Exponential Growth of Positive Initial Energy Solutions for Coupled Nonlinear Klein-Gordon Equations with Degenerate Damping and Source Terms

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ABSTRACT: In this paper we will prove that the positive initial-energy solution for coupled nonlinear Klein-Gordon equations with degenerate damping and source terms grows exponentially.

Key Words: Klein-Gordon equation, Degenerate damping, Local existence, Growth of solution.

Contents

1. Introduction

In this work, we consider the coupled nonlinear Klein-Gordon equations:

$$
\begin{cases}
u_{tt} - \Delta u + m_1^2 u + \left(|u|^k + |v|^l \right) |u_t|^{p-1} u_t = f_1(u,\, v), \\
v_{tt} - \Delta v + m_2^2 v + \left(|v|^\theta + |u|^\varrho \right) |v_t|^{q-1} v_t = f_2(u,\, v),
\end{cases}
\tag{1.1}
$$

where $p,\, q > 1$, $k,\, l,\, \theta,\, \varrho \geq 0$, $m_1,\, m_2 > 0$, $(x,\, t) \in \Omega \times (0,\, T)$ and Ω is a bounded domain with smooth boundary $\partial\Omega$ in $R^n\,(n \geq 1)$, and the two functions $f_1(u,\, v)$ and $f_2(u,\, v)$ given by

$$
\begin{cases}
f_1(u,\, v) = |u + v|^{2(r+1)} (u + v) + |u|^r u |v|^{r+2}, \\
f_2(u,\, v) = |u + v|^{2(r+1)} (u + v) + |u|^{r+2} |v|^r v.
\end{cases}
\tag{1.2}
$$

The system (1.1) is supplemented with the following initial conditions:

$$
((u(0), v(0))) = (u_0, v_0),\ \ ((u_t(0), v_t(0))) = (u_1, v_1),\ x \in \Omega
\tag{1.3}
$$

and boundary conditions

$$
u(x) = v(x) = 0,\ x \in \partial\Omega.
\tag{1.4}
$$

Some special case of the single wave equation with nonlinear damping and nonlinear source terms in the form

$$
u_{tt} - \Delta u + a |u_t|^{p-1} u_t = b |u|^{q-1} u,
\tag{1.5}
$$

with the presence of different mechanisms of dissipation, damping and for more general forms of nonlinearities has been extensively studied and results concerning existence, nonexistence and asymptotic behavior of solutions have been established by several authors and many results appeared in the literature over the past decades. See ([1], [5] − [8], [10], [16]). The absence of the terms $m_1^2 u$ and $m_2^2 u$, equations (1.1) take the form

$$
\begin{cases}
u_{tt} - \Delta u + \left(|u|^k + |v|^l \right) |u_t|^{p-1} u_t = f_1(u,\, v), \\
v_{tt} - \Delta v + \left(|v|^\theta + |u|^\varrho \right) |v_t|^{q-1} v_t = f_2(u,\, v).
\end{cases}
\tag{1.6}
$$

In [13] Rammaha and Sakuntasathien focus on the global well-posedness of the system of nonlinear wave equation (1.6). In [17] Wu studied blow up of solutions of the system (1.1) for $n = 3$ and $k = l = \theta = \varrho = 0$. Agre and Rammaha [3] studied the global existence and the blow up of the solution of problem (1.6) when $k = l = \theta = \varrho$, and also Alves et al [4], investigated the existence, uniform decay rates and blow up of the solution. In [11] Erhen Pişkin prove the blow up of solutions of (1.1) in finite time with negative initial energy and nondegenerate damping terms. In the work [9], authors considered the following nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g\,(t-s)\,\Delta u\,(x,s)\,ds + |u_t|^{p-1}\,u_t = f_1\,(u,\ v), \\ v_{tt} - \Delta v + \int\limits_0^t h\,(t-s)\,\Delta v\,(x,s)\,ds + |v_t|^{q-1}\,v_t = f_2\,(u,\ v). \end{cases} \tag{1.7}$$

and they prove a global nonexistence for certain solutions with positive initial energy, the main tool proof is a method used in [15]. In [14], B. Said-Houari proved that the energy associated to the system (1.8)

$$\begin{cases} |u|^j\,u_{tt} - \Delta u - \Delta u_{tt} + \int\limits_0^t g\,(t-s)\,\Delta u\,(x,s)\,ds + |u_t|^{p-1}\,u_t = f_1\,(u,\ v), \\ |v|^j\,v_{tt} - \Delta v - \Delta v_{tt} + \int\limits_0^t h\,(t-s)\,\Delta v\,(x,s)\,ds + |v_t|^{q-1}\,v_t = f_2\,(u,\ v). \end{cases} \tag{1.8}$$

is unbounded and it grows up as an exponential fonction as time goes to infinity, provided that the initail data are large enough. The key ingredient in his proof is a method used in vitillaro [16] and developed in [15] for a system of wave equations.

Our paper is organized as follows, In section 2, we present the assumptions and some lemmas needed for our result. Section 3 is devoted the proof of the main result.

2. Preliminaries

In this section, we shall give some lemmas which will be used throughout this work.

Lemma 2.1. [2] (*Sobolev-Poincaré inequality*) *Let s be a number with $2 \le s < +\infty$ if $n \le 2$ and $2 \le s \le \frac{2n}{n-2}$ if $n > 2$. Then there is a constant C depending on Ω and s such that*

$$\|u\|_s \le C\,\|\nabla u\|_2\,, \quad u \in H_0^1. \tag{2.1}$$

Lemma 2.2. (*Young's inequality*) *Let $a, b \ge 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $0 < p,\ q < +\infty$, then one has the inequatity*

$$ab \le \delta a^p + c\,(\delta)\,b^q,$$

where $\delta > 0$ is an constant, and $c\,(\delta)$ is a positive constant depending on δ.

We assume that

$$\begin{cases} r > -1 \text{ if } n = 1, 2 \\ -1 < r \le \frac{3-n}{n-2} \text{ if } n \ge 3. \end{cases} \tag{2.2}$$

We can easily verify that

$$u f_1\,(u,\ v) + v f_2\,(u,\ v) = 2\,(r+2)\,F\,(u,\ v) \tag{2.3}$$

where

$$F\,(u,\ v) = \frac{1}{2\,(r+2)}\left[|u+v|^{2(r+2)} + 2\,|uv|^{r+2}\right]. \tag{2.4}$$

Lemma 2.3. [9] *There exist two positive constants c_1 and c_2 such that*

$$c_1\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \le 2\,(r+2)\,F\,(u,\ v) \le c_2\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right), \tag{2.5}$$

is satisfied.

Now, we define the following energy function associated with a solution (u, v) of problem $(1.1) - (1.4)$

$$E\left(t\right) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)$$

$$+ m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 - \int_\Omega F\left(u,\ v\right) dx. \tag{2.6}$$

Lemma 2.4. *Let* (u, v) *be a solution* $(1.1) - (1.4)$ *then* $E\left(t\right)$ *is a nonincreasing function for* $t > 0$ *and*

$$E^{'}\left(t\right) = -\int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{p+1}\, dx - \int_\Omega\left(|v|^\theta + |u|^\varrho\right)|u_t|^{q+1}\, dx \tag{2.7}$$

Proof. By multiplying the first equation of (1.1) by u_t and the second equation by v_t, integrating over Ω, using integration py parts and summing up, we get

$$E\left(t\right) - E\left(0\right) = -\int_0^t\int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{p+1}\, dx - \int_0^t\int_\Omega\left(|v|^\theta + |u|^\varrho\right)|u_t|^{q+1}\, dxds \tag{2.8}$$

$\square$

Next, we state the local existence theorem that can be established combinig arguments of $[12,\ 13]$. We give the definition of a weak solution to problem $(1.1) - (1.4)$.

Definition 2.5. *A pair of function* $(u,\ v)$ *is said to be a weak solution of* $(1.1) - (1.4)$ *on* $[0,\ T]$ *if*

$$u, v \in C\left([0,\ T];\ H_0^1\left(\Omega\right) \cap L^{2(r+2)}\left(\Omega\right)\right), u_t \in C\left([0,\ T];\ L^2\left(\Omega\right)\right) \cap L^{p+1}\left(\Omega \times (0,\ T)\right)$$

$$and$$

$$v_t \in C\left([0,\ T];\ L^2\left(\Omega\right)\right) \cap L^{q+1}\left(\Omega \times (0,\ T)\right).$$

In addition, $(u,\ v)$ *satisfies*

$$\int_\Omega u^{'}\left(t\right)\varphi dx - \int_\Omega u_1\left(t\right)\varphi dx + \int_\Omega \nabla u\nabla\varphi dx + m_1^2\int_\Omega u\varphi dx$$

$$+ \int_0^t\int_\Omega\left(|u|^k + |v|^l\right)\left|u^{'}\right|^{p+1} u^{'}\varphi dxds = \int_0^t\int_\Omega f\left(u,\ v\right)\varphi dxds \tag{2.9}$$

$$\int_\Omega v^{'}\left(t\right)\phi dx - \int_\Omega v_1\left(t\right)\phi dx + \int_\Omega \nabla v\nabla\phi dx + m_2^2\int_\Omega v\phi dx$$

$$+ \int_0^t\int_\Omega\left(|v|^\theta + |u|^\varrho\right)\left|v^{'}\right|^{q+1} v^{'}\phi dxds = \int_0^t\int_\Omega f_2\left(u,\ v\right)\varphi dxds \tag{2.10}$$

for all test function $\varphi \in H_0^1\left(\Omega\right) \cap L^{p+1}\left(\Omega\right),\ \phi \in H_0^1\left(\Omega\right) \cap L^{q+1}\left(\Omega\right)$ *and for almost all* $t \in [0,\ T]$.

Theorem 2.6. *(Local existence) Assume that (2.2) holds. Then, for any initial data $u_0,\ v_0 \in H_0^1\left(\Omega\right) \cap L^{2(r+2)}\left(\Omega\right)$ and $u_1,\ v_1 \in L^2\left(\Omega\right)$. There exists a unique local weak solution $(u,\ v)$ of problem $(1.1) - (1.4)$ (in the sense of definition 2.5) defined in $[0,\ T]$ for some $T > 0$, and satisfies the energy identity*

$$E\left(t\right) + \int_0^t\int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{p+1}\, dxds + \int_0^t\int_\Omega\left(|v|^\theta + |u|^\varrho\right)|v_t|^{q+1}\, dxds = E\left(0\right),\quad t \geq 0. \tag{2.11}$$

3. Exponential growth

In this section, we are going to prove our main result. We need in the sequel the following Lemmas.

Lemma 3.1. [9] *Suppose that* (2.2) *holds. Then there exists $\eta > 0$ such that for any $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ the inequality*

$$2(r+2) \int_\Omega F(u, v) \leq \eta \left(\frac{1}{2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 \right)^{r+2} \tag{3.1}$$

holds.

Proof. Direct computation using Minkowski, Hölder's and Young's inequality and the embedding theorem yields the proof of this Lemma. ∎ □

We introduce the following constants:

$$B = \eta^{\frac{1}{2(r+2)}}, \; \alpha_1 = B^{-\frac{r+2}{r+1}}, \; E_1 = \left(\frac{1}{2} - \frac{1}{2(r+2)} \right) \alpha_1^2, \tag{3.2}$$

where η is the optimal constant in (3.1).

The following lemma is very useful to prove our result for positive initial energy $E(0) > 0$. It is similar to the one the lemma in [9], first used by Vitillaro [16].

Lemma 3.2. *Suppose that* (2.2) *holds. Let (u, v) be a solution of $(1.1) - (1.4)$. Assume further that $E(0) < E_1$ and*

$$\left(\frac{1}{2} \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 \right) + \frac{m_1^2}{2} \|u_0\|_2^2 + \frac{m_2^2}{2} \|v_0\|_2^2 \right)^{\frac{1}{2}} > \alpha_1 \tag{3.3}$$

Then there exists a constant $\alpha_2 > \alpha_1$, such that

$$\left(\frac{1}{2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + \frac{m_1^2}{2} \|u\|_2^2 + \frac{m_2^2}{2} \|\nabla v\|_2^2 \right)^{\frac{1}{2}} > \alpha_2, \tag{3.4}$$

and

$$\left(\|u + v\|_{2(r+2)}^{2(r+2)} + 2 \|uv\|_{r+2}^{r+2} \right)^{\frac{1}{2(r+2)}} \geq B\alpha_2, \quad \forall t \geq 0. \tag{3.5}$$

Theorem 3.3. *Suppose that* (2.2) *holds. Assume further that*

$$2(r+2) > \max \{k+p+1, \; l+p+1, \; \theta+q+1, \; \varrho+q+1\} \tag{3.6}$$

Then any solution of problem $(1.1) - (1.4)$ with initail data satisfying

$$\left(\frac{1}{2} \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 \right) + \frac{m_1^2}{2} \|u_0\|_2^2 + \frac{m_2^2}{2} \|v_0\|_2^2 \right)^{\frac{1}{2}} > \alpha_1, \tag{3.7}$$

$E(0) < E_1$, *where α_1 is defined in (3.2), grows exponentially.*

Proof. Set

$$H(t) = E_1 - E(t) \tag{3.8}$$

By using (2.6) and (3.8) we get

$$0 < H(0) \leq H(t) = E_1 - \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) - \frac{1}{2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)$$

□

$$-\left(\frac{m_1^2}{2}\|u\|_2^2 + \frac{m_2^2}{2}\|v\|_2^2\right) + \int_\Omega F(u,\,v)\,dx$$

From (3.4), we obtain

$$E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)$$

$$-\left(\frac{m_1^2}{2}\|u\|_2^2 + \frac{m_2^2}{2}\|v\|_2^2\right) \le E_1 - \frac{1}{2}\alpha_1^2 \le -\frac{1}{2(r+2)}\alpha_1^2 < 0 \tag{3.9}$$

Hence, by the above inequality and (2.5), we have

$$0 < H(0) \le H(t) \le \frac{c_2}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \tag{3.10}$$

We then define the following Lyaponov function

$$G(t) = H(t) + \epsilon \int_\Omega (uu_t + vv_t)\,dx \tag{3.11}$$

for ϵ small to be chosen later.

Our goal is to show that $G(t)$ satisfies a differential inequality of the form

$$\frac{d}{dt}G(t) \ge \zeta G(t) \tag{3.12}$$

By taking a derivative of (3.11) and using equations (1.1), we obtain

$$\begin{aligned} G'(t) &= H'(t) + \epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \epsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &\quad -\epsilon\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) + \epsilon\int_\Omega (uf_1(u,\,v) + vf_2(u,\,v))\,dx \\ &\quad -\epsilon\int_\Omega \left(|u|^k + |v|^l\right)u_t\,|u_t|^{p-1}\,udx - \epsilon\int_\Omega \left(|v|^\theta + |u|^\varrho\right)v_t\,|v_t|^{q-1}\,vdx. \end{aligned} \tag{3.13}$$

From the definition of $H(t)$, it follows that

$$-\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) = -2E_1 + 2H(t) - 2\int_\Omega F(u,\,v)$$

$$+\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \tag{3.14}$$

Inserting (3.14) into (3.13), lead to

$$G'(t) = H'(t) + 2\epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - 2\epsilon E_1 + 2\epsilon H(t)$$

$$+\epsilon\left(1 - \frac{1}{r+2}\right)\left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right)$$

$$-\epsilon\int_\Omega \left(|u|^k + |v|^l\right)u_t\,|u_t|^{p-1}\,udx - \epsilon\int_\Omega \left(|v|^\theta + |u|^\varrho\right)v_t\,|v_t|^{q-1}\,vdx \tag{3.15}$$

By taking $c_3 = 1 - \frac{1}{r+2} - 2E_1 (B\alpha_2)^{-2(r+2)}$, one can easily check that $c_3 > 0$, since $\alpha_2 > B^{-\frac{r+2}{r+1}}$. Therefore, (3.15) take the form

$$G^{'}(t) \geq H^{'}(t) + 2\epsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right)$$

$$+2\epsilon H(t) + \epsilon c_3 \left(\|u + v\|_{2(r+2)}^{2(r+2)} + 2 \|uv\|_{r+2}^{r+2} \right)$$

$$- \epsilon \int_{\Omega} \left(|u|^k + |v|^l \right) u_t |u_t|^{p-1} u dx - \epsilon \int_{\Omega} \left(|v|^\theta + |u|^\varrho \right) v_t |v_t|^{q-1} v dx \qquad (3.16)$$

In the order to estimate the last two terms in (3.16), we make of the following Young's inequatily

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^{-\beta} Y^\beta}{\beta},$$

$X, Y \geq 0, \delta > 0, \alpha, \beta \in \mathbb{R}^+$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Consequently, we get

$$u_t |u_t|^{p-1} u \leq \frac{p}{p+1} \delta_1^{-\frac{p+1}{p}} |u|^{p+1} + \frac{1}{p+1} \delta_1^{p+1} |u_t|^{p+1},$$

and therefore

$$\int_{\Omega} \left(|u|^k + |v|^l \right) u_t |u_t|^{p-1} u dx \leq \frac{p}{p+1} \delta_1^{-\frac{p+1}{p}} \int_{\Omega} \left(|u|^k + |v|^l \right) |u|^{p+1} dx$$

$$+ \frac{1}{p+1} \delta_1^{p+1} \int_{\Omega} \left(|u|^k + |v|^l \right) |u_t|^{p+1} dx \qquad (3.17)$$

Similarly, for all $\delta_2 > 0$

$$v_t |v_t|^{q-1} v \leq \frac{q}{q+1} \delta_2^{-\frac{q+1}{q}} |v|^{q+1} + \frac{1}{q+1} \delta_2^{q+1} |v_t|^{q+1},$$

then

$$\int_{\Omega} \left(|v|^\theta + |u|^\varrho \right) v_t |v_t|^{q-1} v dx \leq \frac{q}{q+1} \delta_2^{-\frac{q+1}{q}} \int_{\Omega} \left(|v|^\theta + |u|^\varrho \right) |v|^{q+1} dx$$

$$+ \frac{1}{q+1} \delta_2^{q+1} \int_{\Omega} \left(|v|^\theta + |u|^\varrho \right) |v_t|^{q+1} dx \qquad (3.18)$$

Inserting $(3.17), (3.18)$ into (3.16), we obtain

$$G^{'}(t) \geq H^{'}(t) + 2\epsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right)$$

$$+2\epsilon H(t) + \epsilon c_3 \left(\|u + v\|_{2(r+2)}^{2(r+2)} + 2 \|uv\|_{r+2}^{r+2} \right)$$

$$- \epsilon \frac{1}{p+1} \delta_1^{p+1} \int_{\Omega} \left(|u|^k + |v|^l \right) |u_t|^{p+1} dx - \epsilon \frac{1}{q+1} \delta_2^{q+1} \int_{\Omega} \left(|v|^\theta + |u|^\varrho \right) |v_t|^{q+1} dx$$

$$- \epsilon \frac{p}{p+1} \delta_1^{-\frac{p+1}{p}} \int_{\Omega} \left(|u|^k + |v|^l \right) |u|^{p+1} dx - \epsilon \frac{q}{q+1} \delta_2^{-\frac{q+1}{q}} \int_{\Omega} \left(|v|^\theta + |u|^\varrho \right) |v|^{q+1} dx \qquad (3.19)$$

Using Young's inequality, we get

$$\int_{\Omega} \left(|u|^k + |v|^l \right) |u|^{p+1} dx \leq \|u\|_{k+p+1}^{k+p+1} + \int_{\Omega} |v|^l |u|^{p+1} dx$$

$$\leq \|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\gamma_1^{\frac{l+p+1}{l}} \|v\|_{l+p+1}^{l+p+1} + \frac{p+1}{l+p+1}\gamma_1^{-\frac{l+p+1}{p+1}} \|u\|_{l+p+1}^{l+p+1} \tag{3.20}$$

and

$$\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v|^{q+1}\,dx \leq \|v\|_{\theta+q+1}^{\theta+q+1} + \int_\Omega |u|^\varrho\,|v|^{q+1}\,dx$$

$$\leq \|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\varrho}{\varrho+q+1}\gamma_2^{\frac{\varrho+q+1}{\varrho}} \|u\|_{\varrho+q+1}^{\varrho+q+1} + \frac{q+1}{\varrho+q+1}\gamma_2^{-\frac{\varrho+q+1}{q+1}} \|v\|_{\varrho+q+1}^{\varrho+q+1} \tag{3.21}$$

By using (3.20), (3.21) and lemma 3, (3.19) becames

$$G'(t) \geq H'(t) + 2\epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \epsilon\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right)$$

$$+2\epsilon H(t) + \epsilon c_4\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right)$$

$$-\epsilon\frac{1}{p+1}\delta_1^{p+1}\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{p+1}\,dx - \epsilon\frac{1}{q+1}\delta_2^{q+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q+1}\,dx$$

$$-\epsilon\frac{p}{p+1}\delta_1^{-\frac{p+1}{p}}\left[\|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\gamma_1^{\frac{l+p+1}{l}}\|v\|_{l+p+1}^{l+p+1}\right.$$

$$\left.+\frac{p+1}{l+p+1}\gamma_1^{-\frac{l+p+1}{p+1}}\|u\|_{l+p+1}^{l+p+1}\right]$$

$$-\epsilon\frac{q}{q+1}\delta_2^{-\frac{q+1}{q}}\left[\|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\varrho}{\varrho+q+1}\gamma_2^{\frac{\varrho+q+1}{\varrho}}\|u\|_{\varrho+q+1}^{\varrho+q+1}\right.$$

$$\left.+\frac{q+1}{\varrho+q+1}\gamma_2^{-\frac{\varrho+q+1}{q+1}}\|v\|_{\varrho+q+1}^{\varrho+q+1}\right] \tag{3.22}$$

We define the algebric inequality

$$z^\upsilon \leq (z+1) \leq \left(1+\frac{1}{a}\right)(z+a),\ \forall z \geq 0,\ 0 < \upsilon < 1,\ a > 0. \tag{3.23}$$

In the sequel noting by c the various constants.

Since (3.6) holds, by the embedding theorem and the previous inequality, we obtain

$$\begin{cases} \|u\|_{k+p+1}^{k+p+1} \leq c\|u\|_{2(r+2)}^{k+p+1} \leq cd\left(\|u\|_{2(r+2)}^{2(r+2)} + H(0)\right) \leq cd\left(\|u\|_{2(r+2)}^{2(r+2)} + H(t)\right) \\ \|u\|_{l+p+1}^{l+p+1} \leq cd\left(\|u\|_{2(r+2)}^{2(r+2)} + H(t)\right) \\ \|u\|_{\varrho+q+1}^{\varrho+q+1} \leq cd\left(\|u\|_{2(r+2)}^{2(r+2)} + H(t)\right) \\ \|v\|_{l+p+1}^{l+p+1} \leq cd\left(\|v\|_{2(r+2)}^{2(r+2)} + H(t)\right) \\ \|v\|_{\theta+q+1}^{\theta+q+1} \leq cd\left(\|v\|_{2(r+2)}^{2(r+2)} + H(t)\right) \\ \|v\|_{\varrho+q+1}^{\varrho+q+1} \leq cd\left(\|v\|_{2(r+2)}^{2(r+2)} + H(t)\right) \end{cases} \tag{3.24}$$

where $d = 1 + \frac{1}{H(0)}$. Choosing

$$K_1 = \frac{p+1}{l+p+1}\gamma_1^{-\frac{l+p+1}{p+1}},\ \ K_2 = \frac{\varrho}{\varrho+q+1}\gamma_2^{\frac{\varrho+q+1}{\varrho}},$$

$$K_3 = \frac{l}{l+p+1}\gamma_1^{\frac{l+p+1}{l}},\ \ K_4 = \frac{q+1}{\varrho+q+1}\gamma_2^{-\frac{\varrho+q+1}{q+1}}$$

Substituting (3.24) into (3.22) and using the formula of $H'(t)$, we obtain

$$G'(t) \geq (1 - \epsilon K)H'(t) + 2\epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \epsilon\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right)$$

$$+\epsilon\left[c_4 - \frac{p}{p+1}\delta_1^{-\frac{p+1}{p}}cd\left(1 + K_1 + K_2\right)\right]\|u\|_{2(r+2)}^{2(r+2)}$$

$$+\epsilon\left[c_4 - \frac{q}{q+1}\delta_2^{-\frac{q+1}{q}}cd\left(1 + K_3 + K_4\right)\right]\|v\|_{2(r+2)}^{2(r+2)}$$

$$+\epsilon\left[2 - \frac{p}{p+1}\delta_1^{-\frac{p+1}{p}}cd\left(1 + K_1 + K_2\right) - \frac{q}{q+1}\delta_2^{-\frac{q+1}{q}}cd\left(1 + K_3 + K_4\right)\right]H\left(t\right) \tag{3.25}$$

where $K = \max\left(\frac{1}{p+1}\delta_1^{p+1},\ \frac{1}{q+1}\delta_2^{q+1}\right)$. At this point, and for large values of δ_1 and δ_2, and we pick ϵ small enough, we can find positive constants λ, λ_1, λ_2 and λ_3 such that (3.25) becomes

$$G^{'}\left(t\right) \geq \lambda H^{'}\left(t\right) + 2\epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)$$

$$+ \epsilon\lambda_1\|u\|_{2(r+2)}^{2(r+2)} + \epsilon\lambda_2\|v\|_{2(r+2)}^{2(r+2)} + \epsilon\lambda_3 H\left(t\right). \tag{3.26}$$

Therefore

$$G^{'}\left(t\right) \geq M\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} + H\left(t\right)\right). \tag{3.27}$$

where $M = \min\left(2,\ \lambda_1,\ \lambda_2,\ \lambda_3\right)$. Consequently, we have

$$G\left(0\right) = H\left(0\right) + \epsilon\int_{\Omega}\left(u_0 u_1 + v_0 v_1\right)dx > 0.$$

Now, by Hölder's and Young's inequalities, we estimate

$$\int_{\Omega}\left(uu_t + vv_t\right)dx \leq k_1\|u\|_2^2 + \frac{1}{4k_1}\|u_t\|_2^2 + k_2\|v\|_2^2 + \frac{1}{4k_2}\|v_t\|_2^2,\ \ k_1,\ k_2 > 0. \tag{3.28}$$

We can find constant c such that

$$\int_{\Omega}\left(uu_t + vv_t\right)dx \leq c\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} + \|u\|_2^2 + \|v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2\right) \tag{3.29}$$

Again applied (3.23) and the embedding theorem, we get

$$\|u\|_2^2 \leq c\|u\|_{2(r+2)}^2 \leq c\left(\|u\|_{2(r+2)}^{2(r+2)}\right)^{\frac{2}{2(r+2)}} \leq cd\left(\|u\|_{2(r+2)}^{2(r+2)} + H\left(t\right)\right).$$

Similarly,

$$\|v\|_2^2 \leq cd\left(\|v\|_{2(r+2)}^{2(r+2)} + H\left(t\right)\right).$$

Also, noting that

$$G\left(t\right) = H\left(t\right) + \epsilon\int_{\Omega}\left(uu_t + vv_t\right)dx \leq c(H\left(t\right) + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}$$

$$+ \|u_t\|_2^2 + \|v_t\|_2^2). \tag{3.30}$$

And combining with (3.27) and (3.30), we arrive at

$$\frac{d}{dt}G\left(t\right) \geq \xi G\left(t\right),\ \xi > 0,\ \ \forall t \geq 0. \tag{3.31}$$

Integrating of (3.31) between 0 and t gives

$$G\left(t\right) \geq G\left(0\right)e^{\xi t},\ \ \forall t \geq 0. \tag{3.32}$$

This completes the proof. ∎

References

1. M. Abdelli and B. Abbes, *Energy decay of solutions of a degenerate Kirchhoff equation with a weak nonlinear dissipation*, Nonlinear Analysis 69, 1999-2008 (2008).

2. R. A. Adams and J. J. F.Fournier, : *Sobolev Spaces*, Academic Press, New York (2003).

3. K. Agre and M.A. Rammaha, *Systems of nonlinear wave equations with damping and source terms*, Diff. Integral Eqns., 19 (11), 1235–1270 (2006).

4. C. O. Alves, M. M. Cavalcanti, V. N. Domingos Cavalcanti, M. A. Rammaha, and D. Toundykov, *On the existence, uniform decay rates and blow up of solutions to systems of nonlinear wave equations with damping and source terms*, Discrete and Continuous Dynamical Systems-Series S, 2(3), 583–608 (2009).

5. D. D. Ang and A. P. N. Dinh,*Strong solutions of a quasilinear wave equation with non linear damping*, SIAM Journal on Mathematical Analysis, vol. 19, no. 2, pp. 337-347 (1988).

6. V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source term*, J. Diff .Eq., 109, 295-308 (1994).

7. K. Kafini and S. A. Messaoudi, *A blow-up result in a Cauchy viscoelastic problem*, Applied Mathematics Letters 21, 549–553 (2008).

8. H. A. Levine and J. Serrin, *Global nonexistence theorems for quasilinear evolution equations with dissipation, Archive for Rational* Mechanics and Analysis, vo l37, no. 4, pp. 341-361 (1997).

9. S. A. Messaoudi and B. Said-Houari, *Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms*, J. Math. Anal. Appl. 365, 277-287 (2010).

10. S. A. Messaoudi, *Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation*, J. Math. Anal. Appl. 320, 902–915 (2006).

11. E. Pişkin, *Blow up of positive initial-energy solutions for coupled nonlinear wave equations with degenerate damping and source terms*, Boundary Value Problems, 1-11 (2015).

12. E. Pişkin, N. Polat, *Global existence, decay and blow up solutions for coupled nonlinear wave equations with damping and source terms*, Turk J Math, 37: 633 – 651 (2013).

13. M. A. Rammaha, S. Sakuntasathien, *Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms*, Nonlinear Anal., 72, 2658-2683 (2010).

14. B. Said-Houari, *Exponential growth of positive initial-energy solution of a system of nonlinear viscoelastic wave equation with damping and source terms*, Z. Angew. Math. Phys, 62, 115-133 (2011).

15. B. Said-Houari, *Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms*, Diff. Integral Eqns., 23 (1–2), 79–92 (2010).

16. E. Vitillaro, *Global existence theorems for a class of evolution equations with dissipation*, Arch. Rational Mech. Anal., 149, 155–182 (1999).

17. S. T. Wu, *Blow-up of solutions for a system of nonlinear wave equations with nonlinear damping*, Electron. J. Differential Equations, 2009 (105), 1–11 (2009).

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