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Improved Convergence Ball and Error Analysis of Müller's Method

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ABSTRACT: We present an improved convergence analysis of Müller's method for solving nonlinear equation under conditions that the divided differences of order one of the involved function satisfy the Lipschitz conditions. Our result improves the earlier work in literature. Numerical examples are presented to illustrate the theoretical results.

Key Words: Müller's method, Free-derivative method, Local convergence analysis, Lipschitz conditions.

Contents

1. Introduction

In this study we are concerned with the convergence analysis of Müller's method which is used to solve the following equation

$$f(x) = 0, \tag{1.1}$$

where f is defined on an open domain or closed domain D on a real space $\mathbb{R}$.

Many problems in Computational Sciences and other disciplines can be brought in a form like (1.1) using mathematical modelling [1]. The solutions of these equations can be rarely be found in closed form. That is why most solution methods for these equations are usually iterative.

The study about convergence of iterative procedures is normally centered on two types: semi–local and local convergence analysis. The semi–local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure. While the local analysis is based on the information around a solution, to find estimates of the radii of convergence balls.

The famous Müller's method is defined in [2] by

$$x_{n+1} = x_n - \frac{2C_n}{B_n \pm \sqrt{B_n^2 - 4A_nC_n}}, \quad n = 0, 1, 2, \ldots, \quad x_{-2}, x_{-1}, x_0 \in D, \tag{1.2}$$

where,

$$A_n = f[x_n, x_{n-1}, x_{n-2}], \quad B_n = f[x_n, x_{n-1}] + A_n(x_n - x_{n-1}), \quad C_n = f(x_n), \tag{1.3}$$

and $f[\cdot,\cdot]$, $f[\cdot,\cdot,\cdot]$ are divided differences of order one and two, respectively (see [3]). The sign in the denominator of (1.2) is chosen so as to give the larger value.

Müller's method is widely used [2,4]. It is a free–deravative method and has a convergence order $1.839\ldots$ under reasonable conditions [2]. Xie [5] established a semilocal convergence theorem of the method under bounded third and fourth derivatives. Bi et.al [6] presented a new semilocal convergence theorem of the method under γ–condition. Wu et.al [7] gave the convergence ball and error analysis of the method under the hypotheses that the second-order and third–order derivative of function f are bounded.

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In this paper, we provide a new estimate on the radius of convergence ball of Müller's method under the weaker conditions than the coresponding conditions in [7]. In fact, we assume that f is differentiable in D, $f'(x_\star) \neq 0$, and the following Lipschitz conditions are true:

$$|f'(x_\star)^{-1}(f[x,y]-f[u,v])| \leq K(|x-u|+|y-v|), \quad \text{for any} \quad x,y,u,v \in D \tag{1.4}$$

and

$$|f'(x_\star)^{-1}(f[x,x_\star]-f[x_\star,x_\star])| \leq K_\star|x-x_\star|, \quad \text{for any} \quad x \in D, \tag{1.5}$$

where, $K>0$ and $K_\star>0$ are constants, and $x_\star \in D$ is a solution of (1.1).

The paper is organized as follows: Section 2 contains the convergence ball analysis of method (1.2). The numerical examples including favorable comparisons with earlier study [7] are presented in the concluding Section 3.

2. Improved convergence ball analysis of method (1.2)

We present the local convergence of method (1.2) in this section. Denote $U(x,r)$ as an open ball around x with radius r. We have:

Theorem 2.1. *Suppose $x_\star \in D$ is a solution of Eq. (1.1), $f'(x_\star) \neq 0$, conditions (1.4) and (1.5) are satisfied. Denote*

$$R' = \frac{1}{5K+K_\star+2\sqrt{4K^2+2KK_\star}}. \tag{2.1}$$

Assume

$$U(x_\star,R') \subseteq D. \tag{2.2}$$

Then, the sequence $\{x_n\}$ generated by Müller's method (1.2) starting from any three distinct points $x_{-2},x_{-1},x_0 \in U(x_\star,R')$ is well defined, and converges to $x_\star$. Moreover, the following estimates hold:

$$|x_\star-x_{n+1}| \leq (\frac{C}{R'})^{F_{n+1}-1}|x_\star-x_0|^{F_n}|x_\star-x_{-1}|^{F_{n-1}}, \quad \textit{for any} \quad n=0,1,2,\ldots, \tag{2.3}$$

where,

$$C = \frac{2K}{\sqrt{4K^2+2KK_\star}},$$

and $\{F_n\}$ is Fibonacci sequence, and is defined by $F_{-1}=1, F_0=1$ and $F_{n+1}=F_n+F_{n-1}$ for any $n=0,1,2,\ldots$

Proof. We will prove the theorem by induction. Denote $e_n = x_\star - x_n (n=-2,-1,\ldots)$. Let $x_{-2},x_{-1},x_0 \in U(x_\star,R')$ be distinct points. By (1.3) and (1.4), we have

$$|f'(x_\star)^{-1}A_0| = |f'(x_\star)^{-1}f[x_0,x_{-1},x_{-2}]| = |f'(x_\star)^{-1}\frac{f[x_0,x_{-2}]-f[x_{-1},x_{-2}]}{x_0-x_{-1}}| \leq K. \tag{2.4}$$

Using $x_{-1},x_0 \in U(x_\star,R')$, (1.3), (1.5) and (2.4), we have

$$\begin{aligned}
|1-f'(x_\star)^{-1}B_0| &= |1-f'(x_\star)^{-1}(f[x_0,x_{-1}]+A_0(e_{-1}-e_0))| \\
&= |f'(x_\star)^{-1}(f[x_\star,x_\star]-f[x_\star,x_0]+f[x_0,x_\star]-f[x_0,x_{-1}]+A_0(e_0-e_{-1}))| \\
&\leq K_\star|e_0|+K|e_{-1}|+K(|e_0|+|e_{-1}|) \\
&= (K+K_\star)|e_0|+2K|e_{-1}| \\
&< (3K+K_\star)R' = \frac{3K+K_\star}{5K+K_\star+2\sqrt{4K^2+2KK_\star}} < 1,
\end{aligned} \tag{2.5}$$

which means

$$|f'(x_\star)^{-1}B_0| > 1-(3K+K_\star)R' = \tfrac{2K+2\sqrt{4K^2+2KK_\star}}{5K+K_\star+2\sqrt{4K^2+2KK_\star}} > 0 \tag{2.6}$$

and

$$|B_0^{-1}f'(x_\star)| < \frac{1}{1-(3K+K_\star)R'} = \frac{5K+K_\star+2\sqrt{4K^2+2KK_\star}}{2K+2\sqrt{4K^2+2KK_\star}}. \tag{2.7}$$

Using $x_0 \in U(x_\star, R')$, $f(x_\star) = 0$, (1.3) and (1.5), we have

$$\begin{aligned} |f'(x_\star)^{-1}C_0| &= |f'(x_\star)^{-1}(f(x_0)-f(x_\star))| = |f'(x_\star)^{-1}(f[x_\star,x_0]-f[x_\star,x_\star]+f[x_\star,x_\star])e_0| \\ &\le K_\star|e_0|^2+|e_0| < K_\star R'^2+R'. \end{aligned} \tag{2.8}$$

In view of (2.4), (2.6), (2.8) and (2.1), we have

$$\begin{aligned} &(f'(x_\star)^{-1})^2B_0^2-4(f'(x_\star)^{-1})^2A_0C_0 \ge |f'(x_\star)^{-1}B_0|^2-4|f'(x_\star)^{-1}A_0||f'(x_\star)^{-1}C_0| \\ &> (1-(3K+K_\star)R')^2-4K(K_\star R'^2+R') \\ &= [(3K+K_\star)^2-4KK_\star]R'^2-[2(3K+K_\star)+4K]R'+1 \\ &= (9K^2+2KK_\star+K_\star^2)R'^2-2(5K+K_\star)R'+1=0. \end{aligned} \tag{2.9}$$

Hence, x_1 can be defined. Denote $sign(s)$ as an sign function, i.e., $sign(s)=1$ for $s\ge 0$ and $sign(s)=-1$ for $s<0$. By (1.2), we have

$$\begin{aligned} e_1 &= e_0+\frac{2C_0}{B_0+sign(B_0)\sqrt{B_0^2-4A_0C_0}} = e_0+\frac{B_0-sign(B_0)\sqrt{B_0^2-4A_0C_0}}{2A_0} \\ &= \frac{2A_0e_0+B_0-sign(B_0)\sqrt{B_0^2-4A_0C_0}}{2A_0} = \frac{(2A_0e_0+B_0)^2-B_0^2+4A_0C_0}{2A_0(2A_0e_0+B_0+sign(B_0)\sqrt{B_0^2-4A_0C_0})} \\ &= \frac{2(A_0e_0^2+B_0e_0+C_0)}{2A_0e_0+B_0+sign(B_0)\sqrt{B_0^2-4A_0C_0}}, \end{aligned} \tag{2.10}$$

so

$$|e_1| = \frac{2|A_0e_0^2+B_0e_0+C_0|}{|2A_0e_0+B_0+sign(B_0)\sqrt{B_0^2-4A_0C_0}|}. \tag{2.11}$$

Next we shall show $2A_0e_0+B_0$ has the same sign as B_0. In fact, using (2.4), (2.1) and (2.7), we have

$$\begin{aligned} \frac{2A_0e_0+B_0}{B_0} &= 1+2f'(x_\star)^{-1}A_0f'(x_\star)B_0^{-1}e_0 \ge 1-2|f'(x_\star)^{-1}A_0||f'(x_\star)B_0^{-1}||e_0| \\ &> 1-\frac{2K(5K+K_\star+2\sqrt{4K^2+2KK_\star})}{2K+2\sqrt{4K^2+2KK_\star}}R' \\ &= \frac{\sqrt{4K^2+2KK_\star}}{K+\sqrt{4K^2+2KK_\star}} > 0. \end{aligned} \tag{2.12}$$

Therefore, by (2.11), (1.3), (2.4), (1.4), (2.9), (2.6) and (2.12), we have

$$\begin{aligned} |e_1| &= \frac{2|A_0e_0^2+B_0e_0+C_0|}{|2A_0e_0+B_0|+\sqrt{B_0^2-4A_0C_0}} = \frac{2|(A_0e_0+B_0)e_0-f[x_\star,x_0]e_0|}{|2A_0e_0+B_0|+\sqrt{B_0^2-4A_0C_0}} \\ &= \frac{2|A_0e_{-1}+A_0(e_0-e_{-1})+f[x_0,x_{-1}]-A_0(e_0-e_{-1})-f[x_0,x_\star]|}{|2A_0e_0+B_0|+\sqrt{B_0^2-4A_0C_0}}|e_0| \\ &= \frac{2|f'(x_\star)^{-1}(A_0e_{-1}+f[x_0,x_{-1}]-f[x_0,x_\star])|}{|f'(x_\star)^{-1}B_0|\frac{2A_0e_0+B_0}{B_0}+\sqrt{(f'(x_\star)^{-1}B_0)^2-4f'(x_\star)^{-1}A_0f'(x_\star)^{-1}C_0}}|e_0| \\ &\le \frac{4K|e_{-1}|}{\frac{2K+2\sqrt{4K^2+2KK_\star}}{5K+K_\star+2\sqrt{4K^2+2KK_\star}}\frac{\sqrt{4K^2+2KK_\star}}{K+\sqrt{4K^2+2KK_\star}}}|e_0| \\ &= \frac{2K(5K+K_\star+2\sqrt{4K^2+2KK_\star})}{\sqrt{4K^2+2KK_\star}}|e_0||e_{-1}|. \end{aligned} \tag{2.13}$$

So, by the definition of C in this theorem, we have

$$\frac{|e_1|}{R'} \leq \frac{2K}{\sqrt{4K^2+2KK_\star}} \frac{|e_0|}{R'} \frac{|e_{-1}|}{R'} = C\frac{|e_0|}{R'} \frac{|e_{-1}|}{R'} < 1, \tag{2.14}$$

that is, we have $x_1 \in U(x_\star, R')$.

Following now an inductive procedure on $n = 0, 1, 2, \ldots$, we have $x_n \in U(x_\star, R')$, and

$$\frac{|e_{n+1}|}{R'} \leq C\frac{|e_n|}{R'} \frac{|e_{n-1}|}{R'}, \quad \text{for any} \quad n = 0, 1, 2, \ldots \tag{2.15}$$

Denote

$$\rho_n = C\frac{|e_n|}{R'}, \quad n = 0, 1, 2, \ldots, \tag{2.16}$$

then we have

$$\rho_{n+1} \leq \rho_n \rho_{n-1}, \quad n = 0, 1, 2, \ldots \tag{2.17}$$

It is easy to see that the following ralation holds:

$$\rho_{n+1} \leq \rho_0^{F_n} \rho_{-1}^{F_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{2.18}$$

which means estimates (2.3) are true. □

Remark 2.2 (a) It is easy to see that the condition (1.4) used in Theorem 2.1 is weaker than bounded conditions of the second–order and third–order derivative of function f used in [7]. In fact, suppose that f is twice differentiable on D, and

$$|f'(x_\star)^{-1} f''(x)| \leq N, \quad \text{for any} \quad x \in D, \tag{2.19}$$

then, for any $x, y, u, v \in D$, we have

$$\begin{aligned} &|f'(x_\star)^{-1}(f[x, y] - f[u, v])| = |f'(x_\star)^{-1} \int_0^1 (f'(tx + (1-t)y) - f'(tu + (1-t)v))dt| \\ &= |f'(x_\star)^{-1} \int_0^1 \int_0^1 f''\big(s(tx + (1-t)y) + (1-s)(tu + (1-t)v)\big)\big(t(x-u) + (1-t)(y-v)\big)dsdt| \\ &\leq \tfrac{N}{2}(|x-u| + |y-v|), \end{aligned} \tag{2.20}$$

which shows condition (1.4) holds for $K = \frac{N}{2}$. Notice also that we cannot obtain condition (2.19) from condition (1.4), see Example 3.3 in Section 3.

(b) Note that a semilocal convergence theorem for Müller's method is given in [6] under the γ–condition of order two which is a condition of third–order derivative of function f, see [6]. However, we don't use any information of f''' in our theorem.

Remark 2.3 Notice that from (1.4) and (1.5)

$$K_\star \leq K \tag{2.21}$$

holds in general and $\frac{K}{K_\star}$ can be arbitrarily large [1]. Hence, the condition $K_\star \leq K$ in Theorem 2.1 involves no loss of generality. If we use only condition (1.4) in Theorem 2.1, we can establish a similar theorem by replacng (2.1) and (2.3) by

$$\overline{R}' = \frac{1}{2(3+\sqrt{6})K} \tag{2.22}$$

and

$$|x_\star - x_{n+1}| \leq ((4+2\sqrt{6})K)^{F_{n+1}-1} |x_\star - x_0|^{F_n} |x_\star - x_{-1}|^{F_{n-1}}, \ \textit{for any } n = 0, 1, 2, \ldots, \tag{2.23}$$

respectively. Moreover, if (2.21) holds strictly, it is easy to see that the following inequalities are true:

$$\overline{R}' < R' \tag{2.24}$$

and

$$(4+2\sqrt{6})K < \frac{C}{R'} = \frac{2K(5K+K_\star)}{\sqrt{4K^2+2KK_\star}} + 4K, \tag{2.25}$$

which means that we have bigger radius of convergence ball of Müller's method and tighter errors by using conditions (1.4) and (1.5) simultaneously instead of using only condition (1.4).

3. Numerical examples

In this section, we present some examples.

Example 3.1. *Let f be defined on $D = [-\frac{5}{2}, \frac{1}{2}]$ by*

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4 + 4x, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{3.1}$$

Then, we have

$$f'(x) = \begin{cases} 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2 + 4, & x \neq 0, \\ 4, & x = 0, \end{cases} \tag{3.2}$$

$$f''(x) = \begin{cases} 6x \ln x^2 + 20x^3 - 12x^2 + 10x, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{3.3}$$

and

$$f'''(x) = \begin{cases} 6 \ln x^2 + 60x^2 - 24x + 22, & x \neq 0, \\ -\infty, & x = 0, \end{cases} \tag{3.4}$$

which means f'' is continuous on D and f''' is unbounded on D. Then, Theorem 2.1 applies. However, the theorem in [7] cannot apply.

Example 3.2. *Let f be defined on $D = [-1, 1]$ by*

$$f(x) = e^x - 1. \tag{3.5}$$

Then, we have

$$f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x. \tag{3.6}$$

It is obvious that $x_\star = 0$, $f'(x_\star) = 1$ and for any $x, y, u, v \in D$, we have

$$\begin{aligned} &|f'(x_\star)^{-1}(f[x, y] - f[u, v])| = |\int_0^1 (f'(tx + (1-t)y) - f'(tu + (1-t)v))dt| \\ &= |\int_0^1 (e^{tx+(1-t)y} - e^{tu+(1-t)v})dt| \\ &= |\int_0^1 \int_0^1 e^{s(tx+(1-t)y)+(1-s)(tu+(1-t)v)} ds(t(x-u) + (1-t)(y-v))dt| \\ &\leq \tfrac{e}{2}(|x-u| + |y-v|). \end{aligned} \tag{3.7}$$

On the other hand, for any $x \in D$, we have

$$\begin{aligned} &|f'(x_\star)^{-1}(f[x, x_\star] - f[x_\star, x_\star])| = |\int_0^1 (f'(tx) - f'(0))dt| \\ &= |\int_0^1 (e^{tx} - 1)dt| = |\int_0^1 (tx + \tfrac{(tx)^2}{2!} + \cdots)dt| \\ &\leq \int_0^1 t|x|(1 + \tfrac{1}{2!} + \cdots)dt| = \tfrac{e-1}{2}|x - x_\star|. \end{aligned} \tag{3.8}$$

Hence, we can choose $K = \frac{e}{2}$ and $K_\star = \frac{e-1}{2}$ in (1.4) and (1.5), respectively. By (2.1), we have $R' \approx 0.0720$ and (2.2) is true. Therefore, all conditions of Theorem 2.1 hold, so it applies. Moreover, if we use only condition (1.4), we have only $R' \approx 0.0675$. Note also that we can choose constants N (the upper bound for $|f'(x_\star)^{-1}f''(x)|$ on D) and M (the upper bound for $|f'(x_\star)^{-1}f'''(x)|$ on D) of Theorem 1 in [7] as $N = M = e$. However, the other condition $1215N^2 \leq 32M$ of Theorem 1 in [7] is not satisfied, and so Theorem 1 in [7] cannot apply.

Example 3.3. *Let f be defined on $D = [-1, 1]$ by*

$$f(x) = \begin{cases} \dfrac{1}{2}x^2 + 3x, & 0 \leq x \leq 1, \\ -\dfrac{1}{2}x^2 + 3x, & -1 \leq x < 0. \end{cases} \tag{3.9}$$

Then, we have $x_\star = 0$,

$$f'(x) = |x| + 3 = \begin{cases} x + 3, & 0 \leq x \leq 1, \\ -x + 3, & -1 \leq x < 0, \end{cases} \tag{3.10}$$

and

$$f''(x) = \begin{cases} 1, & 0 < x \leq 1, \\ \textit{does not exist}, & x = 0, \\ -1, & -1 \leq x < 0. \end{cases} \tag{3.11}$$

Notice that, for any $x, y, u, v \in D$, we have

$$\begin{array}{rcl} |f'(x_\star)^{-1}(f[x,y] - f[u,v])| & = & \frac{1}{3}|\int_0^1 (f'(tx+(1-t)y) - f'(tu+(1-t)v))dt| \\ & = & \frac{1}{3}|\int_0^1 (|tx+(1-t)y| - |tu+(1-t)v|)dt| \\ & \leq & \frac{1}{3}\int_0^1 ||tx+(1-t)y| - |tu+(1-t)v||dt \\ & \leq & \frac{1}{3}\int_0^1 |tx+(1-t)y-(tu+(1-t)v)|dt \\ & \leq & \frac{1}{3}\int_0^1 (t|x-u| + (1-t)|y-v|)dt \\ & = & \frac{1}{6}(|x-u| + |y-v|). \end{array} \tag{3.12}$$

That is, condition (1.4) holds for $K = \frac{1}{6}$ but $f''(x)$ is not differentiable at $0 \in D$.

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