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Irregular Stable Sampling and Interpolation in Functional Normed Spaces*

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ABSTRACT: We define the concepts of stable sampling set and stable interpolation set, uniqueness set and complete interpolation set for a normed space of functions. In addition we will show some relationships between these concepts. The main relationships arise when one wants to reduce an stable sampling set or to extend an stable interpolation set. We will prove that for Banach spaces verifying certain conditions, the complete interpolation sets are precisely the minimal stable sampling sets and are also the maximal stable interpolation sets. Finally we illustrate these results applying them to Paley-Wiener spaces, where we use a result by B. Matei, Yves Meyer and J. Ortega-Cerdá based on the celebrated Fefferman theorem.

Key Words: Normed spaces, Banach spaces, p-sampling set, p-interpolation set, p-complete interpolation set, uniqueness set, Paley-Wiener spaces.

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1. Introduction

The aim of this paper is to establish general results in irregular stable sampling and interpolation theory in functional normed spaces and the connection between the main concepts of this theory. We also apply some of these results to Paley-Wiener spaces.
We denote by $\mathfrak{F}(\mathbb{R}^n,\ \mathbb{C})$ (respectively, $\mathfrak{F}(\mathbb{C}^n,\ \mathbb{C})$) the set of the complex functions defined in $\mathbb{R}^n$ (respectively $\mathbb{C}^n$).

We also denote by $\mathcal{H}(\mathbb{C}^n)$ (respectively, $\mathcal{M}(\mathbb{C}^n)$) the vector space of holomorphic (respectively, meromorphic) functions whose domain is $\mathbb{C}^n$.

We will work mainly with normed vector subspaces of $\mathfrak{F}(\mathbb{R}^n,\ \mathbb{C})$ and $\Lambda \subseteq \mathbb{R}^n$, but all the results and definitions are analogously extended for normed vector subspaces of $\mathfrak{F}(\mathbb{C}^n,\ \mathbb{C})$ and $\Lambda \subseteq \mathbb{C}^n$.
We also denote the cardinal of a set A by $Card(A)$.

Definition 1.1 (Uniformly discrete set). *Let $\Lambda \subseteq \mathbb{C}^n$ be infinite countable. We say that Λ is uniformly discrete (briefly u.d.) if*

$$\delta(\Lambda) := \inf_{\lambda,\ \lambda' \in \Lambda,\ \lambda \neq \lambda'} \|\lambda - \lambda'\| > 0.$$

The constant $\delta(\Lambda)$ is called the separation constant of Λ.

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Definition 1.2 (Uniqueness set)**.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. We say that Λ is a uniqueness or complete set (briefly, US) for E if for every $f \in E$ we have that*

$$(\forall \lambda \in \Lambda \; f(\lambda) = 0) \Rightarrow f = 0.$$

Definition 1.3 (Sequence space $l^p(\Lambda)$)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ be u.d.*

1. *Let $p \in [1, +\infty)$. We define the set*

$$l^p(\Lambda) := \left\{ (a_\lambda)_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda \mid \sum_{\lambda \in \Lambda} |a_\lambda|^p < \infty \right\}.$$

The mapping $\| \, \|_p : l^p(\Lambda) \to \mathbb{R}$ given by $\left\|(a_\lambda)_{\lambda \in \Lambda}\right\|_p := \left(\sum_{\lambda \in \Lambda} |a_\lambda|^p\right)^{\frac{1}{p}}$, is a norm for $l^p(\Lambda)$. With this norm $l^p(\Lambda)$ is a complete space.

2. $$l^\infty(\Lambda) := \left\{ (a_\lambda)_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda \mid \sup_{\lambda \in \Lambda} |a_\lambda| < \infty \right\}.$$

The mapping $\| \, \|_\infty : l^\infty(\Lambda) \to \mathbb{R}$ defined by $\left\|(a_\lambda)_{\lambda \in \Lambda}\right\|_\infty := \sup_{\lambda \in \Lambda} |a_\lambda|$ is a norm for $l^\infty(\Lambda)$ which make this space a Banach space.

Definition 1.4. *Let $(E, \| \, \|)$ be a normed space, verifying $E \subseteq \mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. Let $p \in [1, +\infty]$ and $\Lambda \subseteq \mathbb{R}^n$ be a uniformly discrete set. Assume that*

$$(f(\lambda))_{\lambda \in \Lambda} \in l^p(\Lambda) \text{ for all } f \in E.$$

- *The $\mathbb{C}$-linear mapping $S : (E, \| \, \|) \to (l^p(\Lambda), \| \, \|_p)$ given by $f \to (f(\lambda))_{\lambda \in \Lambda}$ is called the p-sampling operator of $(E, \| \, \|)$ with respect to Λ. Observe that Λ is a uniqueness set for E if and only if S is injective.*

- *We say that Λ verifies the p-Plancherel-Polya condition (briefly p-P.P.C.) for $(E, \| \, \|)$ if S is continuous, this is, if there exists a constant $C > 0$ such that*

$$\|(f(\lambda))_{\lambda \in \Lambda}\|_p \leq C \, \|f\| \text{ for each } f \in E.$$

  We also say that Λ is p-besselian for $(E, \| \, \|)$.

- *Λ is said to be a p-interpolation set (in short, p-IS) for $(E, \| \, \|)$ if S is surjective. (Thus, to be a p-IS does not depend on neither the norm of E nor on $\| \, \|_p$.). Given $c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$ and $f \in E$, we say that f interpolates c (over Λ) if $f(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$.*

- *Λ is said to be a p-stable interpolation set (briefly, p-SIS) for $(E, \| \, \|)$ if S is surjective, and has a continuous inverse by right.*

- *We say that Λ is a p-stable sampling set (briefly, p-SS) for $(E, \| \, \|)$ if S is a topological isomorphism over its image, this is, if there exist constants $c, C > 0$, $c \leq C$, such that*

$$c \, \|(f(\lambda))_{\lambda \in \Lambda}\|_p \leq \|f\| \leq C \, \|(f(\lambda))_{\lambda \in \Lambda}\|_p$$

  for each $f \in E$. That is, if S is continuous, injective and has a continuous inverse by left.

- *Λ is called a p-complete interpolation set (in short, p-CIS) for $(E, \| \, \|)$ if S is bijective; this is, if Λ is a US and a p-IS for E.*

- *We say that Λ is a p-stable complete interpolation set (abreviado, p-SCIS) for $(E, \| \, \|)$ if S is a topological isomorphism; that is, if Λ is p-IS and p-SS for $(E, \| \, \|)$. Observe that Λ is a p-SCIS for E if and only if Λ is a US and a p-SIS for E.*

Observe that every p-SS for $(E, \| \|)$ *is a US. Besides, the following statements are equivalent:*

1. Λ *is p-SCIS for* $(E, \| \|)$.
2. Λ *is p-SIS and p-SS for* $(E, \| \|)$.
3. *S is continuous,* Λ *is a US and a p-SIS for* $(E, \| \|)$.

Λ is a p-SS for $(E, \| \|)$ if and only if there exists a vector subspace of $(l^p(\Lambda), \| \|_p)$ topologically isomorphic to $(E, \| \|)$ through the sampling mapping, and this is a representation of $(E, \| \|)$ as a subspace of $(l^p(\Lambda), \| \|_p)$.

In addition, Λ is a p-SS for $(E, \| \|)$ if and only if Λ allows to build a norm in E equivalent to $\| \|$. This is exactly the content of the next result.

Lemma 1.5. *Let* $(E, \| \|)$ *be a normed space, being* $E \subseteq \mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. *Let* $p \in [1, +\infty]$ *and* $\Lambda \subseteq \mathbb{R}^n$ *be a uniformly discrete set. The following statements are equivalent:*

1. Λ *is a p-SS for* $(E, \| \|)$.
2. *The function* $\| \|_{\Lambda, p} : E \to \mathbb{R}$ *defined by* $\|f\|_{\Lambda, p} := \|(f(\lambda))_{\lambda \in \Lambda}\|_p$ *for all* $f \in E$*, is a norm in E equivalent to* $\| \|$.

In the rest of this article we will omit the norm of E, except if it was necessary, and will refer to the normed space $(E, \| \|)$ simply as E.

2. Characterizations and transference.

In this section we will obtain some results that allow us to establish certain relationships between the concepts of stable sampling set, stable interpolation set and the rest of the concepts defined in definition 1.4.
First observe that given $p \in [1, +\infty]$ and a normed space E verifying $E \subseteq \mathfrak{F}(\mathbb{R}^n, \mathbb{C})$, if a u.d. set $\Lambda \subseteq \mathbb{R}^n$ is a US (respectively, a p-SS) for E, then Λ is also a US (respectively, a p-SS) for every vector subspace of E. If a u.d. set $\Lambda \subseteq \mathbb{R}^n$ is a p-IS for E, then Λ is also a p-IS for each vector extension of E.

2.1. Characterizations

We have the following result for Banach spaces verifying the Plancherel-Polya condition.

Proposition 2.1. *Let E be a Banach space such that* $E \subseteq \mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. *Let* $p \in [1, +\infty]$ *and* $\Lambda \subseteq \mathbb{R}^n$ *be uniformly discrete. Suppose that* Λ *verifies the p-C.P.P. for E. Then we have:*

- *The following statements are equivalent:*
  1. Λ *is a p-IS for E.*
  2. Λ *is a p-SIS for E.*
- *The next statements are equivalent:*
  1. Λ *is a p-CIS for E.*
  2. Λ *is a p-SCIS for E.*
  3. Λ *is a p-SIS and p-SS for E.*
  4. Λ *is a US and p-SIS for E.*
- *If* $Im(S)$ *is of second category of Baire in* $(l^p(\Lambda), \| \|_p)$*, then* Λ *is a p-SIS for E.*
- *Assume that* Λ *is a US for E. Then the following conditions are equivalent:*
  1. Λ *is a p-SS for E.*

2. $(Im(S), \| \|_p)$ *is complete.*
3. $Im(S)$ *is closed in* $(l^p(\Lambda), \| \|_p)$.
4. $S^{-1} : (Im(S), \| \|_p) \to (E, \| \|)$ *is continuous.*
5. S *is open over its image.*
6. S *is closed over its image.*

- *Suppose that* Λ *is a US for* E. *The two following statements are equivalent:*
  1. Λ *is a p-SIS for* $E \Leftrightarrow \Lambda$ *is a p-SS for* E.
  2. S *is open* $\Leftrightarrow S$ *is open over its image.*

Proof. This result is an immediate consequence of the Banach open mapping and homomorphism theorems and because of $(l^p(\Lambda), \| \|_p)$ is complete. □

2.2. Transference

In this subsection we will show how we may obtain new sampling, interpolation or uniqueness sets based on others.
Let $p \in [1, +\infty]$. Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a normed space such that $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $g : \Omega \to \Omega$ be a function and $\Lambda \subseteq \Omega$ be uniformly discrete. Suppose that:

1. $f \circ g \in E$ for all $f \in E$.
2. $g(\Lambda)$ is uniformly discrete.

Then we have the following results:

Proposition 2.2. *Suppose that g is surjective. If Λ is a US for E, then $g(\Lambda)$ is a US for E.*

Proof. Suppose that Λ is a US for E. Veamos that $g(\Lambda)$ is a US for E. $g(\Lambda) = \{g(\lambda) : \lambda \in \Lambda\}$. Let $f \in E$ such that $f(g(\lambda)) = 0$ for each $\lambda \in \Lambda$. We will show that $f = 0$. Define $h := f \circ g \in E$. We have that $h(\lambda) = 0$ for all $\lambda \in \Lambda$. Since Λ is a US for E, then $h = 0$, that is, $f \circ g = 0$. In addition, since g is surjective, then g has got an inverse by right. Thus: $f = f \circ 1_\Omega = f \circ (g \circ g^{-1}) = (f \circ g) \circ g^{-1} = h \circ g^{-1} = 0 \circ g^{-1} = 0$. So that $f = 0$. □

Lemma 2.3. *Suppose that Λ verifies the p-C.P.P. for E and that the two following conditions are verified:*

1. *There exists a constant $R > 0$ such that $\|f \circ g\| \leq R \cdot \|f\|$ for all $f \in E$.*
2. $g|_\Lambda$ *is injective.*

Then, $g(\Lambda)$ verifies the p-C.P.P. for E.

Proof. Since Λ verifies the p-C.P.P. for E, then there exists a constant $C > 0$ such that

$$\|(f(\lambda))_{\lambda \in \Lambda}\|_p \leq C \; \|f\| \text{ for all } f \in E.$$

Let $f \in E$. Since $g|_\Lambda$ is injective and $f \circ g \in E$, we have:

$$\|(f(\mu))_{\mu \in g(\Lambda)}\|_p = \|(f(g(\lambda)))_{\lambda \in \Lambda}\|_p \leq C \cdot \|f \circ g\| \leq C \cdot R \cdot \|f\|.$$

Hence $g(\Lambda)$ verifies the p-C.P.P. for E. □

Proposition 2.4. *Suppose that $g(\Lambda)$ verifies the p-C.P.P. for E and the next two conditions are verified:*

1. *There exists a constant $D > 0$ such that $\|f\| \leq D \cdot \|f \circ g\|$ for all $f \in E$.*

2. *There exists a constant $M = M(g,\ \Lambda) \in \mathbb{Z}^+$ such that $Card(g^{-1}(\mu)) \leq M$ for each $\mu \in g(\Lambda)$. (Observation: for $p = +\infty$ this condition is not necessary.).*

If Λ is a p-SS for E, then $g(\Lambda)$ is also a p-SS for E.

Proof. Suppose that Λ is a p-SS for E, that is, there exists a constant $C > 0$ such that

$$\|f\| \leq C\ \|(f(\lambda))_{\lambda\in\Lambda}\|_p \text{ for all } f \in E.$$

We will prove that $g(\Lambda)$ is also a p-SS for E. Let $f \in E$. Then $f \circ g \in E$, and therefore:

$$D^{-1}\ \|f\| \leq \|f \circ g\| \leq C\|((f \circ g)(\lambda))_{\lambda\in\Lambda}\|_p = C\|((f(g(\lambda)))_{\lambda\in\Lambda}\|_p \leq C \cdot M^{\frac{1}{p}}\|((f(\mu))_{\mu\in g(\Lambda)}\|_p.$$

Hence

$$\|f\| \leq D \cdot C \cdot M^{\frac{1}{p}}\|((f(\mu))_{\mu\in g(\Lambda)}\|_p\ .$$

Conclusion: $g(\Lambda)$ is a p-SS for E. □

Proposition 2.5. *Suppose that g is bijective. If $g(\Lambda)$ is a p-IS for E, then Λ is also a p-IS for E. Obviously, if $f \circ g^{-1} \in E$ for all $f \in E$, then the reciprocal statement is true.*

Proof. Suppose that $g(\Lambda)$ is a p-IS for E. Let us show that Λ is also a p-IS for E.
Let $b = (b_\lambda)_{\lambda\in\Lambda} \in (l^p(\Lambda), \|\ \|_p)$. We define $a_{g(\lambda)} := b_\lambda$ for all $\lambda \in \Lambda$. Then $a = (a_\gamma)_{\gamma\in g(\Lambda)} \in (l^p(g(\Lambda)), \|\ \|_p)$ (and besides $\|b\|_p = \|a\|_p$). Since $g(\Lambda)$ is a p-IS for E, then there exists $f \in E$ such that $b_\lambda = a_{g(\lambda)} = f(g(\lambda)) = (f \circ g)(\lambda)$ for each $\lambda \in \Lambda$. Since $h := f \circ g \in E$, we have that $h(\lambda) = b_\lambda$ for each $\lambda \in \Lambda$. Conclusion: Λ is a p-IS for E. □

Corollary 2.6. *Let E be a normed space such that $E \subseteq \mathfrak{F}(\mathbb{R}^n,\ \mathbb{C})$. Suppose that E is invariant by translations (that is: $\tau_x f \in E$ for each $f \in E$, $x \in \mathbb{R}^n$, where $\tau_x f(y) := f(y - x)$ for each $y \in \mathbb{R}^n$). Let $p \in [1,\ +\infty]$.*

1. *Let $x \in \mathbb{R}^n$. The following statements are equivalent:*
   (a) *Λ is a US for E.*
   (b) *$x + \Lambda$ is a US for E.*
2. *Let $x \in \mathbb{R}^n$. The following statements are equivalent:*
   (a) *Λ is a p-IS for E.*
   (b) *$x + \Lambda$ is a p-IS for E.*
3. *Suppose that E is isometric by translations (that is, E is invariant by translations and verifies that $\|\tau_x f\| = \|f\|$ for each $f \in E$ and for each $x \in \mathbb{R}^n$). Let $x \in \mathbb{R}^n$. The next statements are equivalent:*
   (a) *Λ is a p-SS for E.*
   (b) *$x + \Lambda$ is a p-SS for E.*

In particular we may apply this result to every invariant by translations subspace of $(L^p(\mathbb{R}^n) \cap \mathbb{C}(\mathbb{R}^n),\ \|\ \|_p)$, with $p \in [1,\ +\infty]$, because these subspaces are isometric by translations.

Finally we will finish this section with a result of transference of uniqueness for meromorphic functions.

Proposition 2.7. *Let E a normed space, with $E \subseteq \mathcal{M}(\mathbb{C}^n)$. Suppose that there exists a non constant function $g \in \mathcal{M}(\mathbb{C}^n)$ such that $f \circ g \in E$ for each $f \in E$. Let $\Lambda \subseteq \mathbb{C}^n$ be uniformly discrete. If Λ is a US for E, then $g(\Lambda)$ is a US for E.*

Proof. Take $f \in E$ such that $f(g(\lambda)) = 0$ for each $\lambda \in \Lambda$. We will prove that $f = 0$. Suppose that $f \neq 0$, and we will obtain a contradiction. f is an meromorphic function and is not constant (if f was constant, it would have to be a constant equals to 0 because of once at least it takes the value 0, but this is not possible because of our assumption), whereupon f is an open mapping. We define $h := f \circ g \in E$ what is also an open mapping because h is a composition of two open mappings. We have that $h(\lambda) = 0$ for each $\lambda \in \Lambda$. Since Λ is a US for E, then $h = 0$. But this is a contradiction with the fact consisting of that h is an open mapping. Conclusion: $f = 0$. □

3. Refinement and extension of SS and IS.

It is evident that if we extract one or several elements of an IS (respectively, SIS), the resultant set is also an IS (respectively, SIS); and if we add one or several elements to a uniqueness set, the resultant set is also a uniqueness one. Similarly, if we add one or several elements to an SS, the set that we obtain is an SS provided that this extended set verifies the Plancherel-Polya condition (which happens if the set of elements what have been added verifies the condition of Plancherel-Polya).
The main question is to know what happens if we do the inverse action. That is, we wish to know what happens if we do a refinement of an SS (respectively, of a US) and an extension of an IS (respectively, SIS). We also wonder what happens if we do whichever action of those ones over a CIS.
In this section we will answer these questions, and it allows us to establish the essential relationship between the interpolation sets, the stable sampling sets and the complete interpolation sets.

3.1. Restrictions on the refinement and on the extension.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a normed space such that $E \subseteq \mathfrak{F}(\Omega,\ \mathbb{C})$. Let $p \in [1,\ +\infty]$ and $\Lambda \subseteq \Omega$ be uniformly discrete.*

1. *Suppose that Λ is a p-SS for E. If Λ may be refined as p-SS (this is, there exists $\Gamma \subset \Lambda$ non empty proper subset such that $\Lambda \setminus \Gamma$ is a p-SS for E), then Λ is not a p-IS for E.*

2. *Suppose that Λ is a US for E. If Λ may be refined as a US, (that is, there exists $\Gamma \subset \Lambda$ non empty proper subset such that $\Lambda \setminus \Gamma$ is a US for E) then Λ is not a p-IS for E.*

3. *Assume that Λ is a p-CIS for E. Then Λ cannot be refined as a p-CIS, that is, no proper subset of Λ is a p-CIS for E. As a consequence, neither may we extend Λ to a p-CIS for E (this is, for every non empty $\Gamma \subseteq \mathbb{R}^n \setminus \Lambda$ we have that $\Lambda \cup \Gamma$ is not a p-CIS).*

4. *Suppose that Λ is a p-SCIS for E. Then Λ cannot be refined as a p-SCIS, this is, no proper subset of Λ is a p-SCIS for E. As a consequence, we may not extend Λ to a p-SCIS for E.*

Proof.
1. Suppose that Λ is a p-SS for E and that Λ may be refined as a p-SS We will show that Λ is not a p-IS for E.

Suppose that Λ is a p-IS for E, and we will obtain a contradiction. By our assumption, there exists $\Gamma \subset \Lambda$ non empty proper subset such that $\Lambda \setminus \Gamma$ is a p-SS for E. Let $\lambda_0 \in \Gamma$. Then $\Lambda \setminus \{\lambda_0\} \supseteq \Lambda \setminus \Gamma$, therefore $\Lambda \setminus \{\lambda_0\}$ is a p-SS for E. Consider $e_{\lambda_0} := (\delta_{\lambda_0\ \lambda})$, being

$$\delta_{\lambda_0\ \lambda} = \begin{cases} 1, & \text{if } \lambda = \lambda_0 \\ 0, & \text{if } \lambda \neq \lambda_0. \end{cases}$$

Since Λ is a p-IS for E, then $\Lambda \setminus \{\lambda_0\}$ is also a p-IS for E. Hence we have that the sampling operators

$$S_\Lambda : (E,\ \|\ \|) \to (l^p(\Lambda), \|\ \|_p)$$

given by: $f \to (f(\lambda))_{\lambda\in\Lambda}$, and

$$S_{\Lambda\setminus\{\lambda_0\}} : (E,\ \|\ \|) \to (l^p(\Lambda \setminus \{\lambda_0\}), \|\ \|_p)$$

given by: $f \to (f(\lambda))_{\lambda\in\Lambda\setminus\{\lambda_0\}}$, are bijective (in fact, are topological isomorphisms). Since $e_{\lambda_0} := (\delta_{\lambda_0\ \lambda}) \in l^p(\Lambda)$ and S_Λ is bijective, we have that there exists an unique $f \in E$ such that $f(\lambda) = e_{\lambda_0}(\lambda) = \delta_{\lambda_0\ \lambda}$ for each $\lambda \in \Lambda$. So that $f(\lambda_0) = 1$, and $f(\lambda) = 0$ for all $\lambda \in \Lambda \setminus \{\lambda_0\}$.

Consider now $0 = (0)_{\lambda\in\Lambda\setminus\{\lambda_0\}} \in l^p(\Lambda \setminus \{\lambda_0\})$. Since $S_{\Lambda\setminus\{\lambda_0\}}$ is bijective, there exists an unique $g \in E$ such that $g(\lambda) = 0$ for each $\lambda \in \Lambda \setminus \{\lambda_0\}$. Obviously $g = 0$. Besides $g(\lambda) = 0 = f(\lambda)$ for every $\lambda \in \Lambda \setminus \{\lambda_0\}$. $\Lambda \setminus \{\lambda_0\}$ is a US for E (because is a p-SS), and thus $f = g = 0$. But this is a contradiction with $f(\lambda_0) = 1 \neq 0$. Hence Λ is not a p-IS for E.

2. It is the same proof as before taking *US* instead of *p-SS*.
3. This is an immediate consequence of the previous item.
4. It is an immediate consequence of the first item (even of the second one). $\square$

Corollary 3.2. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a normed space such that $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $p \in [1, +\infty]$ and $\Lambda \subseteq \Omega$ be uniformly discrete. If a p-IS (respectively, p-SIS) Λ may be extended (strictly, adding an element at least) to a p-IS (respectively p-SIS), then Λ is not a US for E, and thus neither is a p-SS for E.*

Proof. Suppose that Λ is a p-IS (respectively, p-SIS). Assume that Λ is a US. We will obtain a contradiction. The extended set $\Lambda' \supset \Lambda$ is both a p-IS (respectively p-SIS) and a US, whereupon Λ and Λ' are p-CIS (respectively p-SCIS). Hence, Λ is a refinement of Λ', and this is false because of p-CIS (respectively, p-SCIS) cannot be refined (neither extended). Hence, Λ is not a US. $\square$

3.2. Refinement of SS

In this subsection we will show when an stable sampling set may be refined.

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a normed space such that $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be uniformly discrete and $p \in [1, +\infty)$. Suppose that E is complete. Suppose that Λ verifies the p-C.P.P. for E and that it is a p-SS but is not a p-IS for E. Then:*

1. *There exists $\lambda_0 \in \Lambda$ such that $\Lambda \setminus \{\lambda_0\}$ is a p-SS for E.*
2. *Assume that there is no CIS for E. Then for every $r \in \mathbb{Z}^+$ there exist $\lambda_1, ..., \lambda_r \in \Lambda$ such that $\Lambda \setminus \{\lambda_1, ..., \lambda_r\}$ is a p-SS for E.*

Proof.
1. Consider the sampling operator

$$S_\Lambda : (E, \| \|) \to (l^p(\Lambda), \| \|_p)$$

given by: $f \to (f(\lambda))_{\lambda \in \Lambda}$, what is continuous exactly because of Λ verifies the p-C.P.P. for E. In fact S is a topological isomorphism over its image (in particular, S is injective) because Λ is a p-SS. But it is not surjective because Λ is not a p-IS.

In addition, we know that $B = \{e_\lambda = (\delta_{\lambda\, t})_{t \in \Lambda}\}_{\lambda \in \Lambda}$ is a total set for $(l^p(\Lambda), \| \|_p)$, that is, $\overline{span(B)} = l^p(\Lambda)$.

Observe that there exists $\lambda_0 \in \Lambda$ such that $e_{\lambda_0} = (\delta_{\lambda_0\, t})_{t \in \Lambda}$ does not belong to $Im(S)$ because S is not surjective. Let us prove this statement.
Suppose that $e_\lambda \in Im(S)$ for every $\lambda \in \Lambda$. We will obtain a contradiction. $B \subseteq Im(S)$ whereby $\overline{span(B)} \subseteq \overline{Im(S)}$, this is, $\overline{Im(S)} = l^p(\Lambda)$. Since S is a topological isomorphism of E over $(Im(S), \| \|_p)$ and E, is complete, then $(Im(S), \| \|_p)$ is complete, this is, $Im(S)$ is closed in $(l^p(\Lambda), \| \|_p)$. Hence $Im(S) = \overline{Im(S)} = l^p(\Lambda)$, and this is a contradiction with our assumption consisting of that S is not surjective.

Hence there exists $\lambda_0 \in \Lambda$ such that $e_{\lambda_0} = (\delta_{\lambda_0\, t})_{t \in \Lambda}$ is not an element of $Im(S)$. Since $Im(S)$ is a vector subspace of $l^p(\Lambda)$ and $e_{\lambda_0} = (\delta_{\lambda_0\, t})_{t \in \Lambda}$ does not belong to $Im(S)$, then for every $z \in \mathbb{C} \setminus \{0\}$ we have that $z \cdot e_{\lambda_0}$ does not belong to $Im(S)$. Observe that $\Lambda \setminus \{\lambda_0\}$ is a US for E. Let us show this. Indeed, suppose that $\Lambda \setminus \{\lambda_0\}$ is not a US for E. We will obtain a contradiction. There exists $f \in E \setminus \{0\}$ such that $f(\lambda) = 0$ for every $\lambda \in \Lambda \setminus \{\lambda_0\}$. Since Λ is a US for E, then $f(\lambda_0) \neq 0$. Therefore $S(f) = (f(\lambda))_{\lambda \in \Lambda} = f(\lambda_0) \cdot e_{\lambda_0}$ does not belong to $Im(S)$, and this is obviously a contradiction.

Let us prove that $\Lambda \setminus \{\lambda_0\}$ is a p-SS for E. Consider the projection

$$p_{\Lambda \setminus \{\lambda_0\}} : (l^p(\Lambda), \| \|_p) \to (l^p(\Lambda \setminus \{\lambda_0\}), \| \|_p)$$

defined by: $(a_\lambda)_{\lambda \in \Lambda} \to (a_\lambda)_{\lambda \in \Lambda \setminus \{\lambda_0\}}$. Clearly, it is continuous.

Consider the sampling operator

$$S_\Lambda : (E, \| \|) \to (l^p(\Lambda), \| \|_p)$$

defined by: $f \to (f(\lambda))_{\lambda \in \Lambda}$, and

$$S_{\Lambda \setminus \{\lambda_0\}} : (E, \| \|) \to (l^p(\Lambda \setminus \{\lambda_0\}), \| \|_p)$$

given by: $f \to (f(\lambda))_{\lambda\in\Lambda\setminus\{\lambda_0\}}$. We have that

$$S_{\Lambda\setminus\{\lambda_0\}} = p_{\Lambda\setminus\{\lambda_0\}} \circ S.$$

Hence $Im\left(S_{\Lambda\setminus\{\lambda_0\}}\right) = p_{\Lambda\setminus\{\lambda_0\}}\left(Im\left(S_\Lambda\right)\right)$. We also know that $S_{\Lambda\setminus\{\lambda_0\}}$ is linear and continuous, and is injective because $\Lambda\setminus\{\lambda_0\}$ is a US for E. Then, by the Banach homomorphism theorem the following statements are equivalent:

1. $\Lambda\setminus\{\lambda_0\}$ is a p-SS for E.
2. $S^{-1}_{\Lambda\setminus\{\lambda_0\}} : (Im(S_{\Lambda\setminus\{\lambda_0\}}), \|\,\|_p) \to (E, \|\,\|)$ is continuous.
3. $S_{\Lambda\setminus\{\lambda_0\}} : (E, \|\,\|) \to (Im(S_{\Lambda\setminus\{\lambda_0\}}), \|\,\|_p)$ is open.
4. $(Im(S_{\Lambda\setminus\{\lambda_0\}}) = S_{\Lambda\setminus\{\lambda_0\}}(E) = p_{\Lambda\setminus\{\lambda_0\}}(Im\left(S_\Lambda\right), \|\,\|_p)$ is complete.

Since Λ is p-SS for E, we have in the same way as before that $(Im\left(S_\Lambda\right) = S_\Lambda(E), \|\,\|_p)$ is complete. We have to prove that $(p_{\Lambda\setminus\{\lambda_0\}}(Im\left(S_\Lambda\right)), \|\,\|_p)$ is complete. Consider

$$\tilde{p}_{\Lambda\setminus\{\lambda_0\}} : \left(Im\left(S_\Lambda\right), \|\,\|_p\right) \to \left(Im\left(S_{\Lambda\setminus\{\lambda_0\}}\right), \|\,\|_p\right)$$

given by: $(a_\lambda)_{\lambda\in\Lambda} \to p_{\Lambda\setminus\{\lambda_0\}}((a_\lambda)_{\lambda\in\Lambda}) := (a_\lambda)_{\lambda\in\Lambda\setminus\{\lambda_0\}}$, which is a projection and is continuous, linear, surjective and open (by the open mapping theorem). Since $\tilde{p}_{\Lambda\setminus\{\lambda_0\}}$ is surjective, then

$$Im\left(\tilde{p}_{\Lambda\setminus\{\lambda_0\}}\right) = Im\left(S_{\Lambda\setminus\{\lambda_0\}}\right) = p_{\Lambda\setminus\{\lambda_0\}}\left(Im\left(S_\Lambda\right)\right).$$

Since $(Im(S), \|\,\|_p)$ is complete and $\tilde{p}_{\Lambda\setminus\{\lambda_0\}}$ is open, by the Banach homomorphism theorem we obtain

$$\left(Im(\tilde{p}_{\Lambda\setminus\{\lambda_0\}}), \|\,\|_p\right) = \left(p_{\Lambda\setminus\{\lambda_0\}}(Im(S_\Lambda)), \|\,\|_p\right)$$

is complete, and therefore $\Lambda\setminus\{\lambda_0\}$ is a p-SS for E.

2. This is a immediate consequence of the previous item.

□

The following consequence shows that if $p \in [1, +\infty)$, then the p-SCIS of a complete space E are exactly the p-SS of E what cannot be refined, this is, the minimal p-SS of E.

Corollary 3.4. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a Banach space such that $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be uniformly discrete and $p \in [1, +\infty)$. Suppose that Λ verifies the p-C.P.P. for E. Also assume that Λ is a p-SS that cannot be refined as p-SS (that is, for each $\lambda \in \Lambda$ we have that $\Lambda\setminus\{\lambda\}$ is not a p-SS for E). Then Λ is a p-SIS for E, and therefore is a p-SCIS for E too. Besides, the reciprocal statement is true: if Λ is a p-SCIS for E, then Λ is a p-SS for E that cannot be refined as p-SS.*

Proof. This is an immediate consequence of the theorems 3.3 (the direct statement) and 3.1 (the reciprocal one). □

Theorem 3.3 allows to know when there exists a refinement of an SS of E. Now we wonder in what situations *every* refinement of an SS of E is still an SS.

Lemma 3.5. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a normed space such that $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $p \in [1, +\infty]$ and $\Lambda \subseteq \Omega$ be uniformly discrete. Let $\lambda_0, \lambda_1 \in \Lambda$, $\lambda_0 \neq \lambda_1$. Suppose that there exists a surjective function $g_{\lambda_0, \lambda_1} : \Omega \to \Omega$ such that:*

1. *$g_{\lambda_0, \lambda_1}(\Lambda) \subseteq \Lambda$.*
2. *$g_{\lambda_0, \lambda_1}|_\Lambda : \Lambda \to \Lambda$ is bijective.*
3. *$g_{\lambda_0, \lambda_1}(\lambda_0) = \lambda_1$.*

4. $f \circ g_{\lambda_0,\, \lambda_1} \in E$ *for every* $f \in E$.

Then:

1. If $\Lambda \setminus \{\lambda_0\}$ *is a US for* E*, then* $\Lambda \setminus \{\lambda_1\}$ *is a US for* E.

2. Suppose that there exist constants $A = A(\lambda_0,\, \lambda_1),\ D = D(\lambda_0,\, \lambda_1) > 0,\ A \leq B$*, such that*

$$A \cdot \|f \circ g_{\lambda_0,\, \lambda_1}\| \leq \|f\| \leq B \cdot \|f \circ g_{\lambda_0,\, \lambda_1}\|$$

for all $f \in E$*. Then, if* $\Lambda \setminus \{\lambda_0\}$ *is a p-SS for* E*, we have that* $\Lambda \setminus \{\lambda_1\}$ *is also a p-SS for* E.

Proof. $g := g_{\lambda_0,\, \lambda_1} : \Lambda \to \Lambda$ is a bijection such that $g(\lambda_0) = \lambda_1$ and such that $f \circ g \in E$ for every $f \in E$.
1. Suppose that $\Lambda \setminus \{\lambda_0\}$ is a US for E. Let $f \in E$. The following statements are equivalent:

- $f(\lambda) = 0$ for every $\lambda \in \Lambda \setminus \{\lambda_1\}$.
- $f(g(\lambda)) = 0$ for each $\lambda \in \Lambda \setminus \{\lambda_0\}$.
- $(f \circ g)(\lambda) = 0$ for every $\lambda \in \Lambda \setminus \{\lambda_0\}$.
- $f \circ g = 0$
- $f = 0$

The equivalence between the first statement and the last one says that $\Lambda \setminus \{\lambda_1\}$ is a US for E.
2. Suppose that $\Lambda \setminus \{\lambda_0\}$ is a p-SS for E. There exist constants $c,\ C > 0,\ c \leq C$, such that

$$c\, \|(f(\lambda))_{\lambda \in \Lambda \setminus \{\lambda_0\}}\|_p \leq \|f\| \leq C\, \|(f(\lambda))_{\lambda \in \Lambda \setminus \{\lambda_0\}}\|_p$$

for all $f \in E$. Let $f \in E$. Then $f \circ g \in E$, and hence we have:

$$c\, \|((f \circ g)(\lambda))_{\lambda \in \Lambda \setminus \{\lambda_0\}}\|_p \leq \|f \circ g\| \leq C\, \|((f \circ g)(\lambda))_{\lambda \in \Lambda \setminus \{\lambda_0\}}\|_p$$

this is:

$$c\, \|(f(g(\lambda))_{\lambda \in \Lambda \setminus \{\lambda_0\}}\|_p \leq \|f \circ g\| \leq C\, \|(f(g(\lambda))_{\lambda \in \Lambda \setminus \{\lambda_0\}}\|_p.$$

This is equivalent to the inequalities

$$c\, \|(f(\lambda)_{\lambda \in \Lambda \setminus \{\lambda_1\}}\|_p \leq \|f \circ g\| \leq C\, \|(f(\lambda)_{\lambda \in \Lambda \setminus \{\lambda_1\}}\|_p.$$

Considering that $A \cdot \|f \circ g\| \leq \|f\| \leq B \cdot \|f \circ g\|$, with $A,\ B$ independent of f, then $\Lambda \setminus \{\lambda_1\}$ is a p-SS for E.

□

As a consequence, we have the following result.

Corollary 3.6. *Let* $\Omega \subseteq \mathbb{R}^n,\ \Omega \neq \emptyset$*, and let* E *be a Banach space such that* $E \subseteq \mathfrak{F}(\Omega,\, \mathbb{C})$*. Let* $\Lambda \subseteq \Omega$ *be uniformly discrete and* $p \in [1,\, +\infty)$*. Assume that* Λ *verifies the p-C.P.P. for* E *and is a p-SS but is not a p-IS for* E*. Suppose that for all* $\lambda_0,\ \lambda_1 \in \Lambda,\ \lambda_0 \neq \lambda_1$ *there exists a surjective function* $g_{\lambda_0,\, \lambda_1} : \Omega \to \Omega$ *such that:*

1. $g_{\lambda_0,\, \lambda_1}(\Lambda) \subseteq \Lambda$.

2. $g_{\lambda_0,\, \lambda_1}|_\Lambda : \Lambda \to \Lambda$ *is bijective.*

3. $g_{\lambda_0,\, \lambda_1}(\lambda_0) = \lambda_1$.

4. $f \circ g_{\lambda_0,\, \lambda_1} \in E$ *for every* $f \in E$.

Then:

1. *For each* $\lambda \in \Lambda$ *we have that* $\Lambda \setminus \{\lambda\}$ *is a US for* E.

2. *Suppose that for every* $\lambda_0,\ \lambda_1 \in \Lambda,\ \lambda_0 \neq \lambda_1$ *there exist constants* $A = A(\lambda_0,\ \lambda_1),\ B = B(\lambda_0,\ \lambda_1) > 0,\ A \leq B$, *such that*
$$A \cdot \|f \circ g_{\lambda_0,\ \lambda_1}\| \leq \|f\| \leq B \cdot \|f \circ g_{\lambda_0,\ \lambda_1}\|,$$
for every $f \in E$. *Then:*
   (a) $\Lambda \setminus \{\lambda\}$ *is a* p*-SS for* E *for every* $\lambda \in \Lambda$.
   (b) *If* E *has no* p*-CIS, then for every* $r \in \mathbb{Z}^+$ *and every* $\lambda_1, ..., \lambda_r \in \Lambda$ *we have that* $\Lambda \setminus \{\lambda_1, ..., \lambda_r\}$ *is a* p*-SS for* E.

Proof. This result is an immediate consequence of Theorem 3.3 and Lemma 3.5. □

3.3. Extension of SIS

In this subsection we will show when a SIS may be extended to other SIS. In the previous section we proved that if $p \in [1,\ +\infty)$, then the p-SCIS of a complete space E are exactly the p-SS of E that cannot be refined as SS, that is, the minimal p-SS of E. We will finish this section proving that if $p \in [1,\ +\infty]$, the p-CIS of a normed space E are exactly the p-IS of E that cannot be extended as IS, that is, the maximal p-IS of E. Observe that this implies that if $p \in [1,\ +\infty]$, the p-SCIS of a Banach space E are exactly the p-SIS of E that cannot be extended as SIS, that is, the maximal p-SIS of E.

Definition 3.7 (see [1], p. 352)**.** *Let* $\Omega \subseteq \mathbb{R}^n,\ \Omega \neq \emptyset$, *and let* E *be a normed space such that* $E \subseteq \mathfrak{F}(\Omega,\ \mathbb{C})$. *Let* $p \in [1,\ +\infty]$ *and* $\Lambda \subseteq \Omega$ *be uniformly discrete. For each* $x \in \Omega$ *we define*

$$\rho(x,\ \Lambda) := \sup\{|f(x)|\ :\ f \in E,\ f|_\Lambda = 0,\ \|f\| \leq 1\} \geq 0.$$

Clearly $\rho(\ ,\ \Lambda)$ *depends on* Λ *and* E. *Observe that for every* $x \in \Omega$ *we have that the next statements are equivalent:*

- $\rho(x,\ \Lambda) = 0$.
- *For each* $f \in E$ $(f|_\Lambda = 0 \Rightarrow f(x) = 0)$.
- *For each* $f \in E$ $(f|_\Lambda = 0 \Rightarrow f|_{\Lambda \cup \{x\}} = 0)$.

Hence, Λ *is a US for* E *if and only if* $\rho(x,\ \Lambda) = 0$ *for every* $x \in \Omega$.

First consider a property of transitivity.

Lemma 3.8. *Let* $\Omega \subseteq \mathbb{R}^n,\ \Omega \neq \emptyset$, *and let* E *be a normed space such that* $E \subseteq \mathfrak{F}(\Omega,\ \mathbb{C})$. *Let* $p \in [1,\ +\infty]$ *and* $\Lambda \subseteq \Omega$ *be uniformly discrete. Let* $x_0,\ x_1 \in \Omega$. *Suppose that there exists a mapping* $g_{x_0,\ x_1} : \Omega \to \Omega$ *such that:*

1. $g_{x_0,\ x_1}(\Lambda) \subseteq \Lambda$.
2. $g_{x_0,\ x_1}(x_0) = x_1$.
3. $f \circ g_{x_0,\ x_1} \in E$ *for every* $f \in E$.

Then, for every $f \in E$ *such that* $f|_\Lambda = 0$ *there exists* $h_f \in E$ *such that* $h_f|_\Lambda = 0$ *and* $h_f(x_0) = f(x_1)$.

Proof. We define $g := g_{x_0,\ x_1}$. Let $f \in E$ such that $f|_\Lambda = 0$. Define $h_f := f \circ g$. For each $\lambda \in \Lambda$ we have that $h_f(\lambda) = f(g(\lambda)) = 0$ since $f|_\Lambda = 0$. Therefore, $h_f|_\Lambda = 0$. In addition $h_f(x_0) = f(g(x_0)) = f(x_1)$. □

The main result is the next one.

Proposition 3.9. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a normed space such that $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $p \in [1, +\infty]$ and $\Lambda \subseteq \Omega$ be uniformly discrete. Suppose that for every $x_0, x_1 \in \Omega$ there exists a bijective function $g_{x_0, x_1} : \Omega \to \Omega$ such that:*

1. $g_{x_0, x_1}(\Lambda) \subseteq \Lambda$.
2. $g_{x_0, x_1}(x_0) = x_1$.
3. $f \circ g_{x_0, x_1} \in E$ *for every* $f \in E$.

Then:

1. *If Λ is not a US for E, then $\rho(x, \Lambda) > 0$ for every $x \in \Omega \setminus \Lambda$.*
2. *If Λ is a p-IS and for E and is not a US, then $\Lambda \cup \{x_0\}$ is a p-IS for E for every $x_0 \in \Omega \setminus \Lambda$.*
3. *If Λ is a p-IS for E and cannot be extended as p-IS (that is, for each $x \in \Omega\setminus\Lambda$ we have that $\Lambda\cup\{x_0\}$ is not a p-IS), then Λ is a US for E.*
4. *If Λ is a maximal p-IS for E (that is, cannot be extended as p-IS), then Λ is a p-CIS for E. The reciprocal statement is also true.*

Proof.
1. Since Λ is not a US for E, there exists $f \in E$ such that $f|_\Lambda = 0$ and $f \neq 0$. Hence, there exists $x_1 \in \Omega$ such that $f(x_1) \neq 0$. Since $f|_\Lambda = 0$, then x_1 is not an element of Λ. So that $x_1 \in \Omega \setminus \Lambda$ and $f(x_1) \neq 0$.

Let $x \in \Omega \setminus \Lambda$. We will show that $\rho(x, \Lambda) > 0$. By the previous lemma we know that there exists $h_f \in E$ verifying $h_f|_\Lambda = 0$ and $h_f(x) = f(x_1) \neq 0$. Thus $h_f \neq 0$, and consequently $\|h_f\| \neq 0$. We define $\varphi_x := \frac{h_f}{\|h_f\|} \in E$, what verifies $\varphi_x|_\Lambda = 0$, $\varphi_x(x) \neq 0$, $\|\varphi_x\| = 1$. Hence $\rho(x, \Lambda) \geq |\varphi_x(x)| > 0$.

2. Suppose that Λ is a p-IS for E but is not a US. Let $x_0 \in \Omega \setminus \Lambda$. We will prove that $\Lambda \cup \{x_0\}$ is a p-IS for E. Let $a = (a_\lambda)_{\lambda\in\Lambda\cup\{x_0\}} \in l^p(\Lambda \cup \{x_0\})$. By hypothesis, Λ is a p-IS for E and this implies that there exists $f \in E$ such that $f(\lambda) = a_\lambda$ for every $\lambda \in \Lambda$. Since Λ is not a US for E, then there exists $l \in E$ such that $l|_\Lambda = 0$ and $l \neq 0$. Hence there exists $x_1 \in \Omega$ such that $l(x_1) \neq 0$ (and thus $x_1 \in \Omega \setminus \Lambda$). By the previous lemma we know that there exists $h_l \in E$ verifying $h_l|_\Lambda = 0$ and $h_l(x_0) = l(x_1) \neq 0$. Therefore $\|h_l\| \neq 0$. We define $\varphi_x := \frac{h_l}{\|h_l\|} \in E$, what verifies $\varphi_{x_0}|_\Lambda = 0$, $\varphi_{x_0}(x_0) \neq 0$, $\|\varphi_{x_0}\| = 1$.

We define the mapping $g : \Omega \to \mathbb{C}$ by:

$$g(x) := f(x) + \frac{a_{x_0}}{\varphi_{x_0}(x_0)}\, \varphi_{x_0}(x) \text{ for every } x \in \Omega.$$

$g \in E$ because $\varphi_{x_0} \in E$. We have: $g(x_0) = a_{x_0}$ and for every $\lambda \in \Lambda$ we have that $g(\lambda) = f(\lambda) = a_\lambda$. Hence $g \in E$ and g interpolates $a = (a_\lambda)_{\lambda\in\Lambda\cup\{x_0\}}$.

3. This is the contrareciprocal statement of: Λ is not a US $\Rightarrow$ previous item.
4. This is an immediate consequence of the previous item. □

The relationship between SS, SIS and CIS is given by the following result.

Corollary 3.10. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \emptyset$, and let E be a Banach space such that $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $p \in [1, +\infty)$ and $\Lambda \subseteq \Omega$ be uniformly discrete. Suppose that Λ verifies the p-C.P.P. for E and for every $x_0, x_1 \in \Omega$ there exists a bijective function $g_{x_0, x_1} : \Omega \to \Omega$ such that:*

1. $g_{x_0, x_1}(\Lambda) \subseteq \Lambda$.
2. $g_{x_0, x_1}(x_0) = x_1$.
3. $f \circ g_{x_0, x_1} \in E$ *for all* $f \in E$.

Then the next statements are equivalent:

1. *Λ is a minimal p-SS for E.*
2. *Λ is a maximal p-SIS for E.*
3. *Λ is a p-CIS (or what is equivalent, p-SCIS) for E.*

4. Applications to Paley-Wiener spaces

We recall the definitions of Paley-Wiener and Bernstein spaces. The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^n,$$

with the usual extension to tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 4.1. *Let $K \subseteq \mathbb{R}^n$ be a compact set and $p \in (0, +\infty]$. We write*

$$E_K^p := \{f \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp}(\hat{f}) \subseteq K \quad \text{and} \quad \|f\|_p < \infty\}.$$

This is a closed vector subspace of $(L^p(\mathbb{R}^n), \|\cdot\|_p)$, which we call the (p, K)-Paley-Wiener space, and is invariant and isometric by translations. It is well known by the Paley-Wiener theorems that the elements of E_K^p may be seen as entire functions. When $p = \infty$ we just write $B_K := (E_K^\infty, \|\,\|_\infty)$, which we call the (classical) Bernstein space with spectrum K.

We also recall the following result.

Theorem 4.2 (see [3], p. 9)**.** *Let $K \subseteq \mathbb{R}^n$ be a compact set such that K has some point with positive (Gauss) curvature, and $p \in (1, +\infty)$, $p \neq 2$. The space E_K^p has no p-CIS (or what is equivalent, has no p-SCIS).*

Corollary 4.3. *Let $p \in (1, +\infty)$, $p \neq 2$. Let $K \subseteq \mathbb{R}^n$ be a compact set such that K has some point with positive (Gauss) curvature, and $\Lambda \subseteq \mathbb{R}^n$ be u.d. Then:*

1. *Assume that Λ is a p-SS for E_K^p. Then for every $r \in \mathbb{Z}^+$ and every $\lambda_1, ..., \lambda_r \in \Lambda$ we have that $\Lambda \setminus \{\lambda_1, ..., \lambda_r\}$ is a p-SS for E_K^p.*
2. *Assume that Λ is a p-IS for E_K^p. Then for every $s \in \mathbb{Z}^+$ and every $x_1, ..., x_s \in \mathbb{R}^n \setminus \Lambda$ we have that $\Lambda \cup \{x_1, ..., x_s\}$ is a p-IS for E_K^p.*

Proof. This result is an immediate consequence of Theorem 4.2 and of Corollary 3.6. □

Corollary 4.3 is true in general for every Paley-Wiener space E_K^p, with $p \in [1, +\infty]$ and K compact set, provided that there exists no p-CIS for it. For example, if $p = \infty$ and K is an n-dimensional closed interval.

Next we adapt Corollary 3.10 to Paley-Wiener spaces.

Corollary 4.4. *Let $p \in [1, +\infty]$ and $K \subseteq \mathbb{R}^n$ be a compact set such that E_K^p has some p-CIS. Let $\Lambda \subseteq \mathbb{R}^n$ be u.d. Then the next statements are equivalent:*

1. *Λ is a minimal p-SS for E_K^p.*
2. *Λ is a maximal p-SIS for E_K^p.*
3. *Λ is a p-CIS (or what is equivalent, p-SCIS) for E_K^p.*

Proof. This result is an immediate consequence of Corollary 3.10. □

The following result is well known in sampling and interpolation in the Hilbert spaces E_S^2 being $S \subseteq \mathbb{R}^n$ a bounded set (see [4], Proposition 2.8, p. 16; and Proposition 4.6, p. 36).

Theorem 4.5. *Let $S \subseteq \mathbb{R}^n$ be a bounded set and $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. For each $\lambda \in \Lambda$ consider the function $\phi_\lambda : \mathbb{R}^n \to \mathbb{C}$ defined by*

$$x \mapsto \phi_\lambda(x) := \begin{cases} 0, & \text{if } x \notin S \\ e^{i\lambda x}, & \text{if } x \in S \end{cases}.$$

We also define the set $E(\Lambda) := \{\phi_\lambda\}_{\lambda\in\Lambda} \subseteq L^2(S)$*. Then:*

1. Λ *is a 2-SS for* E_S^2 *if and only if* $E(\Lambda)$ *is a frame for* $L^2(S)$*.*
2. Λ *is a 2-SIS for* E_S^2 *if and only if* $E(\Lambda)$ *is a Riesz sequence for* $L^2(S)$*.*
3. Λ *is a 2-CIS for* E_S^2 *if and only if* $E(\Lambda)$ *is a Riesz basis for* $L^2(S)$*.*

Corollary 4.6. *Let* $S \subseteq \mathbb{R}^n$ *be a bounded set such that there exists a 2-CIS for* E_S^2*, and* $\Lambda \subseteq \mathbb{R}^n$ *be uniformly discrete. For each* $\lambda \in \Lambda$ *consider the function* $\phi_\lambda : \mathbb{R}^n \to \mathbb{C}$ *defined by*

$$x \mapsto \phi_\lambda(x) := \begin{cases} 0, & \text{if } x \notin S \\ e^{i\lambda x}, & \text{if } x \in S \end{cases}.$$

Define the set $E(\Lambda) := \{\phi_\lambda\}_{\lambda\in\Lambda} \subseteq L^2(S)$*. Then the following statements are equivalent:*

1. $E(\Lambda)$ *is a minimal frame (this is, an exact frame) for* $L^2(S)$*.*
2. $E(\Lambda)$ *is a maximal Riesz sequence for* $L^2(S)$*.*
3. $E(\Lambda)$ *is a Riesz basis for* $L^2(S)$*.*

Proof. This result is an immediate consequence of Corollary 4.4 (what is consequence of Corollary 3.10) and of Theorem 4.5. $\square$

Observe that we can apply this result to the particular case $S \subseteq \mathbb{R}$ is a closed interval because the space E_S^2 has a CIS (in fact, infinitely many; see [2]).

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