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On Left and Right West-Stampfli Decomposition

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ABSTRACT: In this paper we define and investigate the decomposition of a Hilbert space operator T in the form $T = K + Q$ where K is a compact and the approximate points spectrum (or the surjectivity spectrum) of Q is identical to the set of all accumulation point of the approximate point spectrum (or the surjectivity spectrum) of T. Also, we provide the relation between operators having these decomposition and left (or right) Stampfli operators.

Key Words: Left and right poles of the resolvent, Upper semi-Weyl spectrum, Lower semi-Weyl spectrum.

Contents

1. Introduction and Preliminaries

In keeping with current usage $\mathcal{B}(X)$ (*resp*, $\mathcal{B}(H)$) denotes the algebra of all bounded linear operator on an infinite dimensional complex Banach (*resp*, Hilbert) space and $\mathcal{K}(X)$ the closed ideal of compact operator in $\mathcal{B}(X)$. For $T \in \mathcal{B}(X)$ the spectrum of T is denoted by $\sigma(T)$, the dimension of the null space $N(T)$ is denoted by $\alpha(T)$ and the dimension of the quotient space $X/R(T)$ is denoted by $\beta(T)$. An operator $T \in \mathcal{B}(X)$ is bounded below, if T is injective and has closed range. It is known that in the case of Hilbert space, bounded below operators are left invertible. While, surjective operators are right invertible. The approximate point spectrum of $T \in \mathcal{B}(X)$ is defined by $\sigma_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$ and the surjectivity spectrum is defined by $\sigma_s(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}$.

$\Phi_+(X) = \{T \in B(X) : R(T) \text{ is closed and } \alpha(T) < \infty\}$ is the set of upper semi-Fredholm operators and $\Phi_-(X) = \{T \in B(X) : R(T) \text{ is closed and } \beta(T) < \infty\}$ is the set of lower semi-Fredholm operators. The set of semi-Fredholm operators and the set of Fredholm operators are given respectively by $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ and $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.
The upper semi-Fredholm spectrum, the lower semi-Fredholm spectrum, the semi-Fredholm spectrum, are defined by:

$$\sigma_{uf}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\},$$
$$\sigma_{lf}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_-(X)\},$$
$$\sigma_{sf}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(X)\}.$$

It is known that $\sigma_{uf}(T) = \sigma_a(\pi(T))$ and $\sigma_{lf}(T) = \sigma_s(\pi(T))$, where π is the quotient map from $\mathcal{B}(H)$ onto the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$.
In the sequel we write $\rho_a(T) = \mathbb{C}\backslash\sigma_a(T)$, $\rho_s(T) = \mathbb{C}\backslash\sigma_s(T)$, $\Phi_+(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(X)\}$, $\Phi_-(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_-(X)\}$ and $\Phi_\pm(T) = \Phi_+(T) \cup \Phi_-(T)$.

If $T \in \Phi_\pm(X)$, the index of T is denoted by $\text{ind}(T) = \alpha(T) - \beta(T)$. It is a common knowledge that for any $T \in \Phi_\pm(X)$ and $K \in \mathcal{K}(X)$ we get $T + K \in \Phi_\pm(X)$ and $\text{ind}(T + K) = \text{ind}(T)$.
The upper semi-Weyl region, the lower semi-Weyl region and the Weyl region are defined as:

$$\Phi_0^+(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(X) : ind(\lambda I - T) \leq 0\},$$

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$$\Phi_0^-(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_-(X) : ind(\lambda I - T) \geq 0\},$$

$$\Phi_0(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi(X) : ind(\lambda I - T) = 0\}.$$

$\Phi_0^+(T)$, $\Phi_0^-(T)$ and $\Phi_0(T)$ are open and invariant under compact perturbation. Let $T \in \mathcal{B}(X)$, we recall the upper semi-Weyl spectrum, the lower semi-Weyl spectrum as well as the Weyl spectrum as follows:

$$\sigma_{uw}(T) = \sigma_a(T) \backslash \Phi_0^+(T) = \bigcap_{K \in K(X)} \sigma_a(T+K),$$

$$\sigma_{lw}(T) = \sigma_s(T) \backslash \Phi_0^-(T) = \bigcap_{K \in K(X)} \sigma_s(T+K),$$

$$\sigma_w(T) = \sigma(T) \backslash \Phi_0(T) = \bigcap_{K \in K(X)} \sigma(T+K).$$

If $\lambda I - T$ is semi-Fredholm, the minimal index of $\lambda I - T$ is defined by:

$$\min \operatorname{ind}(\lambda I - T) = \min\{\alpha(\lambda I - T), \beta(\lambda I - T)\}.$$

From [9, Corollary 1.14], the function $\lambda \longmapsto \min\{\alpha(\lambda I - T), \beta(\lambda I - T)\}$ is constant on every component of $\mathbb{C} \backslash \sigma_{sf}(T)$ (except for a denumerable subset without limit points in $\mathbb{C} \backslash \sigma_{sf}(T)$). It is clear that if $(\lambda I - T) \in \Phi_{\pm}(X)$ such that $\min \operatorname{ind}(\lambda I - T) = 0$ then $\lambda I - T$ is bounded below or surjective.
Recall from [11], the upper sub-spectrum and the lower sub-spectrum for $T \in \mathcal{B}(X)$

$$\sigma_{us}(T) = \sigma_{uf}(T) \cup \{\lambda \in \Phi_+(T) : \min \operatorname{ind}(\lambda I - T) \neq 0\},$$

$$\sigma_{ls}(T) = \sigma_{lf}(T) \cup \{\lambda \in \Phi_-(T) : \min \operatorname{ind}(\lambda I - T) \neq 0\}.$$

The ascent of $T \in \mathcal{B}(X)$ is defined by $\operatorname{asc}(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and the descent is defined by $\operatorname{des}(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$. If such n does not exist, then $\operatorname{asc}(T) = \infty$ respectively $\operatorname{des}(T) = \infty$.

We recall that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if $\lambda I - T \in \Phi_+(H)$ and $\operatorname{asc}(\lambda I - T) < \infty$. While, $\lambda \in \sigma_s(T)$ is a right pole of T of finite rank if $\lambda I - T \in \Phi_-(H)$ and $\operatorname{des}(\lambda I - T) < \infty$. It is straightforward that, λ is a left pole of T of finite rank if and only if $\overline{\lambda}$ is a right pole of T^* of finite rank, where $\overline{\lambda}$ is the conjugate of $\lambda \in \mathbb{C}$ and T^* is the Hilbert adjoint of T.

It is worth noticing that according to [6], the set of all left poles and right poles of finite rank are given respectively by:

$$\Pi_l^0(T) = \{\lambda \in \sigma_a(T) \backslash \sigma_{uw}(T) : \lambda \text{ is an isolated point in } \sigma_a(T)\},$$

$$\Pi_r^0(T) = \{\lambda \in \sigma_s(T) \backslash \sigma_{lw}(T) : \lambda \text{ is an isolated point in } \sigma_s(T)\}.$$

Clearly, $\Pi_l^0(T) = \overline{\Pi_r^0(T^*)}$ and $\Pi_r^0(T) = \overline{\Pi_l^0(T^*)}$. So $\Pi^0(T) = \Pi_l^0(T) \cap \Pi_r^0(T)$ is the set of Riesz points for T, it is well known that $\lambda \in \Pi^0(T)$ if and only if λ is isolated in $\sigma(T)$ and $(\lambda I - T) \in \Phi(X)$.

We define the upper semi-Browder spectrum, lower semi-Browder spectrum and the Browder spectrum by

$$\sigma_{ub}(T) = \sigma_a(T) \backslash \Pi_l^0(T),$$

$$\sigma_{lb}(T) = \sigma_s(T) \backslash \Pi_r^0(T),$$

$$\sigma_b(T) = \sigma(T) \backslash \Pi^0(T).$$

Before beginning our discussion we start by gathering together some results, which are all known. Recall that an operator T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP) if for every open neighborhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \to X$ which satisfies the equation

$$(\lambda I - T) f(\lambda) = 0 \text{ for all } \lambda \in U$$

is the function $f = 0$. T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.
If $\lambda I - T \in \Phi_{\pm}(H)$ then

$$T \text{ has the SVEP at } \lambda \Leftrightarrow \text{asc}(\lambda I - T) < \infty,$$
$$T^* \text{ has the SVEP at } \lambda \Leftrightarrow \text{des}(\lambda I - T) < \infty.$$

For a compact subset K of $\mathbb{C}$, we write acc(K), iso(K), int(K) and ∂K to denote the set of accumulation points, the set of isolated points, the set of interior points and the boundary of K.

It is easily checked (cf [15, Theorem 9.8.4]) that

$$\partial\sigma(T) \subseteq \sigma_{sf}(T) \cup \Pi^0(T). \tag{1.1}$$

Following [[9],[10], [11], [12]], an operator $T \in \mathcal{B}(X)$ is called Stampfli if

$$\sigma_b(T) = \text{acc}\sigma(T)$$

i.e. every isolated point of the spectrum is a pole of finite rank. Recall that T has the West-Stampfli decomposition if $T = K + Q$ where K is compact and $\sigma(Q) = \text{acc}\sigma(T)$. Moreover, we say that T obeys Stampfli's theorem if there exists a compact operator K such that $\sigma(T + K) = \sigma_w(T)$.

The concept of Stampfli operators was first introduced by Stampfli who proved that every Stampfli operator T on a Hilbert space H has the West-Stampfli decomposition this follows from Stampfli's Lemma 6 in [12]. In [10] authors have studied a generalization of West-Stampfli decomposition by constructing a Banach space operator. In [11] M. Ó Searcóid characterized Stampfli operators by means of the minimal index.

Among other results, recall from [11, Proposition 3.47] and [13, Theorem 4.4] that for a Hilbert space operators T and $\varepsilon > 0$ there always exists a compact operator $K \in \mathcal{K}(H)$, $\|K\| < \varepsilon$ for which $T + K$ is Stampfli, in this case in this case $\Pi^0(T + K) = \Pi^0(T)$. As a result every Hilbert space operator obeys Stampfli's theorem (see for instance [12], Theorem 4).

In this paper we further generalized the concept of West-Stampfli decomposition by introducing the left (*resp*, right) West-Stampfli decomposition by replacing the condition $\sigma(Q) = \text{acc}\sigma(T)$ with the condition $\sigma_a(Q) = \text{acc}\sigma_a(T)$ (*resp*, $\sigma_s(Q) = \text{acc}\sigma_s(T)$).

Definition 1.1. *We call $T = K + Q$ a left West-Stampfli decomposition if K is compact and $\sigma_a(Q) = acc\sigma_a(T)$. Simultanously, we call $T = K + Q$ a right West-Stampfli decomposition if K is compact and $\sigma_s(Q) = acc\sigma_s(T)$.*

We also extend the class of Stampfli operators to the class of left (*resp*, right) Stampfli operators as follows:

Definition 1.2. *An operator $T \in \mathcal{B}(X)$ is called left Stampfli if*

$$\sigma_{ub}(T) = acc\sigma_a(T).$$

Or equivalently, $iso\sigma_a(T) = \Pi_l^0(T)$.
An operator $T \in \mathcal{B}(X)$ is called right Stampfli if

$$\sigma_{lb}(T) = acc\sigma_s(T).$$

Or equivalently, $iso\sigma_s(T) = \Pi_r^0(T)$.

Often what we call here left (*resp*, right) Stampfli operator is commonly known as finitely left (*resp*, right) Polaroid.

An approximate point version of Stampfli's theorem is given by the so-called a-Stampfli's theorem.

Definition 1.3. *$T \in \mathcal{B}(X)$ is said to satisfy a-Stampfli's theorem if there exists a compact operator K such that $\sigma_a(T + K) = \sigma_{uw}(T)$.*

This paper is organized as follows, the second section of this paper begins by characterizing left and right Stampfli operators by means of the minimal index. We prove that Hilbert space operators having left (*resp*, right) West-Stampfli decomposition are left (*resp*, right) Stampfli operators. We also study the correlation between left or right West-Stampfli decomposition and West-Stampfli decomposition. In the third section we prescribe some conditions for which a left (*resp*, right) Stampfli operator T on a Hilbert space H has the left (*resp*, right) West-Stampfli decomposition.

2. left and right West-Stampfli decomposition

In order to characterize the left and right Stampfli operators with respect to the minimal index, we start this section by the following two lemmas.

Lemma 2.1. *If $T \in \mathcal{B}(X)$ then $\Phi_{\pm}(T) \cap \sigma(T) \subseteq iso\sigma(T)$.*

Proof. Let $\lambda \in \Phi_{\pm}(T) \cap \sigma(T)$ then $\lambda \in \Phi_{\pm}(T)$ and either $\lambda \in \partial\sigma(T)$ or $\lambda \in \text{int}\sigma(T)$. If $\lambda \in \partial\sigma(T)$ then by (1.1), $\lambda \in \text{iso}\sigma(T)$. If $\lambda \in \text{int}\sigma(T)$ then $\lambda \in \text{acc}\sigma(T)$ hence $\lambda \in \text{acc}\partial\sigma(T)$, from (1.1) it then follows that $\lambda \in \sigma_{sf}(T)$ which is a contradiction. So we deduce that $\lambda \in \text{iso}\sigma(T)$. □

From [1, Theorem 1.22] and [1, Corollary 4.13] we have the following lemma.

Lemma 2.2. *If $T \in \mathcal{B}(X)$ is either left or right Stampfli then T is Stampfli.*

Proposition 2.1. *Let $T \in \mathcal{B}(X)$*

(i) If T is Stampfli then $\{\lambda \in \Phi_{\pm}(T) : \text{ min ind } (\lambda I - T) \neq 0\} = \Pi^0(T)$.

(ii) If T is left Stampfli then $\{\lambda \in \Phi_{+}(T) : \text{ min ind } (\lambda I - T) \neq 0\} = \Pi^0_l(T)$.

(iii) If T is right Stampfli then $\{\lambda \in \Phi_{-}(T) : \text{ min ind } (\lambda I - T) \neq 0\} = \Pi^0_r(T)$.

Proof. (i) suppose that T is Stampfli then $\text{iso}\sigma(T) = \Pi^0(T)$. It is clear that

$$\{\lambda \in \Phi_{\pm}(T) : \text{ min ind } (\lambda I - T) \neq 0\} \subseteq \Phi_{\pm}(T) \cap \sigma(T).$$

then by lemma 2.1, $\{\lambda \in \Phi_{\pm}(T) : \text{ min ind } (\lambda I - T) \neq 0\} \subseteq \text{iso}\sigma(T) = \Pi^0(T)$. Conversely, if $\lambda \in \Pi^0(T)$ then $\lambda \in \Phi_{\pm}(T)$ and $\alpha(\lambda I - T) = -\beta(\lambda I - T) \neq 0$ (because $\lambda I - T$ is Weyl and $\lambda \in \sigma(T)$), hence $\lambda \in \{\lambda \in \Phi_{\pm}(T) : \text{ min ind } (\lambda I - T) \neq 0\}$.
(ii) suppose that T is left Stampfli, then by lemma 2.2 T is Stampfli thus

$$\{\lambda \in \Phi_{\pm}(T) : \text{ min ind } (\lambda I - T) \neq 0\} = \Pi^0(T).$$

On the other hand

$$\begin{aligned}\{\lambda \in \Phi_{+}(T) : \text{min ind}(\lambda I - T) \neq 0\} &\subseteq \{\lambda \in \Phi_{\pm}(T) : \text{min ind}(\lambda I - T) \neq 0\} = \Pi^0(T),\\ &\subseteq \Pi^0_l(T).\end{aligned}$$

Conversely, if $\lambda \in \Pi^0_l(T)$ then $\lambda \in \Phi_{+}(T)$ and $\alpha(\lambda I - T) \leq \beta(\lambda I - T)$, because $\lambda I - T$ is upper semi-Weyl. Then $\text{min ind}(\lambda I - T) = \alpha(\lambda I - T) \neq 0$ since $\lambda \in \sigma_a(T)$.
(iii) The case of right Stampfli may be proved in a similar way. □

Remark 2.2. *As a consequence of the proposition 2.1, we have $\sigma_{ls}(T) = \overline{\sigma_{us}(T^*)}$ for every right stampfli operator $T \in \mathcal{B}(H)$.*

The concept of left and right Stampfli operators are dual each other ([1, Theorem 4.8]).

Corollary 2.3. *$T \in \mathcal{B}(H)$ is left Stampfli (resp, right Stampfli) if and only if T^* is right Stampfli (resp, left Stampfli).*

Lemma 2.3. *Let $T \in \mathcal{B}(H)$.*

(i) If T has the left West-Stampfli decomposition then T is left Stampfli.

(ii) If T has the right West-Stampfli decomposition then T is right Stampfli.

Proof. (i) Let $T = K + Q$ with K is compact and $\sigma_a(Q) = acc\sigma_a(T)$, then for every isolated point λ of $\sigma_a(T)$, $\lambda \notin \sigma_a(Q)$ i.e. $\lambda I - Q$ is bounded below. We have $\lambda I - T = \lambda I - Q - K$, since $\pi(\lambda I - T) = \pi(\lambda I - Q)$ is left invertible, then $\lambda I - T \in \Phi_+(H)$. On the other hand T has the SVEP at λ, which implying that $\text{asc}(\lambda I - T) < \infty$, it follows that λ is a left pole of T of finite rank.
(ii) Suppose now that $T = K + Q$ with K compact and $\sigma_s(Q) = acc\sigma_s(T)$, then for every isolated point λ of $\sigma_s(T)$, $\lambda \notin \sigma_s(Q)$ i.e. $\lambda I - Q$ is onto. Since $\pi(\lambda I - T) = \pi(\lambda I - Q)$ is right invertible, then $\lambda I - T \in \Phi_-(H)$, so with a standard duality we have; $\sigma_s(T)$ does not cluster the point λ if and only if $\sigma_a(T^*)$ does not cluster the point $\overline{\lambda}$ if and only if T^* has the SVEP at $\overline{\lambda}$ if and only $\text{des}(\lambda I - T) < \infty$. Consequently, λ is a right pole of T of finite rank . □

Corollary 2.4. *If either $T \in \mathcal{B}(H)$ has left or right West-Stampfli decomposition then T has West-Stampfli decomposition.*

Proof. Let $T \in \mathcal{B}(H)$. Indeed there are implications

$$\begin{aligned} T \text{ has a left West-Stampfli decomposition} &\Rightarrow T \text{ is a left Stampfli operator,}\\ &\Rightarrow T \text{ is a Stampfli operator,}\\ &\Rightarrow T \text{ has a West-Stampfli decomposition.} \end{aligned}$$

Where the first implication can be ascertained by lemma 2.3, the second follows from lemma 2.2, the third is by lemma 6 in [12].

Also, if T has a right West-Stampfli decomposition then T is a right Stampfli operator, equivalently T^* is a left Stampfli operator. By the first part and since $\sigma(T) = \sigma(T^*)$, T has a West-Stampfli decomposition. □

The following example is fitting with the above corollary, it shows that the reverse can not be occur.

Example 1. Let $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral shift and $Q : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is the weighted unilateral shift defined by:

$$Q(x_1, x_2, ...) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_{3,...}), \qquad (x_1, x_2, ...) \in l^2(\mathbb{N})$$

Q is a compact quasi-nilpotent operator and 0 is not an accumulation point of the spectrum of Q.
If we let $T = U \oplus Q \in B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$ then it is clear that

$$\sigma(T) = \sigma(U) \cup \sigma(Q) = \mathbb{D},$$

where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, as result $\text{iso}\sigma(T) = \emptyset$. Thus, T is a Stampfli operator. Let $K \in B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$ be the compact operator $K = 0 \oplus -Q$ then $(U \oplus 0) - K$ is a West-Stampfli decomposition of T. Indeed,

$$\sigma(T + K) = \sigma(U) = \mathbb{D} = \text{acc}\sigma(T).$$

On the other hand, $\sigma_a(T) = \sigma_a(U) \cup \sigma_a(Q) = \partial\mathbb{D} \cup \{0\}$ and 0 is not an accumulation point of the approximate spectrum of Q.
For $e_n = (0, ..., 0, 1, 0, ...)$ where 1 is the n-th term, $e_{n+1} \in \text{Ker } Q^{n+1}$ while $e_{n+1} \notin \text{Ker } Q^n$, so $\text{asc}(Q) = \infty$, and asc(T)=max{asc(U),asc(Q)} $= \infty$.
$0 \in \text{iso}\sigma_a(T)$ but 0 is not a left pole of T. Consequently, T is not a left Stampfli operator.

Suppose now that T has the left West-Stampfli decomposition then by lemma 2.3 T is left Stampfli which leads to a contradiction.

In addition, T^* has not the right West-Stampfli decomposition even if T^* has the West-Stampfli decomposition.

Corollary 2.5. *If $T \in \mathcal{B}(H)$ has both left and right West-Stampfli decomposition then T has the West-Stampfli decomposition.*

Proof. Suppose that $T \in \mathcal{B}(H)$ has both left and right West-Stampfli decomposition then Lemma 2.3 entails that T is both left and right Stampfli. By [2, Theorem 2.6] it follows that T is Stampfli which implies by [12, Lemma 6] that T has West-Stampfli decomposition. □

It makes sens to find condition for which the reverse of lemme 2.3 is also true.

3. a-Stampfli's theorem

To prove the main results of this section, some preparation is needed. In fact, we have the following definition.

Definition 3.1. *An operator $T \in \mathcal{B}(H)$ is said to have the left compact correction if there exists a compact operator $K \in \mathcal{K}(H)$ such that $\sigma_{us}(T+K) = \sigma_{uf}(T)$.*

For a Hilbert space operator T, authors in [7] claim that necessary and sufficient condition for $T+K$ to be left Stampfli (*resp*, right Stampfli) for every compact operator K, is that $\sigma_{uw}(T)$ (*resp*, $\sigma_{lw}(T)$) is connected. The connectedness of $\sigma_{uw}(T)$ implies that either iso$\sigma_{uw}(T) = \emptyset$ or $\sigma_{uw}(T) = \{0\}$.

Lemma 3.1 ([7], Theorem 6.4)**.** *Let $T \in B(H)$ then, the following equivalences hold:*

(i) iso$\sigma_{uw}(T) = \emptyset$ if and only if $T+K$ is left Stampfli for every compact operator K.

(ii) iso$\sigma_{lw}(T) = \emptyset$ if and only if $T+K$ is right Stampfli for every compact operator K.

The following results are based on the foregoing lemma, they generalize corollary 4.5 and corollary 4.6 in [11].

Theorem 3.2. *Suppose that every left Stampfli operator on H has the left West-Stampfli decomposition. Then a-Stampfli's theorem holds for every operator having a connected upper-semi Weyl spectrum.*

Proof. Let $T \in \mathcal{B}(H)$ such that iso$\sigma_{uw}(T) = \emptyset$ and let $K_1 \in \mathcal{K}(H)$. From lemma 3.1, $T+K_1$ is left Stampfli, it follows that $T+K_1$ has the left West-stampfli decomposition i.e. there exists a compact operator K_2 such that

$$\sigma_a(T+K_1+K_2) = \text{acc}\sigma_a(T+K_1) = \sigma_a(T+K_1)\backslash\Pi_l^0(T+K_1).$$

For $K = K_1 + K_2$ we must show that

$$\sigma_{uw}(T) = \sigma_a(T+K),$$

that is,

$$\sigma_a(T+K)\cap\Phi_0^+(T+K) = \sigma_a(T+K)\cap\Phi_0^+(T+K_1) = \emptyset.$$

For the sake of contradiction assume that $\sigma_a(T+K)\cap\Phi_0^+(T+K_1) \neq \emptyset$, so there is $\lambda \in \sigma_a(T+K)\cap\Phi_0^+(T+K_1) = (\sigma_a(T+K_1)\backslash\Pi_l^0(T+K_1))\cap\Phi_0^+(T+K_1)$. Since $T+K_1$ is a left Stampfli operator then $\lambda \in \sigma_a(T+K_1)\backslash\Pi_l^0(T+K_1)$ implies that $\lambda \in \sigma_a(T+K_1)$ and $\lambda \notin \{\lambda \in \Phi_+(T) : \text{ min ind } (T+K-\lambda I) \neq 0\}$, in other word $T+K_1-\lambda I \in \Phi_+(H)$ and min ind$(T+K_1-\lambda I) = 0$.
Besides that $\lambda \in \Phi_0^+(T+K_1)$, thus ind$(T+K_1-\lambda) < 0$, so $T+K_1-\lambda$ is bounded below. Hence $\lambda \notin \sigma_a(T+K_1)$ and $\lambda \in \sigma_a(T+K_1)$ which is a contradiction. □

Theorem 3.3. *Suppose that every left Stampfli operator on H has the left West-Stampfli decomposition. Then every operator with connected upper semi-Weyl spectrum has a left compact correction.*

Proof. Let $T \in \mathcal{B}(H)$ such that iso$\sigma_{uw}(T) = \emptyset$ then according to theorem 3.2 there exists $K \in \mathcal{K}(H)$ such that $\sigma_a(T+K) = \sigma_{uw}(T)$. We want to show that

$$\sigma_{us}(T+K) = \sigma_{uf}(T)$$

i.e. $\{\lambda \in \Phi_+(T+K) : \min \operatorname{ind}(T+K-\lambda I) \neq 0\} = \emptyset$, since iso$\sigma_{uw}(T) = \emptyset$ then from lemma 3.1 $T+K$ is a left Stampfli operator which implying that,

$$\Pi_l^0(T+K) = \{\lambda \in \Phi_+(T+K) : \min \operatorname{ind}(T+K-\lambda I) \neq 0\}.$$

By contradiction assume that there exists $\lambda \in \Pi_l^0(T+K)$ it follows that $\lambda \in \Phi_0^+(T+K)$. Furthermore $\Pi_l^0(T+K) \subset \sigma_a(T+K)$, then also $\lambda \in \sigma_a(T+K)$, that leads to a contradiction of $\sigma_a(T+K) \cap \Phi_0^+(T+K) = \emptyset$. □

Contrary to [9, Proposition 3.47] and [11, Theorem 4.4], there is no relation between $\Pi_l^0(T)$ and $\Pi_l^0(T+K)$ where K is a compact operator even if $\sigma_{uw}(T)$ is connected as it is shown in the following example. So the reverse of Theorem 3.2 is not true in general.

Example 2. Let $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral shift and let $Q : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the compact operator defined by

$$Q(x_1, x_2, x_3, ...) = (-\frac{x_1}{2}, 0, 0, ...), \qquad (x_1, x_2, x_3, ...) \in l^2(\mathbb{N})$$

set $T = U \oplus I$ and $K = 0 \oplus Q$. T and K are operators defined on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$. Then iso$\sigma_{uw}(T) = \emptyset$, and

$$\text{iso}\sigma_a(T) = \Pi_l^0(T) = \emptyset \neq \{\frac{1}{2}\} = \text{iso}\sigma_a(T+K) = \Pi_l^0(T+K).$$

Proposition 3.4. *Suppose that $T \in \mathcal{B}(H)$ is a left Stampfli operator such that iso$\sigma_{uw}(T) = \emptyset$ and $\Pi_l^0(T) = \Pi_l^0(T+K)$ for every compact operator K. If T has the left compact correction then T has the left West-Stampfli decomposition.*

Proof. Let $T \in \mathcal{B}(H)$ be a left Stampfli operator, hypothetically there exists a compact operator K such that $\sigma_{us}(T+K) = \sigma_{uf}(T)$ i.e. there exists a compact operator K such that $\{\lambda \in \Phi_+(T+K) : \min \operatorname{ind}(T+K-\lambda) \neq 0\} = \emptyset$.
Since iso$\sigma_{uw}(T) = \emptyset$ then $T+K$ is a left Stampfli operator. In this case

$$\Pi_l^0(T+K) = \Pi_l^0(T) = \{\lambda \in \Phi_+(T+K) : \min \operatorname{ind}(T+K-\lambda) \neq 0\} = \emptyset.$$

On the other hand

$$\rho_a(T) \subset \Phi_+(T) \backslash \Pi_l^0(T) = \Phi_+(T) \backslash \Pi_l^0(T+K) = \rho_{us}(T+K) \subset \rho_a(T+K).$$

Thus $\sigma_a(T+K) \subset \sigma_a(T) = \sigma_a(T) \backslash \Pi_l^0(T)$.

□

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References

1. P. Aiena, *Fredholm and local spectral theory II with application to Weyl-type theorems.* Springer, (2018).
2. P. Aiena, E. Aponte, E. Bazan, *Weyl type theorems for left and right polaroid operators*, Int. Equa. Oper. Theory. 66, 1-20, (2010).
3. P.Aiena , JE Sanabria, *On left and right poles of the resolvent.* Acta Sci. Math.(Szeged). 74, 669-687, (2008).
4. C. Bensalloua, M. Nadir, *General note on the theorem of Stampfli*, M. J Inequal Appl. 2016:55, (2016).
5. J. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, Springer, (1990).
6. B.P. Duggal, *Isolated eigenvalues, poles and compact perturbations of Banach space operators*, arXiv:1808.03542v2. (2018).
7. B.P. Duggal, H. Kim, *Generalized Browder, Weyl spectra and the polaroid property under compact perturbations*, J. Korean Math. Soc. 54, 281-302, (2017).

8. R.E. Harte, *Invertibility and singularity for bounded linear operators*, Marcel Dekker, New york, (1988).

9. Herrero, DA, *Approximation of Hilbert Space Operators, vol. 1*, Research Notes in Mathematics, vol. 72. Pitman, London (1982).

10. J.J. Koliha, P. W. Poon , *On West and Stampfli decomposition*, Acta Sci.Math. (Szeged). 63, 181-194, (1997).

11. Mícheal Ó Searcóid, *A contribution to the solution of the compact correction problem for operators on a Banach space*, Glasgow Math. J. 31, 219-229, (1989).

12. J. G. Stampfli, *Compact perturbations, normal eigenvalues and a problem of Salinas*, J. London Math. Soc. (2) 9, 165-175, (1974).

13. V. Rakŏcević, *Approximate point spectrum and commuting compact perturbations*, Glasgow Math.J. 28, 193-198, (1986).

14. T.J. Laffey and T.T. West, *Fredholm commutators*, Proc. Roy. Irish Acad. Sect. A, 82, 129-140, (1982).

15. S. Zhu, C.G. Li, *SVEP and compact perturbations*, J. Math. Anal. Appl. 380, 69-75, (2011).

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