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# Galerkin Finite Element Method for a Semi-linear Parabolic Equation with Purely Integral Conditions

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abstract: The present paper is devoted to prove the existence and uniqueness of a weak solution of a semi-linear reaction-diffusion equation with only integral terms in the boundaries by using the finite element method and a priory estimate.

Key Words: Non local conditions, Integral condition, Finite element method, A priori estimates, Semi-linear parabolic problem, Galerkin method.

## Contents



## 1. Introduction

In the recent years, a new attention has been given to reaction-diffusion systems which involve an integral over the spatial domain to a function of the desired solution on the boundary conditions; These nonlocal boundary conditions such as the integral condition, are arisen mainly when the data on the boundary can not be measured directly, but their average values are known.

More precisely, standard (Dirichlet, Neumann and Robin type) conditions which are prescribed pointwise are not always adequate as it depends on the physical context which data can be measured at the boundary of the physical domain. In some cases it is not possible to prescribe the solution u (pressure, temperature, . . .) pointwise, because the average value of the solution solely can be measured along the boundary or along some part of it. Nonlinear reaction-diffusion equations, an important class of parabolic equations, have come from a variety of diffusion phenomena which appear widely in nature. These are suggested as mathematical models of physical problems in many fields, such as filtration, phase transition, electromagnetism, acoustics, electrochemistry, cosmology, biochemistry and dynamics of biological groups; see [[2], [1], [3], [4], [5], [6], [7], [12], [13], [14], [15], [16], [18]]. The theory concerning existence, uniqueness and other properties of solutions of the initial and boundary value problems for nonlinear partial differential equations which are extensively studied by many authors, see for example, DiBenedetto [ [19], [20]], Coleman [21], Bouziani [22], Showalter [23] and Oussaeif [18] and references therein.

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Motivated by these works, we study in this paper the existence and uniqueness of a solution for the following semi linear generalized parabolic equaion with only integral conditions

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial u}{\partial x}\right)+b\left(x,t\right)v=f\left(x,t,u\right); \quad (x,t)\in Q; &(1.1)\\
&u\left(x,0\right)=u_0\left(x\right), \quad x\in(0,1);\\
&\int_\Omega u\left(x,t\right)dx=E\left(t\right); \quad t\in[0,T];\\
&\int_\Omega xu\left(x,t\right)dx=M\left(t\right); \quad t\in[0,T]; &(1.2)
\end{aligned}$$

where $Q=\Omega\times I$, $\Omega=(0,1)$, is an open bounded interval of $\mathbb{R}$, $I=(0,T)$ with $0<T<+\infty$ and $a$, $b$, $f$, $u_0$, $E$ and $M$ are known functions.

And assume that

$(A_1)$ $0\le c_0\le a\left(x,t\right)\le c_1$, $\left|\frac{\partial a}{\partial t}\right|\le c_2$, $\left|\frac{\partial a}{\partial x}\right|\le c_3$, $c_4'\le b\left(x,t\right)\le c_4$ $(i=1,2)$ for all $(x,t)\in\overline{Q}$;

$(A_2)$ The compatibility conditions

$$\int_\Omega u_0\left(x\right)dx=E\left(0\right); \tag{1.3}$$

$$\int_\Omega xu_0\left(x\right)dx=M\left(0\right); \tag{1.4}$$

The plan of this paper is as follows. Section 2 is devoted to the solvability of a linear case of the problem (1.1); in section 2.1 we start by giving the statement of the problem. Then in section 2.2 we give some notations used through out the paper; In section 2.3 we give a variationel formulation and we give a priori estimation for the probmlem in section 2.4. The existence of solution of semi-linear problem is proved in 3, in 3.1 we construct an approximate solution by using the finite element method, and in section 3.2 we give a priori estimates for the approximation solution; The existence and the uniqueness of solution is established in section 3.5.

## 2. Study of the linear problem

### 2.1. Statement of the linear problem

In this section we are allowed to give the position of the linear problem and introduce the different function spaces needed to investigate the following nonlocal linear problem

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial u}{\partial x}\right)+b\left(x,t\right)v=g\left(x,t\right); \quad (x,t)\in Q; &(2.1)\\
&u\left(x,0\right)=u_0\left(x\right), \quad x\in(0,1);\\
&\int_\Omega u\left(x,t\right)dx=E\left(t\right); \quad t\in[0,T];\\
&\int_\Omega xu\left(x,t\right)dx=M\left(t\right); \quad t\in[0,T]. &(2.2)
\end{aligned}$$

We start by reducing problem (2.1) with inhomogenous integral conditions to an equivalent problem with homogeneous conditions. In order to achieve this, we introduce a new unknown function $z$ defined by

$z(x,t) = u(x,t) - U(x,t)\,,$ where

$$U(x,t) = \frac{\left((\beta-\alpha)^3 + 12\left(\beta^2-\alpha^2\right)(x-\alpha) - 18(\beta+\alpha)(x-\alpha)^2\right)}{(\beta-\alpha)^4} E(t)$$
$$+ \frac{12\left(3(x-\alpha)^2 - 2(\beta-\alpha)(x-\alpha)\right)}{(\beta-\alpha)^4} M(t)\,.$$

Therefore problem (2.1) becomes

$$\frac{\partial z}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right) + b(x,t)\,z = \mathfrak{L}z = f(x,t)\,; \tag{2.3}$$

$$z(x,0) = z_0(x)\,,\ x \in \Omega; \tag{2.4}$$

$$\int\limits_{\Omega} x^k z(x,t)\,dx = 0;\ \ (k = 0,1)\,; \tag{2.5}$$

with

$$\int_{\Omega} x^k z_0(x)\,dx = 0; \qquad k = 0,1; \tag{2.6}$$

Where

$$f(x,t) = g(x,t) - \mathfrak{L}z$$

where

$$\mathfrak{L}U = \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial U}{\partial x}\right) + b(x,t)\,U$$

and

$$z_0(x) = u_0(x) - U(x,0)\,.$$

### 2.2. Notation

Let $L^2(\Omega)$ be the usual space of square integrable functions ; its scalar product is denoted by $(.,.)$ and its associated norm by $\|.\|$. We denote by $C_0(\Omega)$ the space of continuous functions with compact support in $\Omega$.

**Definition 2.1.** *We denote by $B_2^m(\Omega)$ the Hilbert space defined of $C_0(\Omega)$ for the scalar product*

$$(z,w)_{B_2^m(\Omega)} = \int_{\Omega} \Im_x^m z.\Im_x^m w dx, \tag{2.7}$$

*where*

$$\Im_x^m z = \int_{\Omega} \frac{(x-\xi)^{m-1}}{(m-1)!} z(\xi)\,d\xi;$$

*by the norm of the function $z$ from $B_2^m(\Omega)\,,$ the non negative number*

$$\|z\|_{B_2^m(\Omega)} = \left(\int_{\Omega} \left(\Im_x^m z\right)^2 dx\right)^{\frac{1}{2}} < \infty \tag{2.8}$$

*Then the inequality*

$$\|z\|^2_{B_2^m(\Omega)} \leq \frac{(\beta-\alpha)^2}{2} \|z\|^2_{B_2^{m-1}(\Omega)}\,,\ m \geq 1, \tag{2.9}$$

*holds for every $z \in B_2^{m-1}(\Omega)\,,$ and the embedding*

$$B_2^{m-1}(\Omega) \hookrightarrow B_2^m \tag{2.10}$$

*is continuous .*

If $m = 0$, the space $B_2^0(\Omega)$ coincides with $L^2(\Omega)$.

**Definition 2.2.** *We denote by $L_0^2(\Omega)$ the space consisting of elements $z(x)$ of the space $L^2(\Omega)$ verifyng $\int_\Omega x^k z(x)\,dx = 0$ $(k = 0, 1)$.*

Let X be a space with a norm denoted by $\|.\|_X$

**Definition 2.3.** *(i) Denote by $L^2(I, X)$ the set of all measurable abstract functions $u(., t)$ from $I$ into $X$ such that*

$$\|u\|_{L^2(I,X)} = \left(\int_I \|u(., t)\|_X^2\,dt\right)^{\frac{1}{2}} < \infty \tag{2.11}$$

*(ii)Let $C(\bar{I}; X)$ be the set of all continuous functions $u(., t) : \bar{I} \longrightarrow X$ with*

$$\|u\|_{C(\bar{I};X)} = \max \|u(., t)\|_X < \infty$$

**Lemma 2.4.** *Let be $v : [0, T] \to H$ be a Bochner integrable function and let $A \subset [0, T]$, any measurable subset . So:*

*i) the function $\|v(.)\|_H : [0, T] \to H$ is Lebesgue integrable and we have*

$$\left\|\int_A v(t)\,dt\right\|_H \le \int_A \|v(t)\|_H\,dt; \tag{2.12}$$

*ii) For each $\varphi \in H$, the function $(v(.), \varphi)_H : [0, T] \to \mathbb{R}$ is Lebesgue integrable and we have*

$$\left(\int_A v(t)\,dt, \varphi\right)_H = \int_A (v(t), \varphi)_H\,dt; \tag{2.13}$$

*iii) For all $t_0 \in [0, T]$, the function $t \longmapsto u(t) = \int_{t_0}^t v(s)\,ds$ is continuous on $[0, T]$, and even differentiable for a.e. $t \in [0, T]$ with $\dfrac{du}{dt}(t) = v(t)$.*

**Lemma 2.5.** *Let $M$ be a linear closed subspace from a hilbert space $H$ . So for every $h \in H$, there exists a unique $u \in M$ such that:*

$$\|h - u\|_H = \min_{v \in M} \|h - v\|_H. \tag{2.14}$$

*The element $u$ is called the orthogonal projection of $h$ on $M$ relatively to the inner product $(., .)$ and we note $u = P_M h$ . Furthermore, we have the following pythagor relation*

$$\|h\|_H^2 = \|P_M h\|_H^2 + \|h - P_M h\|_H^2. \tag{2.15}$$

**Lemma 2.6** (Gronwall lemma)**.** *Let be $f_1(t)$, $f_2(t)$ two integrables functions on $[0, T]$, and $f_2(t)$ is nondecreasing, so from the inequality*

$$f_1(\tau) \le f_2(\tau) + c\int_0^\tau f_1(t)\,dt,\ \forall \tau \in [0, T],$$

*where c is a real positive constante, we have the following inequality*

$$f_1(t) \le f_2(t) \exp ct,\ \forall t \in [0, T].$$

**Theorem 2.7** (Cauchy Schwarz inequality)**.** *Let be $f$ and $g$ two functions of $L^2(\Omega)$; so*

$$f.g \in L^1(\Omega),$$

*and*

$$\int_\Omega |f.g| \le \|f\|_{L^2} . \|g\|_{L^2}. \tag{2.16}$$

**Theorem 2.8** (The Cauchy inequality)**.** *Let be $a, b \in \mathbb{R}$, and every $\varepsilon > 0$ we have*

$$|ab| \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2.$$

### 2.3. Variationnel formulation

Let be the space

$$V=\left\{v\in L^2\left(0,T,L^2\left(\alpha,\beta\right)\right):\int_\alpha^\beta x^k v\left(x\right)dx=0\ ,\ k=0,1\right\}.$$

Considering the scalar product in $B_2^1\left(Q\right)$ of (2.3) and $v\in V$

$$\left(\frac{\partial z}{\partial t},v\right)_{B_2^1(Q)}-\left(\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial z}{\partial x}\right),v\right)_{B_2^1(Q)}+\left(b\left(x,t\right)z,v\right)_{B_2^1(Q)}=\left(f\left(x,t\right),v\right)_{B_2^1(Q)}. \tag{2.17}$$

From integration by parts of the second term

$$\begin{aligned}\left(\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial z}{\partial x}\right),v\right)_{B_2^1(Q)} &= -\left(\left(a\frac{\partial z\left(.,t\right)}{\partial \xi}\right)\frac{\partial}{\partial \xi}\left(a\frac{\partial z\left(.,t\right)}{\partial \xi}\right),\Im_x v\left(.,t\right)\right)\\ &= \Im_x\frac{\partial}{\partial \xi}\left(a\frac{\partial z}{\partial \xi}\right).\Im_x^2 v\Big|_\alpha^\beta+\left(\frac{\partial}{\partial x}\left(a\frac{\partial z\left(.,t\right)}{\partial x}\right),\Im_x^2 v\left(.,t\right)\right)\\ &\quad a\frac{\partial z}{\partial x}.\Im_x^2 v\Big|_\alpha^\beta-\left(a\frac{\partial z\left(.,t\right)}{\partial x},\Im_x v\left(.,t\right)\right)\\ &= -az.\Im_x v|_\beta^\alpha+\left(az\left(.,t\right),v\left(.,t\right)\right)+\left(\frac{\partial a}{\partial x}z\left(.,t\right),\Im_x v\left(.,t\right)\right)\end{aligned}$$

from wich we obtain

$$-\left(\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial z}{\sigma x}\right),v\right)_{B_2^1(Q)}=\left(az,v\right)+\left(\frac{\partial a}{\partial x}z,\Im_x v\right)$$

So, we get

$$\left(\frac{\partial z}{\partial t},v\right)_{B_2^1(\Omega)}+\left(az,v\right)+\left(\frac{\partial a}{\partial x}z,\Im_x v\right)+\left(b\left(x,t\right)z,v\right)_{B_2^1(\Omega)}=\left(f\left(x,t\right),v\right)_{B_2^1(\Omega)} \tag{2.18}$$

### 2.4. A priori estimate

Considering the scalar product in $B_2^1\left(Q\right)$ of (2.3) and $\frac{\partial z(.,t)}{\partial t}$, and integrating the result over $Q^\tau$ with $0\leq\tau\leq T$, we obtain

$$\begin{aligned}&\textstyle\int_0^\tau\left\|\frac{\partial z\left(.,t\right)}{\partial t}\right\|_{B_2^1(\Omega)}^2dt-\int_\tau^0\left(\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial z\left(.,t\right)}{\partial x}\right),\frac{\partial z\left(.,t\right)}{\partial t}\right)_{B_2^1(\Omega)}dt\\ &\textstyle+\int_0^\tau\left(bz\left(.,t\right),\frac{\partial z\left(.,t\right)}{\partial t}\right)_{B_2^1(\Omega)}dt\\ &\textstyle=\int_0^\tau\left(f\left(.,t\right),\frac{\partial z\left(.,t\right)}{\partial t}\right)_{B_2^1(\Omega)}dt\end{aligned}$$

From integration by parts, we know that

$$\begin{aligned}&\textstyle-\int_\tau^0\left(\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial z\left(.,t\right)}{\partial x}\right),\frac{\partial z\left(.,t\right)}{\partial t}\right)_{B_2^1(\Omega)}dt\\ &\textstyle=\frac{1}{2}\int_\Omega a\left(x,\tau\right)\left(z\left(x,\tau\right)\right)^2dx-\frac{1}{2}\int_\Omega a\left(x,0\right)\left(z_0\left(x\right)\right)^2dx\\ &\textstyle-\frac{1}{2}\int_0^\tau\int_\Omega\frac{\partial a\left(.x,t\right)}{\partial t}z^2dxdt+\int_0^\tau\left(\frac{\partial a}{\partial x}z\left(.,t\right),\Im_x\frac{\partial z(.,t)}{\partial t}\right)dt.\end{aligned}$$

So, we get

$$\begin{aligned}
&2\int_0^\tau \left\| \frac{\partial z(.,t)}{\partial t} \right\|^2_{B_2^1(\Omega)} dt + \int_\Omega a(x,\tau)(z(x,\tau))^2 dx = \\
&2\int_0^\tau \left( f(.,t), \frac{\partial z(.,t)}{\partial t} \right)_{B_2^1(\Omega)} dt + \int_\Omega a(x,0)(z_0(x))^2 dx \\
&+ \int_0^\tau \int_\Omega \frac{\partial a(x,t)}{\partial t} z^2 dx dt - 2\int_0^\tau \left( \frac{\partial a}{\partial x} z(.,t), \Im_x \frac{\partial z(.,t)}{\partial t} \right) dt \\
&-2\int_0^\tau \left( bz(.,t), \frac{\partial z(.,t)}{\partial t} \right)_{B_2^1(\Omega)} dt.
\end{aligned} \tag{2.19}$$

By virtue of $(A_1)$ and applying the Cauchy inequality to the first and the last two terms on the right hand side of (2.19) we obtain :

$$\begin{aligned}
&\int_0^\tau \left\| \frac{\partial z(.,t)}{\partial t} \right\|^2_{B_2^1(\Omega)} dt + \| z(.,t) \|^2 \\
&\le c_5 \left( \int_0^\tau \| f(.,t) \|^2 dt + \| z_0 \|^2 \right) + c_6 \int_0^\tau \| z(.,t) \|^2 dt,
\end{aligned} \tag{2.20}$$

Where

$$c_5 = \frac{\max\left( (\beta-\alpha)^2, c_1 \right)}{\min\left(\frac{1}{2}, c_0\right)},$$

and

$$c_6 = \frac{\max\left( c_2, 2c_3^2, (\beta-\alpha)^2 c_4^2 \right)}{\min\left(\frac{1}{2}, c_0\right)}.$$

By using the lemma Gronwall, it yields ;

$$\int_0^\tau \left\| \frac{\partial z(.,t)}{\partial t} \right\|^2_{B_2^1(\Omega)} dt + \| z(.,t) \|^2 \le c_5 \exp(c_6 T) \left( \int_0^T \| f(.,t) \|^2 dt + \| z_0 \|^2 \right) \tag{2.21}$$

The right-hand side of (2.21) independent of $\tau$, we replace it by the upper bound with $\tau$ from 0 to $T$ . Thus the inequality (2.21) becomes

$$\int_0^\tau \left\| \frac{\partial z(.,t)}{\partial t} \right\|^2_{B_2^1(\Omega)} dt + \| z(.,t) \|^2 \le C \left( \int_0^T \| f(.,t) \|^2 dt + \| z_0 \|^2 \right) \tag{2.22}$$

With

$$C = \sqrt{c_5 \exp(c_6 T)} \tag{2.23}$$

**Definition 2.9.** *A weak solution of problem* (2.2)-(2.4) *means a function $z$ such that*

$(i)$ $z \in L^2(0,T,V) \cap C\left(0,T,B_2^1(\Omega)\right)$,
$(ii)$ $\frac{\partial z}{\partial t} \in L^2\left(0,T,B_2^1(\Omega)\right)$,
$(iii)$ $z(0) = z^0$,
$(iv)$ *$z$ satisfy the identity*

$$\left( \frac{\partial z}{\partial t}, v \right)_{B_2^1(Q)} + (az, v) + \left( \frac{\partial a}{\partial x} z, \Im_x v \right) + (b(x,t) z, v)_{B_2^1(Q)} = (f(x,t), v)_{B_2^1(Q)}$$

### 2.5. Construction of an approximate solution

Let $\varphi_1, \varphi_2 .... \varphi_N ...$be a hilbertian basis of $V$, such that we divise $[\alpha, \beta]$ on $N+1$ parts $(N \in \mathbb{N}^*)$ and we pose

$$h = \tfrac{1}{N+1} \quad , \quad t_i = ih \quad , \quad i = 0,1,2.....N+1.$$

We define functions $(\varphi_i)$ by

$$\varphi_i(x) = \begin{cases} \dfrac{\mathbf{x}-\mathbf{x}_{i-1}}{\mathbf{x}_i-\mathbf{x}_{i-1}} & \ldots \quad \boldsymbol{x}_{i-1} \le \boldsymbol{x} \le \boldsymbol{x}_i \\ \dfrac{\mathbf{x}-\mathbf{x}_{i+1}}{\mathbf{x}_i-\mathbf{x}_{i+1}} & \ldots \quad \boldsymbol{x}_i \le \boldsymbol{x} \le \boldsymbol{x}_{i+1} \end{cases}$$

For every each functions $(\varphi_i)$ are of degree 1 with $\varphi_i(x_j) = \delta_{ij}$.

Let $(V_n)$ the subspace from $V$ generated by the first $n$ elements of the basis.

We have to find for each $n \in \mathbb{N}^*$, the approximate solution wich has the following form;

$$u_n(x,t) = \sum g_{in}(t)\,\varphi_i(x)\,, (x,t) \in (\alpha,\beta)\times[0,T] \tag{2.24}$$

Where $g_{in} \in H^1(0,T)$ are unknown for the moment.

As we have that $u^0 \in V$ and $V_n$ is a closed subspace from $V$, we can define in a unique way $u_{n_0}$ by

$$u_{n_0} = P_{V_n} u^0 \tag{2.25}$$

where $P_{V_n}$ is define in lemma 2.5. By the virtue of the density of $\cup V_n$ in $V$ it follows that

$$u_{n_0} \longrightarrow u^0 \text{ in } V \quad \text{if } n \longrightarrow \infty. \tag{2.26}$$

We note by $\left(g^0_{in}\right)$ the coordinates of $u^0_n$ in the basis $(\varphi_i)^n_{i=1}$ of $V_n$ that is

$$u^0_n = \sum g^0_{in}\varphi_i \tag{2.27}$$

so, we must find

$$u_n \in H^1(0,T;V_n) \tag{2.28}$$

solution of the differential system

$$\begin{cases} \left(\dfrac{\partial u_n}{\partial t},\varphi_j\right)_{B^1_2(\Omega)} - (au_n,\varphi_j) + \left(\dfrac{\partial a}{\partial x}u_n, \Im_x\varphi_j\right) + \left(b(x,t)\,u_n,\varphi_j\right)_{B^1_2(\Omega)} \\ = \left(f(x,t),\varphi_j\right)_{B^1_2(\Omega)}; \\ u_n(0) = u^0_n; \end{cases} \tag{2.29}$$

By replacing $u_n$ by (2.24) and by using the notations

$$\begin{array}{ll} \alpha_{ij} = \left(\varphi_i,\varphi_j\right)_{B^1_2(\Omega)}, & A = (\alpha_{ij})_{1\le i,j\le n} \\ B_{ij} = \left(a(x,t)\,\varphi_i,\varphi_j\right), & B = (B_{ij})_{1\le i,j\le n} \\ C_{ij} = \left(\frac{\delta a}{\delta x}\varphi_i, \Im_x\varphi_j\right), & C = (C_{ij})\,1\le i,j\le n \quad . \\ D_{ij} = \left(b\varphi_i,\varphi_j\right), & D = (D_{ij})_{1\le i,j\le n} \\ F_j = \left(f(x,t),\varphi_j\right)_{B^1_2(\Omega),} & F = (F_j)_{1\le j\le n} \end{array}$$

The system (2.29) can be written

$$A\frac{\overrightarrow{dg_n}}{dt} + M\overrightarrow{g_n} = \overrightarrow{F}, \tag{2.30}$$

$$\overrightarrow{g_n}(0) = \overrightarrow{g^0_n}$$

such that

$$M = B + C + D$$

We easily prove that $A$ is a regular matrix, so that the system (2.30) has a unique solution $\overrightarrow{g_n} \in \left[H^1(0,T)\right]^n$.

**Lemma 2.10.** *For every $n \ge 1$ problem (2.29) admits a unique solution*

$$u_n \in H^1(0,T,V_n)$$

*wich has the form (2.24).*

### 2.6. A priori estimates for approximations

**Lemma 2.11.** *For every $n \in \mathbb{N}^*$, functions $z_n \in H^1(0,T;V_n)$ solution of (2.29) verify*

$$\|z_n\|_{L^2(0,T;V)} \leq C \tag{2.31}$$

$$\left\|\frac{\partial z_n}{\partial t}\right\|_{L^2\left(0,T;B_2^1(\Omega)\right)} \leq C \tag{2.32}$$

*With $c$ is define in (2.23).*

*Proof.* Multiplying identity (2.29) by $g_{jn}(t)$ and summing $j$ up for $j = 1,..,n$, and integrating the resulting relation over $(0,t)$, and proceeding as same as the a priori estimates we have inequalities (2.31), (2.32). □

### 2.7. Convergence and existence result

By the abvious lemma can derive the following corollary

**Corollary 2.12.** *There exists $z \in L^2(0,T;V)$ and $\frac{\partial z}{\partial t} \in L^2\left(0,T;B_2^1(\Omega)\right)$ and a subsequence $(z_{nk}) \subseteq (z_n)_n$ such that*

$$z_{nk} \rightharpoonup z \text{ in } L^2(0,T;V); \tag{2.33}$$

$$\frac{\partial z_{nk}}{\partial t} \rightharpoonup \frac{\partial z}{\partial t} \text{ in } L^2\left(0,T;B_2^1(\Omega)\right)\ ; \tag{2.34}$$

*when $n \to \infty$.*

*Proof.* Estimations (2.31) and (2.32) express that the sequence $(z_n)_{n\in\mathbb{N}^*}$ and $\left(\frac{\partial z_n}{\partial t}\right)_{n\in\mathbb{N}^*}$ are bounded in $L^2(0,T;V)$ and $L^2\left(0,T;B_2^1(\Omega)\right)$ respectively from where, by using the weak compactness property of the unit balls of $L^2(0,T;V)$ and $L^2\left(0,T;B_2^1(\Omega)\right)$, we conclude the existence of two subsequences $(z_{nk})$ and $\left(\frac{\partial z_{nk}}{\partial t}\right)$ and two functions $z \in L^2(0,T;V)$ and $w \in L^2\left(0,T;B_2^1(\Omega)\right)$ such that

$$z_{nk} \rightharpoonup z \text{ in } L^2(0,T;V); \tag{2.35}$$

and

$$\frac{\partial z_{nk}}{\partial t} \rightharpoonup w \text{ in } L^2\left(0,T;B_2^1(\Omega)\right)\ ; \tag{2.36}$$

It remains to show that $w = \frac{\partial z}{\partial t}$. To this aim, we will prove that

$$u(t) = z^0 + \int_0^t w(s)\,ds,\ t \in [0,T]. \tag{2.37}$$

Since $V \hookrightarrow B_2^1(\Omega)$ we have also

$$z_{nk} \rightharpoonup z \text{ in } L^2\left(0,T;B_2^1(\Omega)\right). \tag{2.38}$$

So that proving (2.37) is equivalent to show that

$$z_{nk} \rightharpoonup z^0 + \chi \ \text{ in }\ L^2\left(0,T;B_2^1(\Omega)\right), \tag{2.39}$$

that is

$$\lim\left(z_{nk} - z^0 - \chi, v\right)_{L^2\left(0,T;B_2^1(\Omega)\right)} = 0,\ \forall v \in L^2\left(0,T;B_2^1(\Omega)\right), \tag{2.40}$$

Where

$$\chi(t) = \int_0^t w(s)\,ds$$

is the Bochner integral in $B_2^1(\Omega)$ of $w$.

Taking firstly $v(t) \equiv \Phi \in B_2^1(\Omega)$, $\forall t \in [0,T]$ ($\Phi$ is independent of $t$), and using the equality

$$z_{nk} - z_{nk}^0 = \int_0^t \frac{\partial z_{nk}(s)}{\partial s} ds,$$

Wich results from the fact that $z_{nk} \in H^1(0,T;V_{nk})$, we have

$$\left(z_{nk}(t) - z^0 - \int_0^t w(s)\, ds, \Phi\right)_{B_2^1(\Omega)} = \left(\left(z_{nk}(t) - z_{nk}^0 - \int_0^t w(s)\, ds, \Phi\right)_{B_2^1(\Omega)} + \left(z_{nk}^0 - z^0, \Phi\right)_{B_2^1(\Omega)}\right),$$

By the virtue of $(ii)$ of lemma 2.4, it comes to

$$\begin{gathered}\left(z_{nk}(t) - z^0 - \int_0^t w(s)\, ds, \Phi\right)_{B_2^1(\Omega)} = \\ \int_0^t \left(\frac{\partial z_{nk}(s)}{\partial s} - w(s), \Phi\right)_{B_2^1(\Omega)} ds + \left(z_{nk}^0 - z^0, \Phi\right)_{B_2^1(\Omega)}, \text{ a.e. t } \in [0,T].\end{gathered} \tag{2.41}$$

The linear form $v \mapsto \int_0^t (v(s), \Phi)_{B_2^1(\Omega)} ds$ is continuous on $L^2\left(0,T;B_2^1(\Omega)\right)$ because we have

$$\int_0^t (v(s), \Phi)_{B_2^1(\Omega)} ds \leq \sqrt{T} \|\Phi\|_{B_2^1(\Omega)} . \|v\|_{L^2\left(0,T;B_2^1(\Omega)\right)}$$

hence invoking (2.36) we obtain

$$\lim_{k\to\infty} \int_0^t \left(\frac{\partial z_{nk}(s)}{\partial s} - w(s), \Phi\right)_{B_2^1(\Omega)} ds = 0 \qquad \text{a.e. } t \in [0,T],$$

Morever, according to (2.26) we have

$$\lim_{k\to\infty} \left(z_{nk}^0 - z^0, \Phi\right)_{B_2^1(\Omega)} = 0.$$

Therefore, applying Lebesgue's theorem on majorized convergence, yields

$$\begin{aligned}&\lim_{k\to\infty} \left(z_{nk} - z^0 - \chi, v\right)_{L^2\left(0,T;B_2^1(\Omega)\right)} \\ &= \lim_{k\to\infty} \int_0^T \left(z_{nk}(t) - z^0 - \int_0^t w(s)\, ds, \Phi\right)_{B_2^1(\Omega)} dt \\ &= \int_0^T \lim_{k\to\infty} \left(z_{nk}(t) - z^0 - \int_0^t w(s)\, ds, \Phi\right)_{B_2^1(\Omega)} dt \\ &= 0.\end{aligned}$$

Similarly, this result can be derived for cases when $v$ is the step function on $[0,T]$ and extended to every function $v \in L^2\left(0,T;B_2^1(\Omega)\right)$ by a density argument, and accordingly we conclude thanks to lemma 2.4 that $z$ is in $C\left(0,T;B_2^1(\Omega)\right)$, and even differentiable a.e. in $(0,T)$ with $w = \frac{\partial z}{\partial t}$ in $L^2\left(0,T;B_2^1(\Omega)\right)$, which was to be shown. □

**Theorem 2.13.** *The limit function $z$ of corrollary 2.12 is the unique weak solution to problem (2.3) – (2.5).*

*Proof.* One : Existence, we have to show that the limit function $z$ satisfies all the conditions $(i) - (iv)$ of definition 2.11. Obviously, in light of properties of function $z$ listed in corollary 2.12 the first two conditions of definition 2.9 are already seen . On the other hand, from (2.37), we have directly $z(0) = 0$, so the initial condition is also fulfilled, we have to see that the integral identity took place. For this writing (2.29) for $n = n_k$ and integrating on $[0,t]$, it comes

$$\begin{aligned}&\int_0^t \left(\frac{\partial z_{nk}(s)}{\partial t}, \varphi_j\right)_{B_2^1(\Omega)} - \int_0^t \left(a z_{nk}(s), \varphi_j\right) + \\ &\int_0^t \left(\frac{\partial a}{\partial x} z_{nk}(s), \Im_x \varphi_j\right) + \int_0^t \left(b(x,t)\, z_{nk}(s), \varphi_j\right)_{B_2^1(\Omega)} \\ &= \int_0^t \left(f(x,t), \varphi_j\right)_{B_2^1(\Omega)}, \ \forall t \in [0,T], \quad j = 1, ........, n_k.\end{aligned} \tag{2.42}$$

By performing a limit process $k \to \infty$ in (2.42), we get owing to (2.34), (2.35)

$$\begin{aligned}
&\int_0^t \left(\frac{\partial z(s)}{\partial t}, \varphi_j\right)_{B_2^1(\Omega)} - \int_0^t (az(s), \varphi_j) + \\
&\int_0^t \left(\frac{\partial a}{\partial x} z(s), \Im_x \varphi_j\right) + \int_0^t (b(x,t) z(s), \varphi_j)_{B_2^1(\Omega)} \\
&= \int_0^t (f(x,t), \varphi_j)_{B_2^1(\Omega)}, \ \forall t \in [0,T], \quad j = 1, \dots\dots\dots, n_k.
\end{aligned}$$

Differentiating this latter with respect to $t$, we have

$$\begin{aligned}
&\left(\frac{\partial z(t)}{\partial t}, \varphi_j\right)_{B_2^1(\Omega)} - (az(t), \varphi_j) + \left(\frac{\partial a}{\partial x} z(t), \Im_x \varphi_j\right) + (b(x,t) z(t), \varphi_j)_{B_2^1(\Omega)} \\
&= (f(x,t,z))_{B_2^1(\Omega)}, \ \text{a.e. } t \in [0,T], \quad j \geq 1.
\end{aligned}$$

From where the integral identity (2.18) is obtained.

Two : Uniqueness. Let $z_1$ and $z_2$ two solutions of problem $(2.3) - (2.5)$, and let be $w = z_1 - z_2$. Subtracting (2.18) for $z_1$ and $z_2$ and putting $v = w(t)$ for a.e. $t \in [0,T]$, we obtain

$$\begin{aligned}
&\left(\frac{\partial w(t)}{\partial t}, w(t)\right)_{B_2^1(\Omega)} - (aw(t), w(t)) + \\
&\left(\frac{\partial a}{\partial x} w(t), w(t)\right) + (b(x,t) w(t), w(t))_{B_2^1(\Omega)} = 0.
\end{aligned}$$

So, integrating this last equality on $(0,t)$ and using the fact that $w(0) = 0$, it follows from (2.21) that

$$\|w(t)\|^2 = 0 \qquad \forall t \in [0,T],$$

so, we have the uniqueness of the problem. □

## 3. Solvability of semi-linear problem

This section is consecrated to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the problem (1.1). For this, it is enough study this semi-linear problem :

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial u}{\partial x}\right) + b(x,t) u = g(x,t,u); \tag{3.1}$$

$$u(x,0) = 0, \ x \in \Omega; \tag{3.2}$$

$$\int_\Omega x^k u(x,t) \, dx = 0; \ (k = 0,1). \tag{3.3}$$

Where $g$ is known function.

We shall assume that the function $h$ is bounded in $B_2^1(Q)$ and fulfill the Lipschitz conditions, that is, there exists a positive constant $\delta$ such that

$$\|h(x,t,p) - h(x,t,q)\|_{B_2^1(Q)} \leq \delta \|p - q\|_{B_2^1(Q)}, \ \text{ for all } (x,t) \in Q, \ \delta > 0. \tag{3.4}$$

We shall prove that the problem $(3.1) - (3.3)$ has a unique weak solution.

Firstly, we precise the concept of the solution ; we are considering. Let $v = v(x,t)$ be any function from $\widetilde{C^1}(Q)$, the space of functions $v$ belonging to $C^1(Q)$ and $v(0,t) = 0$, $\int_0^1 v(x,t) dx = 0$.

We shall compute the integral $\int_Q g \Im_x v dx dt$, for this we assume $u, v \in \widetilde{C^1}(Q)$. By using conditions on $u$ and $v$, we have:

$$\int_Q u(x,t) . \Im_x v dx dt = - \int_Q v . (\Im_x u) \, dx dt,$$

$$-\int_Q \frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial u}{\partial x}\right).\left(\Im_x v\right)dxdt=\int_Q v.a\left(x,t\right)\frac{\partial u}{\partial x}dxdt,$$

$$\int_Q g\Im_x v dxdt=-\int_Q v\Im_x g dxdt.$$

It then gives

$$A\left(u,v\right)=-\int_Q v\left(\Im_x g\right)dxdt, \tag{3.5}$$

where

$$A\left(u,v\right)=-\int_Q v.\left(\Im_x u\right)dxdt+\int_Q v.a\left(x,t\right)\frac{\partial u}{\partial x}dxdt+\int_Q b\left(x,t\right)u.\left(\Im_x v\right)dxdt$$

**Definition 3.1.** *A function $u$ is called a weak solution of problem (3.1) − (3.3) if satisfies (3.5) and $u\in L^2(0,T;\,H^1(0.1))$.*

Let us construct an iteration sequence in the following way: Starting with $u^{(0)}=0$, the sequence $(\left\{u^{(n)}\right\}_{n\in N}$ is defined as follows: given the element $u^{(n-1)}$, then for $n=1,2,...$ solve the problem:

$$\frac{\partial u^{(n)}}{\partial t}-\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial u^{(n)}}{\partial x}\right)+b\left(x,t\right)u^{(n)}=g(x,t,u^{(n-1)})\qquad (x,t)\in Q, \tag{3.6}$$

$$u^{(n)}(x,0)=h\left(x\right),\qquad x\in\left[0,b\right], \tag{3.7}$$

$$\int_0^1 x^k u^{(n)}\left(x,t\right)dx=0,\qquad t\in\left[0,T\right], \tag{3.8}$$

The previous section asserts that for fixed $n$, each problem (3.6) − (3.8) has a unique solution $u^{(n)}\left(x,t\right)$. If we set $Z^{(n)}(x,t)=u^{(n+1)}(x,t)-u^{(n)}(x,t)$, then we have the new problem

$$\frac{\partial Z^{(n)}}{\partial t}-\frac{\partial}{\partial x}\left(a\left(x,t\right)\frac{\partial Z^{(n)}}{\partial x}\right)+b\left(x,t\right)Z^{(n)}=P^{(n-1)}\left(x,t\right)\qquad (x,t)\in Q, \tag{3.9}$$

$$Z^{(n)}(x,0)=0,\qquad x\in\left[0,b\right], \tag{3.10}$$

$$\int_0^1 x^k Z^{(n)}\left(x,t\right)dx=0,\qquad t\in\left[0,T\right], \tag{3.11}$$

where

$$P^{(n-1)}\left(x,t\right)=g\left(x,t,u^{(n)}\right)-g\left(x,t,u^{(n-1)}\right).$$

**Lemma 3.2.** *Assume that condition (3.4) holds, then for the linearized problem (3.9) − (3.11), we have the a priori estimate*

$$\left\|Z^{(n)}\right\|_{L^2(0,T;\,H^1(0,1))}\leq\lambda\left\|Z^{(n-1)}\right\|_{L^2(0,T;\,H^1(0,1))}, \tag{3.12}$$

*where $\lambda$ is a positive constant given by*

$$\lambda=\sqrt{\frac{5\delta^2}{\varepsilon m}}$$

*where $\varepsilon<<1$, and*

$$m=\min\left\{\left(1+c_4'+a(x,t)-\frac{1}{2}\frac{\partial^2 a}{\partial x^2}-\frac{1}{2}\frac{\partial^2 b}{\partial x^2}-\frac{\varepsilon}{2}\right),\left(a(x,t)-\frac{\varepsilon}{2}\right)\right\}>0$$

*Proof.* Multiplying equation (3.9) by $\int_x^1 \left(\int_0^\xi Z^{(n)}(\eta,t)\, d\eta\right) d\xi$ and integrating over $Q$, we get

$$\int_Q Z^{(n)}(x,t) \cdot \left(\int_x^1 \left(\int_0^\xi Z^{(n)}(\eta,t)\, d\eta\right) d\xi\right) dxdt \tag{3.13}$$
$$-\int_Q \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial Z^{(n)}}{\partial x}\right) \cdot \left(\int_x^1 \left(\int_0^\xi Z^{(n)}(\eta,t)\, d\eta\right) d\xi\right) dxdt$$
$$= \int_Q P^{(n-1)}(x,t) \cdot \left(\int_x^1 \left(\int_0^\xi Z^{(n)}(\eta,t)\, d\eta\right) d\xi\right) dxdt.$$

Standard integration by parts each term in (3.13) with use the condition (3.10) and (3.11), we obtain

$$\int_Q (1+c_4')\left(\int_0^x Z^{(n)}(\xi,t)\, d\xi\right)^2 dxdt + \int_Q a(x,t)\left(Z^{(n)}(x,t)\right)^2 dxdt \tag{3.14}$$
$$\leqslant \int_Q \left(\frac{1}{2}\frac{\partial^2 a}{\partial x^2} + \frac{1}{2}\frac{\partial^2 b}{\partial x^2} + \frac{\varepsilon}{2}\right)\left(Z^{(n)}(x,t)\right)^2 dxdt + \frac{1}{2\varepsilon}\int_Q \left(P^{(n-1)}\right)^2 dxdt.$$

On the other hand, applying operator $\Im_x^\star$ to equation (3.9), we get

$$\left(\Im_x^\star Z^{(n)}\right) + a(x,t)\frac{\partial Z^{(n)}}{\partial x} + \Im_x^\star\left(bZ^{(n)}\right) = \Im_x^\star\left(P^{(n-1)}(x,t)\right)$$

by taking into account condition (3.11), multiplying the obtained equality with $\frac{\partial Z^{(n)}}{\partial x}$ and integrating over $Q = (0,1)\times(0,T)$, where $0 \le \tau \le T$, by using the Cauchy inequality with $\varepsilon$, we obtain

$$\int_Q (1+c_4')\left(Z^{(n)}\right)^2 dxdt + \int_Q a(x,t)\left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt \tag{3.15}$$
$$\leqslant \frac{1}{2\varepsilon}\int_Q \left(\Im_x^\star P^{(n-1)}\right)^2 dxdt + \frac{\varepsilon}{2}\int_Q \left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt$$

Combining the last two previous inequalities (3.14) and (3.15), we obtain

$$\int_Q (1+c_4')\left(\int_0^x Z^{(n)}(\xi,t)\, d\xi\right)^2 dxdt + \int_Q (1+c_4')\left(Z^{(n)}\right)^2 dxdt$$
$$+\int_Q a(x,t)\left(Z^{(n)}(x,t)\right)^2 dxdt + \int_Q a(x,t)\left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt$$
$$\leqslant \frac{1}{2\varepsilon}\int_Q \left(P^{(n-1)}\right)^2 dxdt + \frac{1}{2\varepsilon}\int_Q \left(\Im_x^\star P^{(n-1)}\right)^2 dxdt$$
$$+\int_Q \left(\frac{1}{2}\frac{\partial^2 a}{\partial x^2} + \frac{1}{2}\frac{\partial^2 b}{\partial x^2} + \frac{\varepsilon}{2}\right)\left(Z^{(n)}(x,t)\right)^2 dxdt + \frac{\varepsilon}{2}\int_Q \left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt. \tag{3.16}$$

Eliminating the two first integrals on the left hand side of inequality (3.16). To this end, using the Cauchy inequality with $\varepsilon$, it follows

$$m\int_Q \left[\left(Z^{(n)}(x,t)\right)^2 dxdt + \left(\frac{\partial Z^{(n)}}{\partial x}\right)^2\right] dxdt$$
$$\leqslant \frac{1}{2\varepsilon}\int_Q \left(P^{(n-1)}\right)^2 dxdt + \frac{1}{2\varepsilon}\int_Q \left(\Im_x^\star P^{(n-1)}\right)^2 dxdt \tag{3.17}$$

Therefore, we have these estimates

$$\left\|\Im_x P^{(n-1)}\right\|_{L^2(0,1)}^2 \leq 4\left(1-0\right)^2\left\|P^{(n-1)}\right\|_{L^2(0,1)}^2 \leq 4\left\|P^{(n-1)}\right\|_{L^2(0,1)}^2. \tag{3.18}$$

$$\left(\int_Q \left(P^{(n-1)}\right)^2 dxdt\right) \leq \delta^2 \int_Q \left(\left|Z^{(n-1)}(x,t)\right| + \left|\frac{\partial Z^{(n-1)}(x,t)}{\partial x}\right|\right)^2 dxdt$$
$$\leq 2\delta^2 \int_0^T \left(\left\|Z^{(n-1)}(.,t)\right\|_{L^2(0,1)}^2 + \left\|\frac{\partial Z^{(n-1)}(.,t)}{\partial x}\right\|_{L^2(0,1)}^2\right) dt. \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17), we get

$$\int_0^T \left(\left\|Z^{(n)}(x,\tau)\right\|_{L^2(0,1)}^2 + \left\|\frac{\partial Z^{(n)}}{\partial x}\right\|_{L^2(0,1)}^2\right) dt \tag{3.20}$$
$$\leqslant \frac{5\delta^2}{\varepsilon m}\left[\int_0^T \left(\left\|Z^{(n-1)}(.,t)\right\|_{L^2(0,1)}^2 + \left\|\frac{\partial Z^{(n-1)}(.,t)}{\partial x}\right\|_{L^2(0,1)}^2\right) dt\right].$$

Then, we obtain the desired inequality

$$\left\|Z^{(n)}\right\|_{L^2(0,T;\ H^1(0,1))} \leq \frac{5\delta^2}{\varepsilon m}\left\|Z^{(n-1)}\right\|_{L^2(0,T;\ H^1(0,1))}.$$

□

From the criteria of convergence of series, we see that the series $\sum_{n=1}^{\infty} Z^{(n)}$ converges if $\frac{5\delta^2}{\varepsilon m} < 1$, that is if $\delta < \sqrt{\frac{\varepsilon m}{5}}$. Since

$Z^{(n)}(x,t) = u^{(n+1)}(x,t) - u^{(n)}(x,t)$, then it follows that the sequence $(u^{(n)})_{n\in N}$ defined by

$$u^{(n)}(x,t) = \sum_{i=0}^{n-1} Z^{(i)} + u^{(0)}(x,t),$$

converges to an element $u \in L^2\left(0,T;\ H^1(0,1)\right)$.

Therefore, we have established the following result:

**Theorem 3.3.** *Assume that condition (3.4) are hold and that*

$$\delta < \sqrt{\frac{\varepsilon m}{5}},$$

*then the semi-linear problem (3.1) – (3.3) admits a weak solution in* $L^2\left(0,T;\ H^1(0,1)\right)$.

It remains to prove that problem (3.1) – (3.3) admits a unique solution.

**Theorem 3.4.** *Under the condition (3.4), the solution of the problem (3.1) – (3.3) is unique.*

*Proof.* Suppose that $u_1$ and $u_2$ in $L^2\left(0,T;\ H^1(0,1)\right)$ are two solution of (3.1) – (3.3), then $Z = u_1 - u_2$ satisfies $Z \in L^2\left(0,T;\ H^1(0,1)\right)$ and

$$\frac{\partial Z}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial Z}{\partial x}\right) + b(x,t)Z = \psi(x,t) \qquad (x,t) \in \overline{\Omega},$$

$$Z(x,0) = 0, \qquad x \in [0,1],$$

$$\int_0^1 x^k Z(x,t)\,dx = 0, \qquad k = 0; 1 \text{ and } t \in [0,T],$$

where

$$\psi(x,t) = g(x,t,u_1) - g(x,t,u_2).$$

Following the same procedure done in establishing the proof of Lemma 3.2, we get

$$\|Z\|_{L^2(0,T;\ H^1(0,1))} \leq \lambda \|Z\|_{L^2(0,T;\ H^1(0,1))} \tag{3.21}$$

where $\lambda$ is the same constant of lemma 3.2.

Since $\lambda < 1$, then from (3.21) that

$$(1-\lambda)\|Z\|_{L^2(0,T;\ H^1(0,1))} \leq 0,$$

from which we conclude that $u_1 = u_2$ in $L^2\left(0,T;\ H^1(0,1)\right)$. □

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