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On Smallest (generalized) Ideals and Semilattices of (2,2)-regular Non-associative Ordered Semigroups

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abstract: An ordered $\mathcal{AG}$-groupoid can be referred to as a non-associative ordered semigroup, as the main difference between an ordered semigroup and an ordered $\mathcal{AG}$-groupoid is the switching of an associative law. In this paper, we define the smallest left (right) ideals in an ordered $\mathcal{AG}$-groupoid and use them to characterize a $(2,2)$-regular class of a unitary ordered $\mathcal{AG}$-groupoid along with its semilattices and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals. We also give the concept of an ordered $\mathcal{AG}^{***}$-groupoid and investigate its structural properties by using the generated ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals. These concepts will verify the existing characterizations and will help in achieving more generalized results in future works.

Key Words: Ordered $\mathcal{AG}$-groupoid, Non-associativity, ordered $\mathcal{AG}^{***}$-groupoid, Left invertive law, Smallest ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy-ideals.

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1. Introduction

An $\mathcal{AG}$-groupoid is a non-associative and a non-commutative algebraic structure lying in a grey area between a groupoid and a commutative semigroup. Commutative law is given by $abc = cba$ in ternary operations. By putting brackets on the left of this equation, i.e. $(ab)c = (cb)a$, in 1972, M. A. Kazim and M. Naseeruddin introduced a new algebraic structure called a left almost semigroup abbreviated as an $\mathcal{LA}$-semigroup [6]. This identity is called the left invertive law. P. V. Protic and N. Stevanovic called the same structure an Abel-Grassmann's groupoid abbreviated as an AG-groupoid [11].

This structure is closely related to a commutative semigroup because a commutative $\mathcal{AG}$-groupoid is a semigroup [9]. It was proved in [6] that an $\mathcal{AG}$-groupoid S is medial, that is, $ab \cdot cd = ac \cdot bd$ holds for all $a,b,c,d \in S$. An $\mathcal{AG}$-groupoid may or may not contain a left identity. The left identity of an $\mathcal{AG}$-groupoid permits the inverses of elements in the structure. If an $\mathcal{AG}$-groupoid contains a left identity, then this left identity is unique [9]. In an $\mathcal{AG}$-groupoid S with left identity (unitary $\mathcal{AG}$-groupoid), the paramedial law $ab \cdot cd = dc \cdot ba$ holds for all $a,b,c,d \in S$. By using medial law with left identity, we get $a \cdot bc = b \cdot ac$ for all $a,b,c \in S$. We should genuinely acknowledge that much of the ground work has been done by M. A. Kazim, M. Naseeruddin, Q. Mushtaq, M. S. Kamran, P. V. Protic, N. Stevanovic, M. Khan, W. A. Dudek and R. S. Gigon. One can be referred to [3,4,7,9,10,11,14] in this regard.

An $\mathcal{AG}$-groupoid $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with an $\mathcal{AG}$-groupoid operation, meaning that for $x, y, z \in S$, $x \leq y \Rightarrow zx \leq zy$ and $xz \leq yz$, is called an ordered $\mathcal{AG}$-groupoid [17].

Let us define a binary operation "$\circ_e$" (e-sandwich operation) on an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ with left identity e as follows:

$$a \circ_e b = ae \cdot b, \ \forall \ a, b \in S.$$

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Then $(\mathcal{S}, \circ_e, \leq)$ *becomes an ordered semigroup* [17].

Note that an ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup because if an ordered $\mathcal{AG}$-groupoid has a right identity then it becomes an ordered semigroup.

2. Preliminaries

The concept of fuzzy sets was first proposed by Zadeh [19] in 1965, which has a wide range of applications in various fields such as computer engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. It gives us a tool to model the uncertainty present in a phenomena that does not have sharp boundaries. Many papers on fuzzy sets have been published, showing the importance and their applications to set theory, algebra, real analysis, measure theory and topology etc.

Murali [8] defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. In [12], the idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined. A new type of fuzzy subgroup, that is (α, β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1] by using the notions of "belongingness and quasi-coincidence" of fuzzy points and fuzzy sets. The concepts of an $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroups [13]. It is now natural to investigate similar type of generalizations of existing fuzzy sub-systems of other algebraic structures. The concept of an $(\in, \in \vee q)$-fuzzy sub-near rings of a near ring introduced by Davvaz in [2]. In [5] Kazanchi and Yamak studied $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. In [15] Shabir et. al. characterized regular semigroups by the properties of $(\in, \in \vee q)$-fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals. In [5] Kazanchi and Yamak defined $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals in semigroups. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra. Generalizing the concept of $x_t q f$, Shabir and Jun [16], defined $x_t q_k f$ as $f(x) + t + k > 1$, where $k \in [0, 1)$. In [16], semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals. In the present paper, we introduce and investigate the notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, and study the relationship between these ideals in detail. As an application of our results we get characterizations of a $(2, 2)$-regular class of a unitary ordered $\mathcal{AG}$-groupoid (an ordered $\mathcal{AG}^{***}$-groupoid) in terms of its semilattices, one-sided (two-sided) ideals based on fuzzy sets and its associated fuzzy points.

Let $\emptyset \neq A \subseteq S$, we denote $(A]$ by $(A] := \{x \in S / x \leq a \text{ for some } a \in A\}$. If $A = \{a\}$, then we write $(\{a\}]$. For $\emptyset \neq A, B \subseteq S$, we denote $AB =: \{ab / a \in A, b \in B\}$.

• A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if:

(i) $SA \subseteq A$ $(AS \subseteq A)$;

(ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Equivalently: A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if $(SA] \subseteq A$ $((AS] \subseteq A)$.

• By two-sided ideal or simply ideal, we mean a nonempty subset of an ordered $\mathcal{AG}$-groupoid S which is both left and right ideal of S.

Lemma 2.1. *[17] Let S be an ordered $\mathcal{AG}$-groupoid and $\emptyset \neq A, B \subseteq S$. Then the followings hold*:

(i) $A \subseteq (A]$;

(ii) *If* $A \subseteq B$, *then* $(A] \subseteq (B]$;

(iii) $(A]\,(B] \subseteq (AB]$;

(iv) $(A] = ((A]]$;

(vi) $((A]\,(B]] = (AB]$;

(vii) $(T] = T$, *for every ideal* T *of* S;

$(viii)$ $(SS] = S = SS$, *if* S *has a left identity.*

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \longrightarrow [0,\ 1]$, where $[0,\ 1]$ is the usual closed interval of real numbers [19]. For any two fuzzy subsets f and g of S, $f \subseteq g$ means that, $f(x) \leq g(x)$, $\forall\ x \in S$.

Let f and g be any fuzzy subsets of an ordered $\mathcal{AG}$-groupoid S, then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee_{a \leq bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b,\, c \in S, \text{ such that } a \leq bc \\ 0, & \text{otherwise.} \end{cases}$$

• Let $\mathcal{F}(S)$ denotes the collection of all fuzzy subsets of an ordered $\mathcal{AG}$-groupoid S, then it is easy to see that $(\mathcal{F}(S), \circ, \subseteq)$ becomes an ordered $\mathcal{AG}$-groupoid.

• The characteristic function $\mathcal{X}_A$ for a non-empty A of an ordered $\mathcal{AG}$-groupoid S is defined by

$$\mathcal{X}_A(x) = \begin{cases} 1, \text{ if } x \in A, \\ 0, \text{ if } x \notin A. \end{cases}$$

• A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid S of the form

$$f(y) = \begin{cases} r(\neq 0), & if\ y \leq x \\ 0, & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r, where $r \in (0,1]$.

• In what follows let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, we define $X^{\delta}_{\gamma B}$ be the fuzzy subset of X by $X^{\delta}_{\gamma B}(x) \geq \delta$ and $X^{\delta}_{\gamma B}(x) \leq \gamma$, $\forall\ x \in B$. Otherwise, clearly $X^{\delta}_{\gamma B}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

• For a fuzzy point x_r and a fuzzy subset f of an ordered $\mathcal{AG}$-groupoid S, we say that:

(i) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.

(ii) $x_r q_\delta f$ if $f(x) + r > 2\delta$.

(iii) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

• Now we introduce a new relation on $\mathcal{F}(S)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.

For any $f, g \in \mathcal{F}(S)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$, we mean that $x_r \in_\gamma f \Longrightarrow x_r \in_\gamma \vee q_\delta g$, $\forall\ x \in S$ and $r \in (\gamma, 1]$. Moreover f and g are said to be (γ,δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

Lemma 2.2. *[18] Let $f, g, h \subseteq \mathcal{F}(S)$ and $\gamma, \delta \in [0,1]$, then*

(i) $f \subseteq \vee q_{(\gamma,\delta)} g$ ($f \supseteq \vee q_{(\gamma,\delta)} g$) $\Leftrightarrow \max\{f(x), \gamma\} \leq \min\{g(x), \delta\}$ ($\max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$), $\forall\ x \in S$.

(ii) If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.

Corollary 2.3. *$= \vee q_{(\gamma,\delta)}$ is an equivalence relation on $\mathcal{F}(S)$.*

• By Lemma 2.2, it is also notified that $f = \vee q_{(\gamma,\delta)} g \Leftrightarrow \max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$, $\forall\ x \in S$, where $\gamma, \delta \in [0,1]$.

Lemma 2.4. *[18] Let A and B be any subsets of an ordered $\mathcal{AG}$-groupoid S, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0,1]$, then:*

(1) $A \subseteq B \Leftrightarrow \mathcal{X}^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \mathcal{X}^{\delta}_{\gamma B}$;

(2) $\mathcal{X}^{\delta}_{\gamma A} \cap \mathcal{X}^{\delta}_{\gamma B} =_{(\gamma,\delta)} \mathcal{X}^{\delta}_{\gamma (A \cap B)}$;

(3) $\mathcal{X}^{\delta}_{\gamma A} \circ \mathcal{X}^{\delta}_{\gamma B} =_{(\gamma,\delta)} \mathcal{X}^{\delta}_{\gamma (AB]}$.

Example 2.5. *Let $S = \{a, b, c\}$ be an ordered $\mathcal{AG}$-groupoid with the following multiplication table and two different orders below:*

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

$$\leq := \{(a,a), (b,b), (c,c), (c,a), (c,b)\} \tag{1}$$

$$\leq := \{(a,a),(b,b),(c,c),(a,c),(a,b)\} \tag{2}$$

• A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if for all $a, b \in S$ and $t \in (\gamma, 1]$, the following conditions hold:

(i) If $a \leq b$ and $b_t \in_\gamma f \Longrightarrow a_t \in_\gamma \vee q_\delta f$.

(ii) If $b_t \in_\gamma f \Longrightarrow (ab)_t \in_\gamma \vee q_\delta f$ $(a_t \in_\gamma f \Longrightarrow (ab)_t \in_\gamma \vee q_\delta f)$.

Let us consider an example 2.5 of an ordered $\mathcal{AG}$-groupoid with order (2). Let $\gamma = 0.4$ and $\delta = 0.5$. Define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.7 \text{ for } x = a \\ 0.8 \text{ for } x = b \\ 0.9 \text{ for } x = c \end{cases}.$$

(1) *Let us consider all the possible cases for $t \in (0.4, 1]$ as follows:*

(i) *When $t \in (0.4, 0.7]$, then $x_t \in_\gamma f$ for all $x \in S$. It is easy to see that $x_t \in_\gamma f$ and $y \leq x \Longrightarrow y_t \in_\gamma f$ for all $x \in S$.*

(ii) *When $t \in (0.7, 0.8]$, then $a_t \bar{\in}_\gamma f$ while $c_t \in_\gamma f$ and $b_t \in_\gamma f$. Now $a \leq c$ and $c_t \in_\gamma f \Longrightarrow f(a) \geq t > \gamma$. Proceeding in the same way as in above example we get $a_t q_\delta f$, and Similar solution for $a \leq b$.*

(iii) *When $t \in (0.8, 0.9]$, then $c_t \in_\gamma f$ while $a_t \bar{\in}_\gamma f$ and $b_t \bar{\in}_\gamma f$.It is easy to verify that $c_t \in_\gamma f$ and $a \leq c \Longrightarrow a_t q_\delta f$.*

(iv) *When $t \in (0.9, 1]$, then $\bar{x}_t \in_\gamma f$ for all $x \in S$. Nothing to show in this case.*

(2) *Again considering all possible cases for $t \in (0.4, 1]$*

(i) *When $t \in (0.4, 0.7]$, then $x_t \in_\gamma f$ for all $x \in S$. It is easy see that $(xy)_t \in_\gamma f$ for all $x \in S$ in this case.*

(ii) *When $t \in (0.7, 0.8]$, then $a_t \bar{\in}_\gamma f$ while $c_t \in_\gamma f$ and $b_t \in_\gamma f$. Now $b_t \in_\gamma f \Longrightarrow (ab)_t q_\delta f$, $(bb)_t q_\delta f$ and $(bc)_t q_\delta f$. Similarly $c_t \in f \Longrightarrow (ac)_t q_\delta f$, $(bc)_t \in_\gamma f$ and $(cc)_t q_\delta f$.*

(iii) *When $t \in (0.8, 0.9]$, then $c_t \in_\gamma f$ while $a_t \bar{\in}_\gamma f$ and $b_t \bar{\in}_\gamma f$. Now $c_t \in f \Longrightarrow (ac)_t q_\delta f$, $(bc)_t \in_\gamma f$ and $(cc)_t q_\delta f$.*

(iv) *When $t \in (0.9, 1]$, then $\bar{x}_t \in_\gamma f$ for all $x \in S$. Again nothing to solve in this case.*

Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.

Theorem 2.6. *[18] A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if for all $a, b \in S$ and $\gamma, \delta \in [0,1]$, the following conditions hold:*

(i) $\max\{f(a), \gamma\} \geq \min\{f(b), \delta\}$ *with $a \leq b$.*

(ii) $\max\{f(ab), \gamma\} \geq \min\{f(b), \delta\}$.

Lemma 2.7. *[18] Let f be a fuzzy subset of an ordered $\mathcal{AG}$-groupoid S and $\gamma, \delta \in [0,1]$, then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if and only if f satisfies the following conditions.*

(i) $x \leq y \Rightarrow \max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$, $\forall\, x, y \in S$.

(ii) $S \circ f \subseteq \vee q_{(\gamma,\delta)} f$ $(f \circ S \subseteq \vee q_{(\gamma,\delta)} f)$.

Lemma 2.8. *[18] Let A be a non-empty set of an ordered $\mathcal{AG}$-groupoid S, then A is a left (right) ideal of $S \Leftrightarrow \mathcal{X}_{\gamma A}^{\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S, where $\gamma, \delta \in [0,1]$.*

Remark 2.9. *If S is an ordered $\mathcal{AG}$-groupoid, then $S \circ S = S$.*

• A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime if for all $a \in S$ and $\gamma, \delta \in [0,1]$, if $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$.

Lemma 2.10. *Let A be any right (left) ideal of an ordered $\mathcal{AG}$-groupoid S. Then A is semiprime if and only if $\mathcal{X}_A$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.*

Proof. It is simple. □

3. On $(2,2)$-regular ordered $\mathcal{AG}$-groupoids via $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy one-sided ideals

By a unitary ordered $\mathcal{AG}$-groupoid, we shall mean an ordered $\mathcal{AG}$-groupoid with left identity unless otherwise satisfied.

3.1. Basic Results

This section contains some examples and basic results which will be essential for up coming section.

Example 3.1. *Let us consider an example 2.5 of an ordered $\mathcal{AG}$-groupoid with order* (2). *Define a fuzzy subset $f : S \to [0,1]$ as follows.*

$$f(x) = \begin{cases} 0.9 \text{ for } x = 1 \\ 0.6 \text{ for } x = 2 \\ 0.7 \text{ for } x = 3 \end{cases}.$$

Then by routine calculation it is easy to observe the following:
(i) f is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy two-sided ideal of S.
(ii) f is not an $(\in, \in \vee q_{0.3})$-fuzzy two-sided ideal of S, because $f(12) < f(2) \wedge \frac{1-0.3}{2}$.

Example 3.2. *Let $S = \{w, x, y, z\}$ be an ordered $\mathcal{AG}$-groupoid define in the following multiplication table and ordered below.*

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	w	w	w
y	w	w	w	x
z	w	w	x	y

$$\leq := \{(w,w),(x,x),(y,y),(z,z),(w,x)\}$$

Define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.75 \text{ for } x = w \\ 0.65 \text{ for } x = x \\ 0.7 \text{ for } x = y \\ 0.5 \text{ for } x = z \end{cases}$$

Then clearly f is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy left ideal of S.
Again define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.9 \text{ for } x = w \\ 0.7 \text{ for } x = x \\ 0.6 \text{ for } x = y \\ 0.5 \text{ for } x = z \end{cases}$$

Then f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.5})$-fuzzy two-sided ideal of S.

Lemma 3.3. *Let R be a right ideal and L be a left ideal of a unitary ordered $\mathcal{AG}$-groupoid S. Then $(RL]$ is a left ideal of S.*

Proof. Let R be a right ideal and L be a left ideal of S. Then by using Lemma 2.1, we get $S(RL] = (SS](RL] \subseteq (SS \cdot RL] = (SR \cdot SL] \subseteq (SR \cdot (SL]] = (SR \cdot L] = ((SS]R \cdot L] \subseteq ((SS)R \cdot L] = ((RS)S \cdot L] \subseteq ((RS]S \cdot L] \subseteq (RL]$,which shows that $(RL]$ is a left ideal of S. $\square$

Lemma 3.4. *Let S be a unitary ordered $\mathcal{AG}$-groupoid. If $a = a^2$ for all $a \in S$, then $R_a = (Sa \cup Sa^2]$ is the smallest right ideal of S containing a.*

Proof. Assume that $a = a^2$ for all $a \in S$. Then by using Lemma 2.1, we have

$$\begin{aligned}(Sa \cup Sa^2]S &= (Sa \cup Sa^2](S] \subseteq ((Sa \cup Sa^2)S] = (Sa \cdot S \cup Sa^2 \cdot S] \\ &= (Sa \cdot SS \cup Sa^2 \cdot SS] = (S \cdot aS \cup S \cdot a^2S] = (a \cdot SS \cup a^2 \cdot SS] \\ &= (a^2 \cdot SS \cup a^2 \cdot SS] = (SS \cdot a^2 \cup SS \cdot a^2] = (Sa \cup Sa^2],\end{aligned}$$

which shows that $(Sa \cup Sa^2]$ is a right ideal of S. It is easy to see that $a \in (Sa \cup Sa^2]$. Let R be another right ideal of S containing a. Since

$$(Sa \cup Sa^2] = (SS \cdot a \cup a \cdot Sa] = (aS \cdot S \cup a \cdot Sa] \subseteq (RS \cdot S \cup RS] \subseteq R,$$

Hence $(Sa \cup Sa^2]$ is the smallest right ideal of S containing a. □

Lemma 3.5. *Let S be a unitary ordered $\mathcal{AG}$-groupoid and $a = a^2$ for all $a \in S$. Then S becomes a commutative monoid.*

Proof. Straightforward. □

Corollary 3.6. $R_a = (Sa \cup Sa^2]$ *is the smallest right ideal of an ordered commutative monoid S containing a.*

Lemma 3.7. *Let S be a unitary ordered $\mathcal{AG}$-groupoid and $a \in S$. Then $L_a = (Sa]$ is the smallest left ideal of S containing a.*

Proof. It is simple. □

• Recall that an ordered $\mathcal{AG}^{**}$-groupoid is an ordered $\mathcal{AG}$-groupoid in which $a \cdot bc = b \cdot ac$, $\forall\, a, b, c \in S$. Note that an ordered $\mathcal{AG}^{**}$-groupoid also satisfies the paramedial law as well.

Now let us introduce the concept of an ordered $\mathcal{AG}^{***}$-groupoid as follows:

• An ordered $\mathcal{AG}^{**}$-groupoid S is called an ordered $\mathcal{AG}^{***}$-*groupoid* if $S = S^2$.

Lemma 3.8. *Let S be an ordered $\mathcal{AG}^{***}$-groupoid and $a \in S$. Then $\langle R \rangle_{a^2} = (Sa^2 \cup a^2]$ $(\langle L \rangle_a = (Sa \cup a])$ is the right (left) ideal of S.*

Proof. Let $a \in S$, then by using Lemma 2.1, we get

$$\begin{aligned}(Sa^2 \cup a^2]S &= (Sa^2 \cup a^2](S] = ((Sa^2 \cup a^2)S] = (Sa^2 \cdot S \cup a^2S] \\ &= (SS \cdot a^2S \cup SS \cdot aa] = (S \cdot a^2S \cup Sa^2] \\ &= (a^2 \cdot SS \cup Sa^2] = (Sa^2] \subseteq (Sa^2 \cup a^2],\end{aligned}$$

which is what we set out to prove. Similarly we can prove that $S(Sa \cup a] \subseteq (Sa \cup a]$. □

Lemma 3.9. *Let S be a unitary ordered $\mathcal{AG}$-groupoid (an ordered $\mathcal{AG}^{***}$-groupoid) and $\emptyset \neq E \subseteq S$. Then the following assertions hold:*

(i) E forms a semilattice, where $E = \{x \in S : x = x^2\}$;
(ii) E is a singleton set, if $a = ax \cdot a$, $\forall\, a, x \in S$.

Proof. It is simple. □

3.2. Characterization Problems

In this section, we generalize the results of an ordered semigroup and get some interesting characterizations which we usually do not find in other algebraic structures.

• An element a of an ordered $\mathcal{AG}$-groupoid S is called a $(2,2)$-regular element of S, if there exists some x in S such that $a \leq a^2x \cdot a^2$, and S is called $(2,2)$-regular ordered $\mathcal{AG}$-groupoid if all elements of S are $(2,2)$-regular.

Let us characterize a $(2,2)$-regular element of an ordered $\mathcal{AG}$-groupoid in the presence of a left identity (an ordered $\mathcal{AG}^{***}$-groupoid) as follows:

Theorem 3.10. *Let S be a unitary ordered $\mathcal{AG}$-groupoid (an ordered $\mathcal{AG}^{***}$-groupoid). An element a of S is $(2,2)$-regular if and only if for all $a \in S$, $a \leq ay \cdot az$ for some $y, z \in S$ ($a \leq at \cdot a$, $at = ta$ for some $t \in S$).*

Proof. Necessity. Let $a \in S$ is $(2,2)$-regular, then $a \leq a^2x \cdot a^2 = a^2 \cdot xa^2 = aa \cdot a(xa) = aa \cdot ay$, where $xa = z \in S$. Thus $a \leq ay \cdot az$ for some $y, z \in S$. It is easy to see that $a \leq a^2x \cdot a^2 = aa \cdot xa^2 = (xa^2 \cdot a)a = ta \cdot a$, where $xa^2 \cdot a = t \in S$. Thus $ta \leq t(ta \cdot a) = ta \cdot ta = (ta \cdot a)t \leq at$, and $a \leq ta \cdot a \leq at \cdot a$.

Sufficiency. Let $a \in S$ such that $a \leq ax \cdot ay$ for some $x, y \in S$, then $a \leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = (a^2 \cdot xy)x \cdot (a^2 \cdot xy)y = (x \cdot xy)a^2 \cdot (a^2 \cdot xy)y = a^2(xy \cdot x) \cdot (a^2 \cdot xy)y = ((a^2 \cdot xy)y \cdot (xy \cdot x))a^2 = ((y \cdot xy)a^2 \cdot (xy \cdot x))a^2 = (a^2(y^2x) \cdot (xy \cdot x))a^2 = ((x \cdot xy) \cdot (y^2x)a^2)a^2 = ((x \cdot xy) \cdot a^2(xy^2))a^2 = (a^2 \cdot (x \cdot xy)(xy^2))a^2$, where $(x \cdot xy)(xy^2) = u \in S$. The remaining part is simple. Hence S is $(2,2)$-regular. □

Now let us characterize a $(2,2)$-regular class of a unitary ordered $\mathcal{AG}$-groupoid (an ordered $\mathcal{AG}^{***}$-groupoid) in terms of its semilattice E as follows:

From now onward, R (*resp.* L) will denote any right (*resp.* left) ideal of an ordered $\mathcal{AG}$-groupoid S; R_a (*resp.* L_a) will denote any smallest right (*resp.* smallest left) ideal of S containing a. Any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right (*resp.* $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left) ideal of an ordered $\mathcal{AG}$-groupoid S will be denoted by f (*resp.* g) unless otherwise specified.

Lemma 3.11. *Let f be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right (left) ideal of a $(2,2)$-regular unitary ordered $\mathcal{AG}$-groupoid (an ordered $\mathcal{AG}^{***}$-groupoid). Then the following assertions hold:*

(i) $f =_{(\gamma,\delta)} f \circ S$ ($f =_{(\gamma,\delta)} S \circ f$);
(ii) f *is* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy semiprime.*

Proof. It is simple. □

Theorem 3.12. *Let f, g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of a unitary ordered $\mathcal{AG}$-groupoid S. Then the following conditions are equivalent:*

(i) S *is* $(2,2)$*-regular;*
(ii) S *is* $(2,2)$*-regular commutative monoid;*
(iii) $(R_aL_a] \cap L_a = ((R_a \cdot R_aL_a)L_a \cdot L_a]$, $(a = a^2, \forall\, a \in S)$;
(iv) $(RL] \cap L = ((R \cdot RL)L \cdot L]$;
(v) $f \cap g =_{(\gamma,\delta)} (f \circ g) \circ f$;
(vi) S *is* $(2,2)$*-regular and* $|E| = 1$, $(a = ax \cdot a, \forall\, a, x \in E)$;
(vii) S *is* $(2,2)$*-regular and* $\emptyset \neq E \subseteq S$ *is semilattice.*

Proof. $(i) \Longrightarrow (vii)$: It can be followed from Lemma 3.9 (i).

$(vii) \Longrightarrow (vi)$: It can be followed from Lemma 3.9 (ii).

$(vi) \Longrightarrow (v)$: Let f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of a $(2,2)$-regular S. Now for $a \in S$, there exist some $x, y \in S$ such that $a \leq ax \cdot ay = ya \cdot xa \leq y(ax \cdot ay) \cdot xa = (ax)(y \cdot ay) \cdot xa =$

$(ay\cdot y)(xa)\cdot xa=(y^2a\cdot xa)(xa)$. Thus $(y^2a\cdot xa,xa)\in A_a$. Therefore

$$\begin{aligned}
\max\{((f\circ g)\circ f)(a),\gamma\} &= \max\left[\bigvee_{(y^2a\cdot xa,xa)\in A_a}\left\{(f\circ g)(y^2a\cdot xa)\wedge f(xa)\right\},\gamma\right]\\
&\geq \max\left[\min\left\{(f\circ g)(y^2a\cdot xa), f(xa)\right\},\gamma\right]\\
&= \min[\max\left\{(f\circ g)(y^2a\cdot xa),\gamma\right\},\max\{f(xa),\gamma\}]\\
&= \min\left[\begin{array}{c}\max\left\{\bigvee_{(y^2a\cdot xa,xa)\in A_a}\left\{f(y^2a\cdot xa)\wedge g(xa),\gamma\right\}\right\},\\ \max\{f(xa),\gamma\}\end{array}\right]\\
&\geq \min\left[\max\left\{f(y^2a\cdot xa)\wedge g(xa),\gamma\right\},\max\{f(xa),\gamma\}\right]\\
&= \min\left[\begin{array}{c}\max\left\{\min\{f(y^2a\cdot xa),g(xa)\},\gamma\right\},\\ \max\{f(xa),\gamma\}\end{array}\right]\\
&= \min\left[\begin{array}{c}\max\left\{\max\{f(y^2a\cdot xa),\gamma\right\},\max\{g(xa),\gamma\}\},\\ \max\{f(xa),\gamma\}\end{array}\right]\\
&\geq \min[\min\{f(a)\wedge g(a),\delta\},\min\{f(xa),\delta\}]\\
&= \min\{(f\cap g)(a),\delta\},
\end{aligned}$$

which shows that $(f\circ g)\circ f\supseteq_{(\gamma,\delta)} f\cap g$. By using Lemmas 2.7 and 3.11, it is easy to show that $(f\circ g)\circ f\subseteq_{(\gamma,\delta)} f\cap g$. Thus $f\cap g=_{(\gamma,\delta)}(f\circ g)\circ f$.

$(v)\Longrightarrow(iv)$: Let R and L be any right and left ideals of S respectively. Then by using Lemmas 2.8 and 3.3, $\mathcal{X}_{(RL]}$ and $\mathcal{X}_L$ are the $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideals of S. Now by using Lemma 2.4, we get $\mathcal{X}_{(RL]\cap L}=\mathcal{X}_{(RL]}\cap\mathcal{X}_L=(\mathcal{X}_{(RL]}\circ\mathcal{X}_L)\circ\mathcal{X}_{(RL]}=\mathcal{X}_{((RL]L\cdot(RL]]}$, which give us $(RL]\cap L=((RL]L\cdot(RL]]$. Now by using Lemma 2.1, we get

$$\begin{aligned}
((RL]L\cdot(RL]] &= ((RL)L\cdot RL]=(L^2R\cdot RL]=(LR\cdot RL^2]=(R(LR\cdot L^2)]\\
&= (R(L^2\cdot RL)]=(R(R\cdot L^2L)]=(R\cdot RL^3]=(R(R\cdot L^2L)]\\
&= (R(L^2\cdot RL)]=((R\cdot RL)L\cdot L].
\end{aligned}$$

$(iv)\Longrightarrow(iii)$: It is simple.

$(iii)\Longrightarrow(ii)$: Since $(Sa\cup Sa^2]$ is the smallest right ideal of S containing a and $(Sa]$ is the smallest left ideal of S containing a, where $a=a^2$, $\forall\ a\in S$. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned}
a &\in ((Sa\cup Sa^2](Sa]]\cap(Sa]=(((Sa\cup Sa^2]\cdot(Sa\cup Sa^2](Sa])(Sa]\cdot(Sa]]\\
&= (((Sa\cup Sa^2)\cdot(Sa\cup Sa^2)(Sa))(Sa)\cdot(Sa)]\subseteq(S(Sa)\cdot(Sa)]\\
&= (S^2a\cdot Sa]=(Sa\cdot Sa]=(aS\cdot aS].
\end{aligned}$$

Hence by using Lemma 3.9, S is $(2,2)$-regular commutative monoid.

$(ii)\Longrightarrow(i)$: It is obvious. □

Theorem 3.13. *Let S be a unitary ordered $\mathcal{AG}$-groupoid. Then the following conditions are equivalent:*

- *(i)* S *is* $(2,2)$*-regular;*
- *(ii)* S *is* $(2,2)$*-regular commutative monoid;*
- *(iii)* $R_a\cap L_a=(R_a(L_aR_a\cdot R_a)]$, $(a=a^2,\ \forall\ a\in S)$;
- *(iv)* $R\cap L=(R(LR\cdot R)]$;
- *(v)* $f\cap g=_{(\gamma,\delta)} f^3\circ g$;
- *(vi)* S *is* $(2,2)$*-regular and* $|E|=1$, $(a=ax\cdot a,\ \forall\ a,x\in E)$;
- *(vii)* S *is* $(2,2)$*-regular and* $\emptyset\neq E\subseteq S$ *is semilattice.*

Proof. $(i) \Longrightarrow (vii)$: It can be followed from Lemma 3.9 (i).

$(vii) \Longrightarrow (vi)$: It can be followed from Lemma 3.9 (ii).

$(vi) \Longrightarrow (v)$: Let f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of a $(2,2)$-regular S respectively. From Lemma 2.7, it is easy to show that $f^3 \circ g \subseteq_{(\gamma,\delta)} f \cap g$. Now for $a \in S$, there exist some x,$y \in S$ such that

$$\begin{aligned} a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = y(ax \cdot ay) \cdot x(ax \cdot ay) \\ &= (ax)(y \cdot ay) \cdot (ax)(x \cdot ay) = (ax)(ay^2) \cdot (ax)(a \cdot xy) \\ &= (y^2a)(xa) \cdot (ax)(a \cdot xy) = ((ax)(a \cdot xy))(xa) \cdot y^2a \\ &= ((ax)(a \cdot xy))(ex \cdot a) \cdot y^2a = ((ax)(a \cdot xy))(ax \cdot e) \cdot y^2a \\ &= bc \cdot y^2a = d \cdot y^2a, \text{ where } d = bc = ((ax)(a \cdot xy))(ax \cdot e). \end{aligned}$$

Thus

$$\begin{aligned} \max\{((f \circ f) \circ f)(d), \gamma\} &= \max\left[\bigvee_{d \leq bc} \{(f \circ f)(b) \wedge f(c)\}, \gamma\right] \\ &\geq \max[\min\{(f \circ f)(b), f(c)\}, \gamma] \\ &= \min[\max\{(f \circ f)(b), \gamma\}, \max\{f(c), \gamma\}] \\ &= \min\left[\begin{array}{c} \max\left\{\bigvee_{b \leq (ax)(a \cdot xy)} \{f(ax) \wedge f(a \cdot xy), \gamma\}\right\} \\ , \max\{f(c), \gamma\} \end{array}\right] \\ &\geq \min[\max\{f(ax) \wedge f(a \cdot xy), \gamma\}, \max\{f(c), \gamma\}] \\ &= \min[\max\{\min\{f(ax), f(a \cdot xy)\}, \gamma\}, \max\{f(c), \gamma\}] \\ &= \min\left[\begin{array}{c} \max\{\max\{f(ax), \gamma\}, \max\{f(a \cdot xy), \gamma\}\} \\ , \max\{f(ax \cdot e), \gamma\} \end{array}\right] \\ &\geq \min[\min\{f(a) \wedge f(a), \delta\}, \min\{f(a), \delta\}] \\ &= \min\{f(a), \delta\}. \end{aligned}$$

Therefore

$$\begin{aligned} \max\{(f^3 \circ g)(a), \gamma\} &= \max\left[\begin{array}{c} \bigvee_{a \leq d \cdot y^2a} \{((f \circ f) \circ f)(((ax)(a \cdot xy))(ax \cdot e)) \\ \wedge g(y^2a)\}, \gamma \end{array}\right] \\ &\geq \min\{(f \cap g)(a), \delta\}, \end{aligned}$$

which shows that $f \cap g \subseteq_{(\gamma,\delta)} f^3 \circ g$. Thus $f \cap g =_{(\gamma,\delta)} f^3 \circ g$.

$(v) \Longrightarrow (iv)$: Let R and L be any right and left ideals of S respectively. Then by using Lemma 2.8, $\mathcal{X}_R$ and $\mathcal{X}_L$ are the $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S respectively. Now by using Lemma 2.4, we get

$$\mathcal{X}_{R \cap L} = \mathcal{X}_R \cap \mathcal{X}_L = ((\mathcal{X}_R \circ \mathcal{X}_R) \circ \mathcal{X}_R) \circ \mathcal{X}_L = \mathcal{X}_{(R^3]} \circ \mathcal{X}_L = \mathcal{X}_{((R^3]L]},$$

which implies that $R \cap L = ((R^3]L]$. Now by using Lemma 2.1, we get $R \cap L = ((R^3]L] = (R^3L] = (R^2R \cdot L] = (LR \cdot R^2] = (R^2 \cdot RL] = (R \cdot R^2L] = (R(LR \cdot R)]$.

$(iv) \Longrightarrow (iii)$: It is simple.

$(iii) \Longrightarrow (ii)$: Since $(Sa \cup Sa^2]$ is the smallest right ideal of S containing a and $(Sa]$ is the smallest left ideal of S containing a. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned} a &\in (Sa \cup Sa^2] \cap (Sa] = ((Sa \cup Sa^2]((Sa](Sa \cup Sa^2] \cdot (Sa \cup Sa^2]))] \\ &= ((Sa \cup Sa^2)((Sa)(Sa \cup Sa^2) \cdot (Sa \cup Sa^2))] \subseteq (S(S(Sa \cup Sa^2) \cdot (Sa \cup Sa^2))] \\ &= (S((S^2a \cup S^2a^2)(Sa \cup Sa^2))] = ((S^2a \cup S^2a^2)(S(Sa \cup Sa^2))] \\ &= ((S^2a \cup S^2a^2)(S^2a \cup S^2a^2)] = ((Sa \cup a^2S^2)(Sa \cup a^2S^2)] \\ &= ((Sa \cup S^2a \cdot a)(Sa \cup S^2a \cdot a)] \subseteq ((Sa \cup Sa)(Sa \cup Sa)] \\ &= (Sa \cdot Sa] = (aS \cdot aS]. \end{aligned}$$

Hence by using Lemma 3.9, S is $(2,2)$-regular commutative monoid.

$(ii) \Longrightarrow (i)$: It is obvious. □

Let S be an ordered $\mathcal{AG}^{***}$-groupoid. From now onward, R (*resp.* L) will denote any right (*resp.* left) ideal of S; $\langle R\rangle_{a^2}$ will denote a right ideal $(Sa^2 \cup a^2]$ of S containing a^2 and $\langle L\rangle_a$ will denote a left ideal $(Sa \cup a]$ of S containing a; f (*resp.* g) will denote any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right (left) ideal of S unless otherwise specified.

Theorem 3.14. *Let S be an ordered $\mathcal{AG}^{***}$-groupoid. Then S is $(2,2)$-regular if and only if $\langle R\rangle_{a^2} \cap \langle L\rangle_a = (\langle R\rangle_{a^2}^2 \langle L\rangle_a^2]$ and $\langle R\rangle_{a^2}$ is semiprime.*

Proof. Necessity: Let S be $(2,2)$-regular. It is easy to see that $(\langle R\rangle_{a^2}^2 \langle L\rangle_a^2] \subseteq \langle R\rangle_{a^2} \cap \langle L\rangle_a$. Let $a \in \langle R\rangle_{a^2} \cap \langle L\rangle_a$. Then there exist some $x, y \in S$ such that

$$\begin{aligned} a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = (x \cdot ay)(ax) \cdot (y \cdot ay)(ax) \\ &= (a \cdot xy)(ax) \cdot (ay^2)(ax) = (a \cdot xy)(ax) \cdot (xa)(y^2a) \\ &\in (\langle R\rangle_{a^2} S \cdot \langle R\rangle_{a^2} S)(S \langle L\rangle_a \cdot S \langle L\rangle_a) \subseteq \langle R\rangle_{a^2}^2 \langle L\rangle_a^2, \end{aligned}$$

which shows that $\langle R\rangle_{a^2} \cap \langle L\rangle_a = (\langle R\rangle_{a^2}^2 \langle L\rangle_a^2]$. It is easy to see that $\langle R\rangle_{a^2}$ is semiprime.

Sufficiency: Since $(Sa^2 \cup a^2]$ and $(Sa \cup a]$ are the right and left ideals of S containing a^2 and a respectively. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned} a &\in (Sa^2 \cup a^2] \cap (Sa \cup a] = ((Sa^2 \cup a^2]^2 (Sa \cup a]^2] \\ &= ((Sa^2 \cup a^2)(Sa^2 \cup a) \cdot (Sa \cup a)(Sa \cup a)] \subseteq (S(Sa^2 \cup a) \cdot S(Sa \cup a)] \\ &= ((S \cdot Sa^2 \cup Sa)(S \cdot Sa \cup Sa)] = ((a^2S \cdot S \cup Sa)(aS \cdot S \cup Sa)] \\ &= ((a^2S \cdot S \cup Sa)(aS \cdot S \cup Sa)] = ((Sa^2 \cup Sa)(Sa \cup Sa)] \\ &= ((a^2S \cup Sa)(Sa \cup Sa)] = ((Sa \cdot a \cup Sa)(Sa \cup Sa)] \subseteq ((Sa \cup Sa)(Sa \cup Sa)] \\ &= (Sa \cdot Sa] = (aS \cdot aS]. \end{aligned}$$

This implies that S is $(2,2)$-regular. □

Corollary 3.15. *Let S be an ordered $\mathcal{AG}^{***}$-groupoid. Then S is $(2,2)$-regular if and only if $\langle R\rangle_{a^2} \cap \langle L\rangle_a = (\langle L\rangle_a^2 \langle R\rangle_{a^2}^2]$ and $\langle R\rangle_{a^2}$ is semiprime.*

Theorem 3.16. *Let S be an ordered $\mathcal{AG}^{***}$-groupoid. Then the following conditions are equivalent:*

(i) S *is* $(2,2)$*-regular;*
(ii) $\langle R\rangle_{a^2} \cap \langle L\rangle_a = (\langle L\rangle_a^2 \langle R\rangle_{a^2}^2]$ *and* $\langle R\rangle_{a^2}$ *is semiprime;*
(iii) $R \cap L = (L^2R^2]$ *and* R *semiprime;*
(iv) $f \cap g =_{(\gamma,\delta)} (f \circ g) \circ (f \circ g)$ *and* f *is* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy semiprime;*
(v) S *is* $(2,2)$*-regular and* $|E| = 1$, $(a = ax \cdot a, \forall\, a, x \in E)$;
(vi) S *is* $(2,2)$*-regular and* $\emptyset \neq E \subseteq S$ *is semilattice.*

Proof. $(i) \Longrightarrow (vi)$: It can be followed from Lemma 3.9 (i).

$(vi) \Longrightarrow (v)$: It can be followed from Lemma 3.9 (ii).

$(v) \Longrightarrow (iv)$: Let f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of a $(2,2)$-regular S respectively. From Lemma 2.7, it is easy to show that $(f \circ g) \circ (f \circ g) \subseteq_{(\gamma,\delta)} f \cap g$. Now for $a \in S$, there exist some $x, y \in S$ such that

$$\begin{aligned} a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = (ax \cdot ay) \cdot ((ax \cdot ay)x)y \\ &= (ax \cdot ay) \cdot (yx)(ax \cdot ay) = (ax \cdot ay) \cdot (ax)(yx \cdot ay) \\ &= (ax \cdot ay) \cdot (ay \cdot yx)(xa) = (ax \cdot ay) \cdot ((yx \cdot y)a)(xa) \\ &= (ax)((yx \cdot y)a) \cdot (ay)(xa) = (ax)(ba) \cdot (ay)(xa), \text{ where } yx \cdot y = b, \end{aligned}$$

which implies that $(ax \cdot ba, ay \cdot xa) \in A_a$. Thus it is to see that $\max\{((f \circ g) \circ (f \circ g))(a), \gamma\} \geq \min\{(f \cap g)(a), \delta\}$, which shows that $(f \circ g) \circ (f \circ g) \supseteq_{(\gamma,\delta)} f \cap g$. Hence $f \cap g =_{(\gamma,\delta)} (f \circ g) \circ (f \circ g)$. Also by using Lemma 3.11, f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

$(iv) \Longrightarrow (iii)$: Let R and L be any left and right ideals of S. Then by using Lemma 2.8, $\mathfrak{X}_R$ and $\mathfrak{X}_L$ are the $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S respectively. Now by using Lemma 2.4, we get $\mathfrak{X}_{R\cap L} = \mathfrak{X}_R \cap \mathfrak{X}_L = (\mathfrak{X}_R \circ \mathfrak{X}_L) \circ (\mathfrak{X}_R \circ \mathfrak{X}_L) = (\mathfrak{X}_R \circ \mathfrak{X}_R) \circ (\mathfrak{X}_L \circ \mathfrak{X}_L) = \mathfrak{X}_{(R^2]} \circ \mathfrak{X}_{(L^2]} = \mathfrak{X}_{(R^2L^2]} = \mathfrak{X}_{(L^2R^2]}$, which implies that $R \cap L = (L^2R^2]$. Also by using Lemma 2.10, R is semiprime.

$(iii) \Longrightarrow (ii)$: It is simple.

$(ii) \Longrightarrow (i)$: It can be followed from Corollary 3.15. □

Theorem 3.17. *Let S be an ordered $\mathcal{AG}^{***}$-groupoid. Then the following conditions are equivalent*:

(i) *S is $(2,2)$-regular*;
(ii) *$\langle R\rangle_{a^2} \cap \langle L\rangle_a = (\langle R\rangle_{a^2} \langle L\rangle_a \cdot \langle R\rangle_{a^2}]$ and $\langle R\rangle_{a^2}$ is semiprime*;
(iii) *$R \cap L = (RL \cdot R]$ and R is semiprime*;
(iv) *$f \cap g =_{(\gamma,\delta)} (f \circ g) \circ f$ and f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime*;
(v) *S is $(2,2)$-regular and $|E| = 1$, $(a = ax \cdot a, \forall\, a, x \in E)$*;
(vi) *S is $(2,2)$-regular and $\emptyset \neq E \subseteq S$ is semilattice.*

Proof. $(i) \Longrightarrow (vi)$: It can be followed from Lemma 3.9 (i).

$(vi) \Longrightarrow (v)$: It can be followed from Lemma 3.9 (ii).

$(v) \Longrightarrow (iv)$: Let f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of a $(2,2)$-regular S over U. Now for $a \in S$, there exist some $x, y \in S$ such that $a \leq ax \cdot ay \leq ax \cdot (ax \cdot ay)y = ((ax \cdot ay)y \cdot x)a = (xy \cdot (ax \cdot ay))a = (ax \cdot (xy \cdot ay))a = (ax \cdot (a \cdot (xy)y))a$.

Thus $(ax \cdot (a \cdot (xy)y), a) \in A_a$. One can easily see that $\max\{((f \circ g) \circ f)(a), \gamma\} \geq \min\{(f \cap g)(a), \delta\}$, which shows that $(f \circ g) \circ f \supseteq_{(\gamma,\delta)} f \cap g$ By using Lemmas 2.7 and 3.11, it is easy to show that $(f \circ g) \circ f \subseteq_{(\gamma,\delta)} f \cap g$. Hence $f \cap g =_{(\gamma,\delta)} (f \circ g) \circ f$. Also by using Lemma 3.11, f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

$(iv) \Longrightarrow (iii)$: Let R and L be any left and right ideals of S. Then by Lemma 2.8, $\mathfrak{X}_R$ and $\mathfrak{X}_L$ are the $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S respectively. Now by using Lemmas 2.4, 3.3 and 2.1, we get $\mathfrak{X}_{R\cap L} = \mathfrak{X}_R \cap \mathfrak{X}_L = (\mathfrak{X}_R \circ \mathfrak{X}_L) \circ \mathfrak{X}_L = \mathfrak{X}_{((RL] \cdot R]} = \mathfrak{X}_{(RL \cdot R]}$, which shows that $R \cap L = (RL \cdot R]$. Also by using Lemma 2.10, R is semiprime.

$(iii) \Longrightarrow (ii)$: It is simple.

$(ii) \Longrightarrow (i)$: Since $(Sa^2 \cup a^2]$ and $(Sa \cup a]$ are the right and left ideals of S containing a^2 and a respectively. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned} a &\in (Sa^2 \cup a^2] \cap (Sa \cup a] = ((Sa^2 \cup a^2](Sa \cup a] \cdot (Sa^2 \cup a^2]] \\ &= ((Sa^2 \cup a^2)(Sa \cup a) \cdot (Sa^2 \cup a^2)] \subseteq (S(Sa \cup a) \cdot (Sa^2 \cup a^2)] \\ &= ((S^2a \cup Sa)(Sa^2 \cup a^2)] = ((S^2a \cdot Sa^2) \cup (S^2a \cdot a^2) \cup (Sa \cdot Sa^2) \cup (S^2a \cdot a^2)] \\ &\subseteq ((Sa \cdot a^2S) \cup (Sa \cdot Sa) \cup (Sa \cdot a^2S) \cup (Sa \cdot Sa)] \\ &\subseteq ((Sa \cdot Sa) \cup (Sa \cdot Sa) \cup (Sa \cdot Sa) \cup (Sa \cdot Sa)] = (Sa \cdot Sa] = (aS \cdot aS]. \end{aligned}$$

Hence S is $(2,2)$-regular. □

4. Conclusions

This paper will give us the extension of the work carried out in [18] in a more generalized way. We have considered the following problems in detail:

i) Compare($\in_\gamma, \in_\gamma \vee q_\delta$)-fuzzy left/right ideals of an ordered $\mathcal{AG}$-groupoid and respective examples are provided.

ii) Introduce the concept of an ordered $\mathcal{AG}^{***}$-groupoid and characterize it by using ($\in_\gamma, \in_\gamma \vee q_\delta$)-fuzzy left/right ideals.

iii) Study the structural properties of a unitary ordered $\mathcal{AG}$-groupoid and ordered $\mathcal{AG}^{***}$-groupoid in terms of its semilattices, $(2,2)$-regular class and generated commutative monoids.

This paper generalized the theory of an $\mathcal{AG}$-groupoid in the following ways:

i) In an $\mathcal{AG}$-groupoid (without order) by using the ($\in_\gamma, \in_\gamma \vee q_\delta$)-fuzzy ideals.

ii) In an $\mathcal{AG}$-groupoid (with and without order) by using fuzzy ideals instead of ($\in_\gamma, \in_\gamma \vee q_\delta$)-fuzzy ideals.

Some important issues for future work are:

i) To develop strategies for obtaining more valuable results in related areas.

ii) To apply these notions and results for studying ($\in_\gamma, \in_\gamma \vee q_\delta$)-fuzzy ideals in $\mathcal{LA}$-semihypergroups and soft $\mathcal{LA}$-semigroups.

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