.*

The attempted improvement in this paper is four-fold:

(1) to introduce the concepts of $(\mathcal{F}^*,\varphi)$-contraction and $(\mathcal{F}^*,\varphi)$-expansion mappings;

(2) to establish some φ-fixed point theorems in metric spaces;

(3) to deduce some related results in the partial metric spaces;

(4) to examine the existence and uniqueness of a solution for nonlinear integral equation.

2. Main Results

Let $\mathbb{F}_{\mathbb{H}}$ be the family of all continuous functions $\mathcal{F}^* : (0,\infty)\times[0,\infty)^2 \to \mathbb{R}$ which satisfy the condition:

(H) for all sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ of positive numbers,

$$\lim_{n\to\infty} \mathcal{F}^*(\alpha_n,\beta_n,\gamma_n) = -\infty \ \text{ if and only if } \ \lim_{n\to\infty} \alpha_n+\beta_n+\gamma_n = 0.$$

Remark 2.1. *If $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,\infty)$, then*

$$\lim_{n\to\infty} \alpha_n+\beta_n+\gamma_n = 0 \ \textit{ if and only if } \ \lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\beta_n = 0 \textit{ and } \lim_{n\to\infty}\gamma_n = 0.$$

Example 2.2. *Let $\mathcal{F}_1^*,\mathcal{F}_2^*,\mathcal{F}_3^* : (0,\infty)\times[0,\infty)^2 \to \mathbb{R}$ be functions defined as*

1. $\mathcal{F}_1^*(\alpha,\beta,\gamma) = \ln(\alpha+\beta+\gamma)$;
2. $\mathcal{F}_2^*(\alpha,\beta,\gamma) = -1/(\alpha+\beta+\gamma)$;
3. $\mathcal{F}_3^*(\alpha,\beta,\gamma) = 1/1 - e^{(\alpha+\beta+\gamma)}$.

Then $\mathcal{F}_1^*,\mathcal{F}_2^*,\mathcal{F}_3^* \in \mathbb{F}_{\mathbb{H}}$.

Now, we introduce the notion of $(\mathcal{F}^*,\varphi)$-contraction mappings followed by an auxiliary tool for proving our results as under:

Definition 2.3. *Let (X,d) be a metric space and $\varphi : X \to [0,\infty)$ be a function. A mapping $T : X \to X$ is called $(\mathcal{F}^*,\varphi)$-contraction if there exist $\mathcal{F}^* \in \mathbb{F}_{\mathbb{H}}$ and $\tau > 0$ such that (for all $u,v \in X$)*

$$Tu \neq Tv \quad \textit{implies} \quad \tau + \mathcal{F}^*(d(Tu,Tv),\varphi(Tu),\varphi(Tv)) \leq \mathcal{F}^*(d(u,v),\varphi(u),\varphi(v)). \tag{2.1}$$

Lemma 2.4. *Let (X,d) be a metric space and $T : X \to X$ be a mapping. Define a sequence $\{u_n\}$ by $u_n = Tu_{n-1}$ with initial point $u_0 \in X$. If $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}_0$ and T is an $(\mathcal{F}^*,\varphi)$-contraction with $\varphi : X \to [0,\infty)$ and $\mathcal{F}^* \in \mathbb{F}_{\mathbb{H}}$, then*

(i) $\lim_{n\to\infty} d(u_n,u_{n+1}) = 0 = \lim_{n\to\infty}\varphi(u_n)$;

(ii) $\{u_n\}$ *is Cauchy.*

Proof. (i) In view of (2.1), we have

$$\begin{aligned}
\mathcal{F}^*(d(Tu_{n-1},Tu_n),\varphi(Tu_{n-1}),\varphi(Tu_n)) &\leq \mathcal{F}^*(d(u_{n-1},u_n),\varphi(u_{n-1}),\varphi(u_n)) - \tau \\
&\leq \mathcal{F}^*(d(u_{n-2},u_{n-1}),\varphi(u_{n-2}),\varphi(u_{n-1})) - 2\tau \\
&\vdots \\
&\leq \mathcal{F}^*(d(u_0,Tu_0),\varphi(u_0),\varphi(Tu_0)) - n\tau.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{F}^*(d(Tu_{n-1}, Tu_n), \varphi(Tu_{n-1}), \varphi(Tu_n)) = -\infty$$

which together with (H) imply that

$$\lim_{n\to\infty} \varphi(u_n) = \lim_{n\to\infty} d(u_n, u_{n+1}) = 0.$$

(ii) Suppose that $\{u_n\}$ is not Cauchy in X. Then (in view of Lemma 1.8) there exist $\epsilon > 0$ and subsequences $\{u_{n_q}\}$ and $\{u_{m_q}\}$ of $\{u_n\}$ such that

$$m_q > n_q > q, \qquad d(u_{m_q}, u_{n_q}) \geq \epsilon \quad \text{and} \quad d(u_{m_q-1}, u_{n_q}) < \epsilon, \text{ for all } q \in \mathbb{N}. \tag{2.2}$$

Now, we have

$$\begin{aligned} \epsilon &\leq d(u_{m_q}, u_{n_q}) \\ &\leq d(u_{m_q}, u_{m_q-1}) + d(u_{m_q-1}, u_{n_q}) \\ &\leq d(u_{m_q}, u_{m_q-1}) + \epsilon. \end{aligned}$$

Letting $q \to \infty$ and using part (i), we obtain

$$\lim_{q\to\infty} d(u_{m_q}, u_{n_q}) = \epsilon. \tag{2.3}$$

Also, we can find $n_0 \in \mathbb{N}$ such that

$$d(u_{m_q}, u_{m_q+1}) < \epsilon/4 \text{ and } d(u_{n_q}, u_{n_q+1}) < \epsilon/4, \text{ for all } q \geq n_0. \tag{2.4}$$

Next, we show that $d(u_{m_q+1}, u_{n_q+1}) > 0$, for all $q \geq n_0$. For the sake of contradiction suppose there exists $l \geq n_0$ such that

$$d(u_{m_l+1}, u_{n_l+1}) = 0. \tag{2.5}$$

Using (2.2), (2.4) and (2.5), we obtain

$$\begin{aligned} \epsilon \leq d(u_{m_l}, u_{n_l}) &\leq d(u_{m_l}, u_{m_l+1}) + d(u_{m_l+1}, u_{n_l}) \\ &\leq d(u_{m_l}, u_{m_l+1}) + d(u_{m_l+1}, u_{n_l+1}) + d(u_{n_l+1}, u_{n_l}) \\ &< \epsilon/4 + 0 + \epsilon/4 = \epsilon/2, \end{aligned}$$

a contradiction. So, we have

$$d(Tu_{m_q}, Tu_{n_q}) = d(u_{m_q+1}, u_{n_q+1}) > 0, \text{ for all } q \geq n_0. \tag{2.6}$$

Therefore, from (2.6) and (2.1), we have (for all $q \geq n_0$)

$$\tau + \mathcal{F}^*(d(u_{m_q+1}, u_{n_q+1}), \varphi(u_{m_q+1}), \varphi(u_{n_q+1})) \leq \mathcal{F}^*(d(u_{m_q}, u_{n_q}), \varphi(u_{m_q}), \varphi(u_{n_q})).$$

As $\mathcal{F}^*$ is continuous, on letting $q \to \infty$ and using (2.3) and part (i), we obtain $\tau + \mathcal{F}^*(\epsilon, 0, 0) \leq \mathcal{F}^*(\epsilon, 0, 0)$, a contradiction. Hence, $\{u_n\}$ must be a Cauchy sequence. $\square$

Now, we state and prove our first main result as follows:

Theorem 2.5. *Let (X, d) be a complete metric space and $\varphi : X \to [0, \infty)$ be a lower semi-continuous function. Assume that $T : X \to X$ satisfies the following conditions:*

(a) T is an $(\mathcal{F}^*, \varphi)$-contraction mapping, where $\mathcal{F}^* \in \mathbb{F}_{\mathbb{H}}$,

(b) $F_T \subseteq Z_\varphi$.

Then T has a unique φ-fixed point.

Proof. Choose any point u_0 in X and define $\{u_n\}$ by

$$u_n = T^n u_0 = T u_{n-1}, \text{ for all } n \in \mathbb{N}.$$

If $u_{n_0} = Tu_{n_0}$ for some $n_0 \in \mathbb{N}$, then u_{n_0} is φ-fixed point of T (due to the condition (b)). Assume that $d(u_n, Tu_n) > 0$, for all $n \in \mathbb{N}$. In view of Lemma 2.4 (ii), we have $\{u_n\}$ is a Cauchy sequence. The completeness of the metric space (X, d) ensures the existences of a point $u \in X$ such that

$$\lim_{n\to\infty} u_n = u. \tag{2.7}$$

Now, due to the lower semi-continuity of φ and Lemma 2.4 (i), we have

$$0 \le \varphi(u) \le \lim_{n\to\infty} \inf \varphi(u_n) = 0,$$

which implies that

$$\varphi(u) = 0. \tag{2.8}$$

Next, let $P = \{n \in \mathbb{N}_0 : u_n = Tu\}$. Depending on P, We distinguish two cases. Firstly, if P is infinite set, then there exists a subsequence $\{u_{n_q}\} \subseteq \{u_n\}$ such that $\lim_{q\to\infty} u_{n_q} = Tu$ and hence $Tu = u$. Secondly, if P is finite set, then $d(u_n, Tu) > 0$ for infinitely many $n \in \mathbb{N}_0$. This ensures the existence of a subsequence $\{u_{n_q}\} \subseteq \{u_n\}$ such that $d(u_{n_q}, Tu) > 0$ for all $q \in \mathbb{N}_0$. As T is an $(\mathcal{F}^*, \varphi)$-contraction mapping, we have

$$\tau + \mathcal{F}^*(d(Tu_{n_q}, Tu), \varphi(Tu_{n_q}), \varphi(Tu)) \le \mathcal{F}^*(d(u_{n_q}, u), \varphi(u_{n_q}), \varphi(u)),$$

so that

$$\tau + \mathcal{F}^*(d(u_{n_q+1}, Tu), \varphi(u_{n_q+1}), \varphi(Tu)) \le \mathcal{F}^*(d(u_{n_q}, u), \varphi(u_{n_q}), \varphi(u)).$$

Taking the limit as $q \to \infty$ in the both sides of the above inequality, using (2.7), (2.8), condition (H), the continuity of $\mathcal{F}^*$ and the Lemma 2.4 (i), we get

$$\lim_{q\to\infty} d(u_{n_q+1}, Tu) = 0,$$

which implies that

$$d(u, Tu) = 0. \tag{2.9}$$

Therefore, in view of (2.8) and (2.9), we conclude that u is φ-fixed point of T. It remains to show that the φ-fixed point of T is unique. Suppose $u_1, u_2 \in X$ are two distinct φ-fixed points of T. Then $d(u_1, u_2) = d(Tu_1, Tu_2) > 0$ and the contractive condition implies that

$$\tau \le \mathcal{F}^*(d(u_1, u_2), 0, 0) - \mathcal{F}^*(d(u_1, u_2), 0, 0) = 0,$$

a contradiction. Hence, the φ-fixed point of the mapping T is unique. □

The following example shows the utility of Theorem 2.5.

Example 2.6. *Let $X = [0, 3]$ be endowed with the usual metric d. Define $T : X \to X$ as*

$$T(u) = \begin{cases} 0, & 0 \le u < 2.5; \\ k \ln(\frac{u}{2}), & 2.5 \le u \le 3, \end{cases}$$

where $k \in (0, 1)$. Observe that T is discontinuous at $u = 2.5$.
Now, we show that Theorem 2.5 is applicable in the context of the present example. To see this, we need the essential functions: $\mathcal{F}^* : (0, \infty) \times [0, \infty)^2 \to \mathbb{R}$ and $\varphi : X \to [0, \infty)$ which are given by:

$$\mathcal{F}^*(\alpha, \beta, \gamma) = \ln(\alpha + \beta + \gamma), \text{ for all } \alpha, \beta, \gamma \in [0, \infty) \text{ and } \alpha \ne 0.$$

and

$$\varphi(u) = u, \text{ for all } u \in X$$

It is obvious that φ is lower semi-continuous and $\mathcal{F}^* \in \mathbb{F}_{\mathbb{H}}$. Now, we are going to show the $(\mathcal{F}^*, \varphi)$-contractivity condition of the mapping T for some $\tau > 0$. To do so, it is enough to show that

$$d(Tu, Tv) + \varphi(Tu) + \varphi(Tv) \leq e^{-\tau}(d(u, v) + \varphi(u) + \varphi(v)).$$

for all $u, v \in X$ and $Tu \neq Tv$.
We consider the following cases:
Case 1: *If $u, v \in [0, 2.5)$, then it is obvious.*
Case 2: *If $u, v \in [2.5, 3]$, then we have*

$$\begin{aligned} d(Tu, Tv) + \varphi(Tu) + \varphi(Tv) &= 2\max\{Tu, Tv\} \\ &\leq 2\max\{ku, kv\} \\ &= k(2\max\{u, v\}) \\ &= k(d(u, v) + \varphi(u) + \varphi(v)). \end{aligned}$$

Case 3: *If $u \in [2.5, 3]$ and $v \in [0, 2.5)$, we have*

$$\begin{aligned} d(Tu, Tv) + \varphi(Tu) + \varphi(Tv) &= 2\max\{Tu, Tv\} \\ &= 2\max\{Tu, 0\} \\ &\leq 2\max\{ku, 0\} \\ &= k(2\max\{u, v\}) \\ &= k(d(u, v) + \varphi(u) + \varphi(v)). \end{aligned}$$

Easily, one can observe that all the requirements of Theorem 2.5 are fulfilled with $\tau = -\ln k > 0$. Therefore, the mapping T has a unique φ-fixed point (namely $u = 0$).

Remark 2.7. *Setting $\mathcal{F}^*(\alpha, \beta, \gamma) = \ln(\alpha + \beta + \gamma)$, for all $\alpha, \beta, \gamma \in [0, \infty)$, $\alpha \neq 0$ and $\varphi(u) = 0$, for all $u \in X$ in Theorem 2.5, we deduce Banach contraction principle [1].*

Next, we define the concept of $(\mathcal{F}^*, \varphi)$-expansion mappings followed by our second main result as follows:

Definition 2.8. *Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a function. A mapping $T : X \to X$ is called an $(\mathcal{F}^*, \varphi)$-expansion if there exist $\mathcal{F}^* \in \mathbb{F}_{\mathbb{H}}$ and $\tau > 0$ such that (for all $u, v \in X$)*

$$Tu \neq Tv \text{ implies } \mathcal{F}^*(d(Tu, Tv), \varphi(Tu), \varphi(Tv)) \geq \mathcal{F}^*(d(u, v), \varphi(u), \varphi(v)) + \tau.$$

Theorem 2.9. *Let (X, d) be a complete metric space and $\varphi : X \to [0, \infty)$ be a lower semi-continuous function. Assume that $T : X \to X$ is a surjective mapping and satisfies the following conditions:*

(a) T is an $(\mathcal{F}^*, \varphi)$-expansion mapping, where $\mathcal{F}^* \in \mathbb{F}_{\mathbb{H}}$,

(b) $F_{T^\star} \subseteq Z_\varphi$, where $T^\star : X \to X$ such that $T \circ T^\star = I_X$ the identity mapping on X.

Then T has a unique φ-fixed point.

Proof. Since the mapping T is surjective then, there exists a mapping $T^\star : X \to X$ such that $T \circ T^\star$ is the identity mapping on X. Let $u, v \in X$ be arbitrary points such that $d(u, v) > 0$, and let $z = T^\star u$ and $w = T^\star v$. Observe that $z \neq w$, otherwise $Tz = Tw$ which implies that $d(u, v) = 0$, a contradiction. As T is an $(\mathcal{F}^*, \varphi)$-expansion mapping, we get

$$\mathcal{F}^*(d(Tz, Tw), \varphi(Tz), \varphi(Tw)) \geq \mathcal{F}^*(d(z, w), \varphi(z), \varphi(w)) + \tau.$$

Since $Tz = u$ and $Tw = v$, the above inequality can be written as follows

$$\tau + \mathcal{F}^*(d(T^\star u, T^\star v), \varphi(T^\star u), \varphi(T^\star v)) \leq \mathcal{F}^*(d(u,v), \varphi(u), \varphi(v)),$$

which implies that T^* is an $(\mathcal{F}^*, \varphi)$-contraction. Then by Theorem 2.5, a φ-fixed point of the mapping $T^\star$ exists and unique (say u_1) that is $T^\star u_1 = u_1$ and $\varphi(u_1) = 0$. Since $T(u_1) = T(T^\star u_1) = u_1$ and $\varphi(u_1) = 0$, then u_1 is also a φ-fixed point of T.

Now, to show that the mapping T has a unique φ-fixed point, suppose u_1, u_2 are two φ-fixed point of T such that $u_1 \neq u_2$. As T is an $(\mathcal{F}^*, \varphi)$-expansion mapping, we obtain

$$\begin{aligned} \mathcal{F}^*(d(u_1,u_2), \varphi(u_1), \varphi(u_2)) &= \mathcal{F}^*(d(Tu_1, Tu_2), \varphi(Tu_1), \varphi(Tu_2)) \\ &\geq \mathcal{F}^*(d(u_1,u_2), \varphi(u_1), \varphi(u_2)) + \tau, \end{aligned}$$

a contradiction. This completes the proof. $\square$

Remark 2.10. *Setting $\mathcal{F}^*(\alpha, \beta, \gamma) = \ln(\alpha + \beta + \gamma)$, for all $\alpha, \beta, \gamma \in [0, \infty)$, $\alpha \neq 0$ and $\varphi(u) = 0$, for all $u \in X$ in Theorem 2.9, we get the well known result due to Wang et al. [16].*

Remark 2.11. *Theorem 2.9 does not hold if T is not surjective. To substantiate this consider $X = [0, \infty)$ with the usual metric. Let $T : X \to X$ be a mapping given by*

$$Tu = 2u + 1, \text{ for all } u \in X.$$

Define two functions $\varphi : X \to [0, \infty)$ and $\mathcal{F}^* : (0, \infty) \times [0, \infty)^2 \to \mathbb{R}$ by

$$\varphi(u) = u, \text{ for all } u \in X$$

and

$$\mathcal{F}^*(\alpha, \beta, \gamma) = \ln(\alpha + \beta + \gamma), \text{ for all } \alpha, \beta, \gamma \in [0, \infty) \text{ and } \alpha \neq 0.$$

Then T satisfies the condition $d(Tu, Tv) + \varphi(Tu) + \varphi(Tv) \geq 2[d(u,v) + \varphi(u) + \varphi(v)]$, for all $u, v \in X$ and T is fixed point free.

3. Results in partial metric spaces

In this section, as application of our main result, we derive results in partial metric spaces.

Let $\mathbb{G}$ be the class of continuous functions $G : (0, \infty) \to \mathbb{R}$ which satisfy the condition:

(G) for any positive real numbers sequence $\{\alpha_n\}$,

$$\lim_{n\to\infty} G(\alpha_n) = -\infty \text{ if and only if } \lim_{n\to\infty} \alpha_n = 0.$$

Example 3.1. *Let $G_1, G_2 : (0, \infty) \to \mathbb{R}$ be a functions defined by*

1. $G_1(\alpha) = \ln \alpha$;
2. $G_2(\alpha) = -1/\alpha$.

Then $G_1, G_2 \in \mathbb{G}$.

Now, we use Theorem 2.5 to deduce a fixed point theorem of Piri and Kumam [7] type in partial metric spaces.

Theorem 3.2. *Let (X, p) be a complete partial metric space and $T : X \to X$. Assume that there exist $G \in \mathbb{G}$ and $\tau > 0$ such that*

$$Tu \neq Tv \text{ implies } \tau + G(p(Tu, Tv)) \leq G(p(u,v)), \text{ for all } u, v \in X. \tag{3.1}$$

Then T has unique fixed point $w \in X$, with $p(w, w) = 0$.

Proof. Consider the metric d_p on X as given in Remark 1.5. Obvious (X, d_p) forms a complete metric space (due to Lemma 1.6). Define $\varphi : X \to [0, \infty)$ as

$$\varphi(u) = p(u, u), \text{ for all } u \in X,$$

then φ is lower semi-continuous (in view of Lemma 1.9). Now, using (3.1) and Remark 1.5, we obtain (for all $u, v \in X, Tu \neq Tv$)

$$\tau + G(\frac{1}{2}(d_p(Tu, Tv) + p(Tu, Tu) + p(Tv, Tv))) \leq G(\frac{1}{2}(d_p(u, v) + p(u, u) + p(v, v)))$$

or

$$\tau + G(\frac{1}{2}(d_p(Tu, Tv) + \varphi(Tu) + \varphi(Tv))) \leq G(\frac{1}{2}(d_p(u, v) + \varphi(u) + \varphi(v))). \tag{3.2}$$

Define $\mathcal{F}^* : (0, \infty) \times [0, \infty)^2 \to \mathbb{R}$ by $\mathcal{F}^*(\alpha, \beta, \gamma) = G(\frac{1}{2}(\alpha + \beta + \gamma))$, for all $\alpha, \beta, \gamma \in [0, \infty)$ *and* $\alpha \neq 0$. Observe that the continuity of G implies the continuity of $\mathcal{F}^*$. Also, if $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, \infty)$, then

$$\lim_{n\to\infty} \mathcal{F}^*(\alpha_n, \beta_n, \gamma_n) = \lim_{n\to\infty} G(\frac{1}{2}(\alpha_n + \beta_n + \gamma_n)) = -\infty \text{ if and only if } \lim_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) = 0.$$

Therefore, $\mathcal{F}^* \in \mathbb{F}_{\mathbb{H}}$. Using the definition of $\mathcal{F}^*$ and (3.2), we obtain

$$Tu \neq Tv \text{ implies } \tau + \mathcal{F}^*(d_p(Tu, Tv), \varphi(Tu), \varphi(Tv)) \leq \mathcal{F}^*(d_p(u, v), \varphi(u), \varphi(v)).$$

for all $u, v \in X$. Therefore, the conditions of Theorem 2.5 are fulfilled. Hence T has a unique φ-fixed point say w. This implies that w is a unique fixed point of T such that $p(w, w) = 0$. Which completes the proof. □

Setting $G(\alpha) = \ln \alpha$, for all $\alpha \in (0, \infty)$ in Theorem 3.2, we deduce Matthews's Theorem [23] as under:

Corollary 3.3. *[23] Let (X, p) be a complete partial metric space and T be a self mapping. Assume that there exists $k \in (0, 1)$ such that*

$$\forall\ u, v \in X,\ Tu \neq Tv \ \textit{ implies } \ p(Tu, Tv) \leq kp(u, v). \tag{3.3}$$

Then the mapping T has a unique fixed point $w \in X$, with $p(w, w) = 0$.

Setting $G(\alpha) = \frac{-1}{\alpha}$, $\forall\ \alpha \in (0, \infty)$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.4. *Let (X, p) be a complete partial metric space and T be a self mapping. Assume that there exists $k \in (0, 1)$ such that*

$$\forall\ u, v \in X.\ Tu \neq Tv \quad \textit{implies} \ \ \tau + \frac{1}{p(u, v)} \leq \frac{1}{p(Tu, Tv)}.$$

Then the mapping T has a unique fixed point $w \in X$, with $p(w, w) = 0$.

Similarly, from Theorem 2.9, we deduce the following result in partial metric spaces.

Theorem 3.5. *Let (X, p) be a complete partial metric space and $T : X \to X$ be a surjective mapping. Suppose that there exist $G \in \mathbb{G}$ and $\tau > 0$ such that*

$$\forall\ u, v \in X,\ u \neq v \quad \textit{implies} \ \ G(p(Tu, Tv)) \geq G(p(u, v)) + \tau$$

Then T has unique fixed point $w \in X$. Moreover, $p(w, w) = 0$.

Proof. The proof follows on the similar lines of proof of Theorem 3.2. □

Taking $G(\alpha) = \ln \alpha$, for all $\alpha \in (0, \infty)$ in Theorem 3.5, we deduce the following well known result due to Wang et al. [16].

Corollary 3.6. *[16] Let (X,p) be a complete partial metric space and T be a surjective self mapping. Suppose that there exists $\lambda > 1$ such that*

$$\forall\ u,v \in X,\ \ u \neq v \ \ \text{implies} \ \ p(Tu,Tv) \geq \lambda p(u,v).$$

Then T has a unique fixed point $w \in X$, with $p(w,w)=0$.

Setting $G(\alpha)=\frac{-1}{\alpha}$, for all $\alpha \in (0,\infty)$ in Theorem 3.5, we obtain The following corollary.

Corollary 3.7. *Let (X,p) be a complete partial metric space and T be a surjective self mapping. Suppose that there exists $\tau > 0$ such that*

$$\forall\ u,v \in X,\ \ u \neq v \ \ \text{implies} \ \ \frac{1}{p(u,v)} \geq \frac{1}{p(Tu,Tv)} + \tau.$$

Then the mapping T has a unique fixed point $w \in X$, with $p(w,w)=0$.

4. An application to integral equations

In this section, we apply theorem 2.5 to show that the solution of the following nonlinear integral equation is exist and unique:

$$\eta(t) = \psi(t) + \int_0^t M(t,s,\eta(s))ds, \quad t \in [0,L], \tag{4.1}$$

where $L>0,\ M:[0,L]\times[0,L]\times\mathbb{R}\to\mathbb{R},\ \psi:[0,L]\to\mathbb{R}$. Let $X=C([0,L],\mathbb{R})$ be the set of all continuous functions, $\eta:[0,L]\to\mathbb{R}$ together with a Bielecki's norm

$$\parallel \eta \parallel = \sup_{t\in[0,L]} e^{-t}\left|\eta(t)\right|.$$

Theorem 4.1. *Let M be a continuous function such that*

$$|M(t,s,\eta(s)) - M(t,s,\mu(s))| \leq \frac{|\eta(s)-\mu(s)|}{\tau \parallel \eta(s)-\mu(s) \parallel +1},$$

for all $\eta,\mu \in C([0,L],\mathbb{R})$, $s,t\in[0,L]$ and for some $\tau>0$. Then the integral equation (4.1) has a unique solution.

Proof. Let $T:X\to X$ be a mapping defined as

$$T(\eta(t)) = \psi(t) + \int_0^t M(t,s,\eta(s))ds, \quad \forall\ \eta \in X,$$

and d is a metric on X defined by

$$d(\eta,\mu) = \sup_{t\in[0,L]} e^{-t}\left|\eta(t)-\mu(t)\right|.$$

Hence (X,d) forms a complete metric space.
Now, define two functions $\mathcal{F}^*:(0,\infty)\times[0,\infty)^2\to\mathbb{R}$ and $\varphi: X\to[0,\infty)$ such that $\mathcal{F}^*(\alpha,\beta,\gamma)=\frac{-1}{\alpha+\beta+\gamma},\ \forall\ \alpha,\beta,\gamma\in[0,\infty), \alpha\neq 0$ and $\varphi(\eta)=0,\ \forall\ \eta\in X$. Then T is an $(\mathcal{F}^*,\varphi)$-contraction. Indeed, for $\eta,\mu\in X$ with $T\eta\neq T\mu$, we have (for all $t\in[0,L]$)

$$\begin{aligned}
|T(\eta(t)) - T(\mu(t))| &= \left|\int_0^t M(t,s,\eta(s))ds - \int_0^t M(t,s,\mu(s))ds\right| \\
&\leq \int_0^t \left|(M(t,s,\eta(s)) - M(t,s,\mu(s)))\right| ds \\
&\leq \int_0^t \frac{1}{\tau \parallel \eta(s)-\mu(s) \parallel +1}(\left|\eta(s)-\mu(s)\right| e^{-s})e^{s}ds \\
&\leq \frac{d(\eta,\mu)\ e^{t}}{\tau d(\eta,\mu)+1},
\end{aligned}$$

so that

$$\sup_{t\in[0,L]} |(T\eta)(t) - (T\mu)(t)|\, e^{-t} \leq \frac{d(\eta,\mu)}{\tau d(\eta,\mu)+1}, \quad \text{for all } t \in [0,L],$$

or

$$\tau + \frac{1}{d(\eta,\mu)} \leq \frac{1}{d(T\eta,T\mu)},$$

or

$$\tau + \frac{1}{d(\eta,\mu)+\varphi(\eta)+\varphi(\mu)} \leq \frac{1}{d(T\eta,T\mu)+\varphi(T\eta)+\varphi(T\mu)},$$

or

$$\tau + \frac{-1}{d(T\eta,T\mu)+\varphi(T\eta)+\varphi(T\mu)} \leq \frac{-1}{d(\eta,\mu)+\varphi(\eta)+\varphi(\mu)},$$

or

$$\tau + \mathcal{F}^*(d(T\eta,T\mu),\varphi(T\eta),\varphi(T\mu)) \leq \mathcal{F}^*(d(\eta,\mu),\varphi(\eta),\varphi(\mu)),$$

for all $\eta,\mu \in X$ with $T\eta \neq T\mu$, which shows that T is an $(\mathcal{F}^*,\varphi)$-contraction. Therefore the requirement of of Theorem 2.5 are fulfilled. Hence, the integral equation (4.1) has a unique solution. □

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