Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.44096

Existence of Weak Solutions for Second-order Boundary-value Problems of Kirchhoff-type with Variable Exponents *

Iman E. Eskandarkolaei, Somayeh Khademloo and Ghasem A. Afrouzi

ABSTRACT: In this paper, we investigate the existence of multiple solutions for a second-order boundary value problems of Kirchhoff-type equation involving a $p(x)$-Laplacian.

Key Words: Three solutions, Kirchhoff-type problem, Neumann problem, Variable exponent Sobolev spaces.

Contents

1. Introduction

In this article, we consider the following boundary value problem of Kirchhoff-type equation involving an ordinary differential equation with $p(x)$-Laplacian operator and nonhomogeneous Neumann conditions

$$\begin{cases} T(u)=\lambda f(x,u), & \text{in } (0,1),\\ |u'(0)|^{p(0)-2}u'(0)=-\mu g(u(0)), & \\ |u'(1)|^{p(1)-2}u'(1)=\mu h(u(1)), & \end{cases} \tag{1.1}$$

where $T(u) := M\left(\int_0^1 \frac{1}{p(x)}\left(|u'|^{p(x)}+\alpha(x)|u|^{p(x)}\right)dx\right)\left[-\left(|u'|^{p(x)-2}u'\right)'+\alpha(x)|u|^{p(x)-2}u\right]$. Here, $M : [0,+\infty[\to\mathbb{R}$ is a continuous function such that there exist positive numbers m_0 and m_1 with

$$m_0 \le M(t) \le m_1 \quad \text{for all } t \ge 0, \tag{1.2}$$

$p \in C([0,1];\mathbb{R})$, $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function, $g, h : \mathbb{R}\to\mathbb{R}$ are non-negative continuous functions, λ and μ are real parameters with $\lambda > 0$ and $\mu \ge 0$, $\alpha \in L^\infty([0,1])$, with $ess\inf_{[0,1]}\alpha > 0$.

Problem (1.1) is a general version of a model presented by Kirchhoff [16]. More precisely Kirchhoff introduced a model

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.3}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Equation (1.3) was developed to form $u_{tt}(x) - M\left(\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u(x) = f(x,u(x))$. Latter , several authors studied the following nonlocal elliptic boundary value problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u(x)=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

* The second author acknowledges the funding support of Babol Noshirvani University of Technology through Grant program No. BNUT/370240/97.
2010 *Mathematics Subject Classification:* 35J35, 35J60.
Submitted August 13, 2018. Published May 15, 2019

Typeset by $\mathcal{B}^{\mathcal{S}}_{\mathcal{M}}\mathcal{P}$ style.
© Soc. Paran. de Mat.

Problems like (1.4) can be used for modeling several physical and biological systems and also for describing the dynamics of an axially moving string ([1]).

Due to this, many authors have investigated the existence and multiplicity of solutions for such problems by using variational methods like the symmetric mountain pass theorem and critical point theorems, lower and upper solution method, fixed point theorems, degree theory and Morse theory ([4,5,7,8]).

Differential equations with variable exponents arise from the nonlinear elasticity theory and the theory electroheological fluids. The study of such problems has received considerable attention in recent years. For background, we refer the readers to [12,17,10,20,22,23].

For example, Zhikov in [26] via Leray-Schauder degree theory, obtained sufficient conditions for the existence of one solution for a weighted $p(x)$-Laplacian system. Wang and Yuan in [25] have been studied the periodic solutions for a class of systems of equation coupled with non-standard $p(x)$-growth. Bonanno and Chinnì in [2] by using a multiple critical points theorem for non-differentiable functionals, investigated the existence and multiplicity of solutions for the following problem

$$\begin{cases} -\Delta_{p(x)}u(x) = \lambda f(x,u(x)) + \mu g(x,u(x)), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

After that, in [3] the existence of three weak solutions for the following problem

$$\begin{cases} -\Delta_{p(x)}u(x) + a(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u(x)) + \mu g(x,u(x)), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

has been obtained by using a three critical points theorem due to Ricceri. D'Aguì in [9], by using variational methods, obtained the existence of an unbounded sequence of weak solutions for the problem

$$\begin{cases} -(|u'|^{p(x)-2}u')' + \alpha(x)|u|^{p(x)-2}u = \lambda f(x,u), & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\mu g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \mu h(u(1)). \end{cases} \tag{1.5}$$

Recently, Heidarkhani et al. in [14] studied the existence of three solutions for the second order boundary value problems with variable exponent (1.5).
Motivated by the papers [6,13,14], in the present paper, we introduce a Kirchhoff $p(x)$-Laplacian problem with nonhomogeneous Neumann condition.

Inspired by the above results, in the present paper, we study the existence of at least three weak solutions for the problem (1.1) for appropriate values of the parameters λ and μ belonging to real intervals. Precisely, employing variational methods and a three critical point theorem due to Ricceri [24], we establish the existence result for problem (1.1) requiring an algebraic condition on f. An example is presented to illustrate our main results.

This paper is organized as follows: In Section 2 we shall recall our main tool and some properties of variable exponent spaces and basic notations. Whereas, in Section 3 we formulate the main result and prove it, in order to discuss the existence of three weak solutions for the problem (1.1). We also list some consequences of the main result and present an example to illustrate the result.

2. Preliminary results

We shall prove the existence of at least three weak solutions to the problem (1.1) applying the following three critical points theorem obtained by Ricceri [24]

For a real Banach space X, we denoted by $\mathcal{W}_X$, the class of all functionals $\Phi : X \to \mathbb{R}$ possess the following property:

If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

Remark 2.1. *[15, Remark 2.1]* If X is uniformly convex and $g : [0,+\infty) \to \mathbb{R}$ is continuous and strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Theorem 2.2. *Let X be a separable and reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$

$$\sigma = \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{J(u)}{\Phi(u)},$$

assume that $\rho < \sigma$. Then, for each compact interval $[c, d] \subseteq (\frac{1}{\sigma}, \frac{1}{\rho})$ (with the conventions $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation $\Phi'(u)-\lambda J'(u)-\mu\Psi'(u) = 0$ has at least three solutions in X whose norms are less than R.

Now we state some properties of variable exponent Sobolev spaces.

For $p \in C([0, 1], \mathbb{R})$ we assume that

$$1 < p^- := \min_{x\in[0,1]} p(x) \le p^+ := \max_{x\in[0,1]} p(x). \tag{2.1}$$

The variable exponent Lebesgue space is defined as follows

$$L^{p(x)}([0, 1]) := \left\{u : [0, 1] \to \mathbb{R} : \ u \text{ is measurable and } \int_0^1 |u|^{p(x)}\, dx < +\infty\right\},$$

endowed with the norm

$$\|u\|_{L^{p(x)}([0,1])} := \inf\left\{\lambda > 0 : \int_0^1 |\frac{u(x)}{\lambda}|^{p(x)}\, dx \le 1\right\}.$$

The generalized Lebesgue-Sobolev space $W^{1,p(x)}([0, 1])$ is defined by

$$W^{1,p(x)}([0, 1]) := \left\{u : \ u \in L^{p(x)}([0, 1]), u' \in L^{p(x)}([0, 1])\right\},$$

which is endowed with the norm

$$\|u\|_{W^{1,p(x)}([0,1])} := \|u\|_{L^{p(x)}([0,1])} + \||u'|\|_{L^{p(x)}([0,1])}. \tag{2.2}$$

From (2.1), both $L^{p(x)}([0, 1])$ and $W^{1,p(x)}([0, 1])$ are separable, reflexive uniformly convex Banach spaces (for more details, see [12]). Moreover, since $\alpha \in L^\infty([0, 1])$, with $\alpha_- := ess\inf_{x\in[0,1]} \alpha(x) > 0$

$$\|u\|_\alpha := \inf\left\{\sigma > 0 : \int_0^1 \left(|\frac{u'(x)}{\sigma}|^{p(x)} + \alpha(x)|\frac{u(x)}{\sigma}|^{p(x)}\right) dx \le 1\right\},$$

on $W^{1,p(x)}([0, 1])$ is an equivalent to that introduced in (2.2).

Proposition 2.3. *(see [9, Proposition 2.1]) For all $u \in W^{1,p(x)}([0, 1])$, one has*

$$\|u\|_{C^0([0,1]} \le m\|u\|_\alpha \quad \text{and} \quad \|u\|_{C^0([0,1]} \le 2\|u\|_{W^{1,p(x)}([0,1])} \tag{2.3}$$

and in particular, $\|u\| := \|u\|_{W^{1,p(x)}([0,1])} \le \frac{m}{2}\|u\|_\alpha$, where

$$m = \begin{cases} 2\left[\dfrac{1}{1+\alpha_-^{\frac{p^+}{p^-(1-p^+)}}}\right]^{\frac{1}{p^+}} + \left[1-\dfrac{1}{1+\alpha_-^{\frac{p^+}{p^-(1-p^+)}}}\right]^{\frac{1}{p^+}} \dfrac{2}{\alpha_-^{\frac{1}{p^-}}}, & \text{if } \alpha_- < 1, \\ 2\left[\dfrac{1}{1+\alpha_-^{\frac{1}{(1-p^+)}}}\right]^{\frac{1}{p^+}} + \left[1-\dfrac{1}{1+\alpha_-^{\frac{1}{(1-p^+)}}}\right]^{\frac{1}{p^+}} \dfrac{2}{\alpha_-^{\frac{1}{p^+}}}, & \text{if } \alpha_- \ge 1. \end{cases}$$

Proposition 2.4. *[11] For $u, u_k \in W^{1,p(x)}(]0,1[)$; $k = 1, 2, ...,$ we have*

(1) $\|u\| \geq 1$ *if and only if* $\|u\|^{p^-} \leq \rho_\alpha(u) \leq \|u\|^{p^+}$;

(2) $\|u\| \leq 1$ *if and only if* $\|u\|^{p^-} \geq \rho_\alpha(u) \geq \|u\|^{p^+}$;

(3) $\|u_k\| \to 0$ *as* $k \to +\infty$ *if and only if* $\rho_\alpha(u_k) \to 0$ *as* $k \to +\infty$;

(4) $\|u_k\| \to +\infty$ *as* $k \to +\infty$ *if and only if* $\rho_\alpha(u_k) \to +\infty$ *as* $k \to +\infty$.

Throughout this article, assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, that is,

(a) $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;

(b) $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;

(c) for every $s > 0$ there is a function $l_s \in L^1([0,1])$ such that

$$\sup_{|\xi| \leq s} |f(x,\xi)| \leq l_s(x), \quad \text{for a.e. } x \in [0,1],$$

the functions $g, h : \mathbb{R} \to \mathbb{R}$ are non-negative continuous, and parameters λ and μ are real.

Corresponding to the functions f, g, h and M, we introduce the functions $F : [0,1] \times \mathbb{R} \to \mathbb{R}$, $G, H : \mathbb{R} \to [0, +\infty[$ and $\tilde{M} : [0, +\infty[\to \mathbb{R}$ defined as follows

$$F(x,t) = \int_0^t f(x,\xi)\, d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R},$$

$$G(t) = \int_0^t g(\xi)\, d\xi, \;\; H(t) = \int_0^t h(\xi)\, d\xi \quad \text{for all } t \in \mathbb{R},$$

$$\tilde{M}(t) = \int_0^t M(\xi)\, d\xi \quad \text{for all } t \geq 0.$$

Definition 2.5. $u : [0,1] \to \mathbb{R}$ *is a weak solution of the problem* (1.1) *if*

$$M\left(\int_0^1 \frac{1}{p(x)}\left(|u'|^{p(x)} + \alpha(x)|u|^{p(x)}\right) dx\right) \times \left[\int_0^1 |u'|^{p(x)-2} u'v'dx + \int_0^1 \alpha(x)|u|^{p(x)-2} uvdx\right]$$

$$-\lambda \int_0^1 f(x,u)v\, dx - \mu\left[g(u(0))v(0) + h(u(1))v(1)\right] = 0,$$

for all $v \in W^{1,p(x)}([0,1])$.

3. Main results

In this section, we establish the main abstract results of this paper. Before introducing our results, for $\theta > 0$ and $\eta \in C([0,1])$ with $1 < \eta^-$, we put

$$[\theta]^\eta := \max\left\{\theta^{\eta^-}, \theta^{\eta^+}\right\} \quad \text{and} \quad [\theta]_\eta := \min\left\{\theta^{\eta^-}, \theta^{\eta^+}\right\}.$$

Let

$$\lambda_1 := \inf\left\{\frac{\tilde{M}\left(\int_0^1 \frac{1}{p(x)}\left(|u'(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\right)\, dx\right)}{\int_0^1 F(x,u(x))\, dx} : \; u \in X, \int_0^1 F(x,u(x))\, dx > 0\right\}$$

and $\lambda_2 := \frac{1}{\max\{0, \lambda_0, \lambda_\infty\}}$, where

$$\lambda_0 := \limsup_{|u| \to 0} \frac{\int_0^1 F(x,u(x))\, dx}{\tilde{M}\left(\int_0^1 \frac{1}{p(x)}\left(|u'(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\right)\, dx\right)}$$

and

$$\lambda_\infty := \limsup_{\|u\|\to+\infty} \frac{\int_0^1 F(x,u(x))\,dx}{\tilde{M}\left(\int_0^1 \frac{1}{p(x)}\left(|u'(x)|^{p(x)}+\alpha(x)|u(x)|^{p(x)}\right)\,dx\right)}.$$

Our main result reads as follows:

Theorem 3.1. *Assume that*

(f_1) *there exists a constant $\epsilon > 0$ such that*

$$\max\left\{\limsup_{u\to 0}\frac{\max_{x\in[0,1]}F(x,u(x))}{|u|^{p^-}},\limsup_{|u|\to+\infty}\frac{\max_{x\in[0,1]}F(x,u(x))}{|u|^{p^-}}\right\}<\epsilon;$$

(f_2) *there exists a function $w \in W^{1,p(x)}([0,1])$ such that*

$$K_w := \tilde{M}\left(\int_0^1 \frac{1}{p(x)}\left(|w'(x)|^{p(x)}+\alpha(x)|w(x)|^{p(x)}\right)\,dx\right)\neq 0$$

and

$$\epsilon < \frac{m_0\int_0^1 F(x,w(x))\,dx}{p^+2^{p^+}K_w}.$$

Then, for each compact interval $[c,d]\subset(\lambda_1,\lambda_2)$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every two non-negative continuous functions $g,h:\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, the problem (1.1) has at least three weak solutions whose norms in $W^{1,p(x)}([0,1])$ are less than R.

Proof. Take $X = W^{1,p(x)}([0,1])$. It is well known that, in view of (2.1), both $L^{p(x)}([0,1])$ and X are separable and reflexive uniformly convex Banach spaces (see [12]). Let the functionals Φ, J and Ψ be defined as fallow:

$$\Phi(u)=\tilde{M}\left(\int_0^1 \frac{1}{p(x)}\left(|u'|^{p(x)}+\alpha(x)|u|^{p(x)}\right)\,dx\right), \tag{3.1}$$

$$J(u)=\int_0^1 F(x,u(x))\,dx, \tag{3.2}$$

$$\Psi(u)=G(u(0))+H(u(1)), \tag{3.3}$$

for every $u\in X$. The functional Φ is of C^1, and by [2, Theorem 3.1] and [9, Theorem 3.1], Φ is sequentially weakly lower semi-continuous and continuously Gâteaux differentiable functional whose Gâteaux derivative $\Phi':X\to X^*$ defined as

$$\Phi'(u)(v)=M\left(\int_0^1\frac{1}{p(x)}\left(|u'|^{p(x)}+\alpha(x)|u|^{p(x)}\right)dx\right)\times\Big[\int_0^1|u'|^{p(x)-2}u'v'\,dx+\int_0^1\alpha(x)|u|^{p(x)-2}u\,v\,dx\Big],$$

for every $u,v\in X$, is a homeomorphism, in particular Φ' admits a continuous inverse on $X^*:=\left(W^{1,p(x)}([0,1])\right)^*$. Moreover, since $m_0\le M(t)\le m_1$ for all $t\in[0,+\infty[$, from (3.1) and using Proposition 2.4, we have

$$\frac{m_0}{p^+}\,[\|u\|]_p\le\Phi(u)\le\frac{m_1}{p^-}\,[\|u\|]^p, \tag{3.4}$$

for all $u\in X$, which follows $\lim_{\|u\|\to+\infty}\Phi(u)=+\infty$, namely, the functional Φ is coercive. Moreover, let A be a bounded subset of X. That is, there exists a constant $m_2>0$ such that $\|u\|\le m_2$ for each $u\in A$. Then, by (3.4), we have $|\Phi(u)|\le\frac{m_1}{p^-}[m_2]^p$. Hence Φ is bounded on each bounded subset of X. Furthermore, by Remark 2.1, $\Phi\in\mathcal{W}_X$. The functionals J and Ψ are two C^1-functionals and using the compact embedding $W^{1,p(x)}(]0,1[)\hookrightarrow L^{p(x)}(]0,1[)$ and considering in fact that the functions g and h are non-negative, It can be obtained that J and Ψ have compact derivatives (for more details see [2, Theorem 3.1]) defined as

$$J'(u)(v) = \int_0^1 f(x, u(x))v(x)\,dx$$

and

$$\Psi'(u)(v) = g(u(0))v(0) + h(u(1))v(1),$$

for every $u, v \in X$. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. In view of (f_1), there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(x, u) \le \epsilon |u|^{p^-} \tag{3.5}$$

for every $x \in (0, 1)$ and every u with $|u| \in [0, \tau_1) \cup (\tau_2, +\infty)$. Since $F(x, u)$ is continuous on $(0, 1) \times \mathbb{R}$, it is bounded on $x \in]0, 1[$ and $|u| \in [\tau_1, \tau_2]$. Thus we can choose $\eta > 0$ and $\nu > p^+$ such that $F(x, u) \le \epsilon|u|^{p^+} + \eta|u|^{\nu}$ for all $(x, u) \in]0, 1[\times\mathbb{R}$. So, by Proposition 2.3, we have

$$J(u) = \int_0^1 F(x, u)\,dx \le \epsilon 2^{p^+}\|u\|^{p^+} + \eta\ 2^{\nu}\|u\|^{\nu} \tag{3.6}$$

for all $u \in X$. Hence, from (3.4) and (3.6) we obtain

$$\begin{aligned}\limsup_{|u|\to 0} \frac{J(u)}{\Phi(u)} &\le \limsup_{|u|\to 0} \frac{\epsilon 2^{p^+}\|u\|^{p^+} + \eta 2^{\nu}\|u\|^{\nu}}{\frac{m_0}{p^+}\|u\|^{p^+}} \\ &= \frac{p^+ 2^{p^+}\epsilon}{m_0}.\end{aligned} \tag{3.7}$$

Moreover, by using (3.4), (3.5) and Proposition 2.3, for each $u \in X\backslash\{0\}$, we obtain

$$\begin{aligned}\frac{J(u)}{\Phi(u)} &= \frac{\int_{|u|\le\tau_2} F(x, u)\,dx}{\Phi(u)} + \frac{\int_{|u|>\tau_2} F(x, u)\,dx}{\Phi(u)} \\ &\le \frac{p^+ \sup_{x\in(0,1),|u|\in[0,\tau_2]} F(x, u)}{m_0[\|u\|]_p} + \frac{2^{p^+}p^+\epsilon[\|u\|]_p}{m_0[\|u\|]_p}.\end{aligned}$$

So

$$\limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)} \le \frac{2^{p^+}p^+\epsilon}{m_0}. \tag{3.8}$$

In view of (3.7) and (3.8), we have

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{J(u)}{\Phi(u)}\right\} \le \frac{p^+ 2^{p^+}\epsilon}{m_0}. \tag{3.9}$$

Assumption (f_2) in conjunction with (3.9) yields

$$\begin{aligned}\sigma &= \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{J(u)}{\Phi(u)} = \sup_{X\backslash\{0\}} \frac{J(u)}{\Phi(u)} \\ &\ge \frac{\int_0^1 F(x, w(x))\,dx}{\Phi(w(x))} = \frac{\int_0^1 F(x, w(x))\,dx}{K_w} \\ &> \frac{p^+ 2^{p^+}\epsilon}{m_0} \ge \rho.\end{aligned}$$

Thus, all the hypotheses of Theorem 2.2 are satisfied. Clearly, $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = \frac{1}{\rho}$. Then, using Theorem 2.2, for each compact interval $[c, d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ with the following property: for every

$\lambda \in [c,d]$ and every two non-negative continuous functions $g,h:\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that for each $\mu\in[0,\delta]$, the problem (1.1) has at least three weak solutions whose norms in X are less than R (standard arguments show that $I =: \Phi-\lambda J-\mu\Psi$ is a Gâteaux differentiable functional and a vector $u\in X$ is a solution of the problem (1.1) if and only if u be a critical point of the function I). $\square$

Another announced application of Theorem 2.2 reads as follows:

Theorem 3.2. *Assume that*

$$\max\left\{\limsup_{u\to 0}\frac{\max_{x\in[0,1]}F(x,u(x))}{|u|^{p^-}},\limsup_{|u|\to+\infty}\frac{\max_{x\in[0,1]}F(x,u(x))}{|u|^{p^-}}\right\}\le 0 \tag{3.10}$$

and

$$\sup_{u\in X}\frac{m_0\int_0^1 F(x,u(x))\,dx}{p^+2^{p^+}\tilde{M}\left(\int_0^1\frac{1}{p(x)}\left(|u'(x)|^{p(x)}+\alpha(x)|u(x)|^{p(x)}\right)dx\right)}>0. \tag{3.11}$$

Then, for each compact interval $[c,d]\subset(\lambda_1,+\infty)$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every two non-negative continuous functions $g,h:\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, the problem (1.1) has at least three weak solutions whose norms in X are less than R.

Proof. In view of (3.10), there exist an arbitrary $\epsilon>0$ and τ_1,τ_2 with $0<\tau_1<\tau_2$ such that $F(x,u)\le\epsilon|u|^{p^-}$ for every $x\in]0,1[$ and every u with $|u|\in[0,\tau_1)\cup(\tau_2,+\infty)$. Since $F(x,u)$ is continuous on $[0,1]\times\mathbb{R}$, it is bounded on $x\in]0,1[$ and $|u|\in[\tau_1,\tau_2]$. Thus we can choose $\eta>0$ and $\nu>p^+$ in a manner that $F(x,u)\epsilon|u|^{p^+}+\eta|u|^\nu$ for all $(x,u)\in]0,1[\times\mathbb{R}$. So, by the same process as that in the proof of Theorem 3.1, we have relations (3.7) and (3.8). Since ϵ is arbitrary, (3.7) and (3.8) give

$$\max\left\{0,\limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)},\limsup_{u\to 0}\frac{J(u)}{\Phi(u)}\right\}\le 0.$$

Then, with the notation of Theorem 2.2, we have $\rho=0$. By (3.11), we also obtain $\sigma>0$. In this case, clearly $\lambda_1=\frac{1}{\sigma}$ and $\lambda_2=+\infty$. Thus, by using Theorem 2.2, the result is achieved. $\square$

Put

$$\vartheta_1 := \int_{(\frac16,\frac56)\setminus(\frac13,\frac23)}\frac{1}{p(x)}\left|324|x-\frac12|(x-\frac12)-216(x-\frac12)+\frac{27}{|x-\frac12|}\right|^{p(x)}dx,$$

$$\vartheta_2 := \int_{\frac13}^{\frac23}\frac{\alpha(x)}{p(x)}\,dx+\int_{(\frac16,\frac56)\setminus(\frac13,\frac23)}\frac{\alpha(x)}{p(x)}\left|108|x-\frac12|^3-108|x-\frac12|^2+27|x-\frac12|-1\right|^{p(x)}dx,$$

and

$$L := \vartheta_1+\vartheta_2. \tag{3.12}$$

The next theorem provides sufficient conditions for applying Theorem 3.1, which does not require to know a test function w satisfying (f_2).

Theorem 3.3. *Assume that assumption (f_1) in Theorem 3.1* holds and there exists a positive constant d such that

(f_3) $F(x,t)\ge 0$ for each $x\in(\frac16,\frac13]\cup[\frac23,\frac56)$;

(f_4) $\tilde{M}\left([d]_p\vartheta_1+[d]_p\vartheta_2\right)\neq 0$ and $\epsilon<\frac{m_0\int_{\frac13}^{\frac23}F(x,d)\,dx}{p^+2^{p^+}\tilde{M}([d]^p\vartheta_1+[d]^p\vartheta_2)}$.

Then, for each compact interval $[c,d]\subset(\lambda_1,\lambda_2)$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every two non-negative continuous functions $g,h:\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, the problem (1.1) has at least three weak solutions whose norms in X are less than R.

Proof. We claim that all the assumptions of Theorem 3.1 are fulfilled by choosing w as follows:

$$w(x) := \begin{cases} 0, & \text{if } x \in [0, \frac{1}{6}] \cup [\frac{5}{6}, 1], \\ d\left(108|x - \frac{1}{2}|^3 - 108|x - \frac{1}{2}|^2 + 27|x - \frac{1}{2}| - 1\right), & \text{if } x \in (\frac{1}{6}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{5}{6}), \\ d, & \text{if } x \in (\frac{1}{3}, \frac{2}{3}). \end{cases} \tag{3.13}$$

We have

$$w'(x) := \begin{cases} 0, & \text{if } x \in [0, \frac{1}{6}] \cup [\frac{5}{6}, 1] \cup (\frac{1}{3}, \frac{2}{3}), \\ d\left(324|x - \frac{1}{2}|(x - \frac{1}{2}) - 216(x - \frac{1}{2})^2 + \frac{27}{|x - \frac{1}{2}|}\right), & \text{if } x \in (\frac{1}{6}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{5}{6}). \end{cases}$$

In particular

$$\int_0^1 \frac{1}{p(x)} |w'(x)|^{p(x)}\, dx \begin{cases} \geq [d]_p \displaystyle\int_{I_1} \frac{1}{p(x)} \Big|324|x - \frac{1}{2}|(x - \frac{1}{2}) - 216(x - \frac{1}{2}) + \frac{27}{|x - \frac{1}{2}|}\Big|^{p(x)} dx, \\ \leq [d]^p \displaystyle\int_{I_1} \frac{1}{p(x)} \Big|324|x - \frac{1}{2}|(x - \frac{1}{2}) - 216(x - \frac{1}{2}) + \frac{27}{|x - \frac{1}{2}|}\Big|^{p(x)} dx, \end{cases}$$

where $I_1 := (\frac{1}{6}, \frac{5}{6}) \backslash (\frac{1}{3}, \frac{2}{3})$.
So

$$[d]_p \vartheta_1 \leq \int_0^1 \frac{1}{p(x)} |w'(x)|^{p(x)}\, dx \leq [d]^p \vartheta_1$$

and similarly

$$[d]_p \vartheta_2 \leq \int_0^1 \frac{\alpha(x)}{p(x)} |w(x)|^{p(x)}\, dx \leq [d]^p \vartheta_2.$$

It is easy to see that $w \in X$, and one has

$$\begin{aligned} m_0[d]_p L &= m_0[d]_p(\vartheta_1 + \vartheta_2) \\ &\leq \tilde{M}([d]_p \vartheta_1 + [d]_p \vartheta_2) \\ &\leq \Phi(w) = \tilde{M}\left(\int_0^1 \frac{1}{p(x)} \left(|w'|^{p(x)} + \alpha(x)|w|^{p(x)}\right) dx\right) \\ &\leq \tilde{M}\left([d]^p \vartheta_1 + [d]^p \vartheta_2\right) \\ &\leq m_1[d]^p(\vartheta_1 + \vartheta_2) = m_1[d]^p L. \end{aligned} \tag{3.14}$$

Thus from (3.14), taking into account we derive that if $\tilde{M}([d]_p(\vartheta_1 + \vartheta_2)) \neq 0$ than $\tilde{M}([d]^p(\vartheta_1 + \vartheta_2)) \neq 0$. So from (f_3) and (f_4), it is easy to see that the assumption (f_2) of Theorem 3.1 is satisfied. Hence, Theorem 3.1 follows the results. □

Remark 3.4. The statements of Theorem 3.3 depend upon the test function w defined by (3.13). If we choose the other type of w, we observe another statement. For example, we pick

$$\vartheta_1' := \int_{(\frac{1}{6}, \frac{5}{6}) \backslash (\frac{1}{3}, \frac{2}{3})} \frac{1}{p(x)} \left|(\frac{1}{3} - |x - \frac{1}{2}|)(\frac{1}{3} - 2|x - \frac{1}{2}|)(x - \frac{1}{2})\right|^{p(x)} dx,$$

$$\vartheta_2' := \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\alpha(x)}{p(x)}\, dx + \int_{(\frac{1}{6}, \frac{5}{6}) \backslash (\frac{1}{3}, \frac{2}{3})} \frac{\alpha(x)}{p(x)} \left|(x - \frac{1}{2})(\frac{1}{3} - |x - \frac{1}{2}|)\right|^{2p(x)} dx$$

and $L' := \vartheta_1' + \vartheta_2'$ and we take

$$w(x) := \begin{cases} 0, & \text{if } x \in [0, \frac{1}{6}] \cup [\frac{5}{6}, 1], \\ 6^4 d|x - \frac{1}{2}|^2(\frac{1}{3} - |x - \frac{1}{2}|)^2, & \text{if } x \in (\frac{1}{6}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{5}{6}), \quad \text{(see [21] for } x^0 = \frac{1}{2}, s = \frac{1}{3}) \\ d, & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \end{cases} \tag{3.15}$$

then we obtain

$$w'(x) := \begin{cases} 0, & \text{if } x \in [0, \frac{1}{6}] \cup [\frac{5}{6}, 1] \cup (\frac{1}{3}, \frac{2}{3}), \\ 2d\ 6^4(\frac{1}{3} - |x - \frac{1}{2}|)(\frac{1}{3} - 2|x - \frac{1}{2}|)(x - \frac{1}{2}), & \text{if } x \in (\frac{1}{6}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{5}{6}). \end{cases}$$

So,

$$[d]_p \vartheta_1' \leq \int_0^1 \frac{1}{p(x)} |w'(x)|^{p(x)}\, dx \leq \left(2 \times 6^4\right)^{p^+} [d]^p \vartheta_1'$$

and similarly

$$[d]_p \vartheta_2' \leq \int_0^1 \frac{\alpha(x)}{p(x)} |w(x)|^{p(x)}\, dx \leq \left(2 \times 6^4\right)^{p^+} [d]^p \vartheta_2'.$$

It is easy to see that $w \in X$ and in particular, one has

$$\begin{aligned} m_0 [d]_p L' &= m_0 [d]_p (\vartheta_1' + \vartheta_2') \\ &\leq \tilde{M}([d]_p \vartheta_1' + [d]_p \vartheta_2') \\ &\leq \Phi(w) = \tilde{M}\left(\int_0^1 \frac{1}{p(x)} \left(|w'|^{p(x)} + \alpha(x)|w|^{p(x)}\right) dx\right) \\ &\leq \left(2 \times 6^4\right)^{p^+} \tilde{M}\left([d]^p \vartheta_1' + [d]^p \vartheta_2'\right) \\ &\leq M^+ [d]^p L', \end{aligned}$$

where $M^+ m_1 \left(2 \times 6^4\right)^{p^+}$. Therefore, condition (f_4) in Theorem 3.3 takes the following form:

(f_5) there exists a positive constant d such that

$$\tilde{M}\left([d]_p \vartheta_1' + [d]_p \vartheta_2'\right) \neq 0 \quad \text{and} \quad \epsilon < \frac{m_0 \int_0^1 F(x,d)\, dx}{p^+ 2^{p^+} \tilde{M}\left([d]^p \vartheta_1' + [d]^p \vartheta_2'\right)},$$

where w is given by (3.15).

Also, by choosing w as given in [21, Remark 3.4] (for $x^0 = \frac{1}{2}, r_2 = \frac{1}{3}$ and $r_1 = \frac{1}{6}$), which is as follows:

$$w(x) := \begin{cases} 0, & \text{if } x \in [0, \frac{1}{6}] \cup [\frac{5}{6}, 1], \\ 432\ d|x - \frac{1}{2}|^2 \left[3|x - \frac{1}{2}|)^2 - 2|x - \frac{1}{2}| + \frac{1}{3}\right], & \text{if } x \in (\frac{1}{6}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{5}{6}), \quad \text{(see [18,19])} \\ d, & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \end{cases}$$

one has another form of condition (f_4).

Now, we point out some results in which the function f has separated variables. Consider the following problem

$$\begin{cases} T(u) = \lambda \theta(x) f(u), & \text{in } (0,1), \\ |u'(0)|^{p(0)-2} u'(0) = -\mu g(u(0)), \\ |u'(1)|^{p(1)-2} u'(1) = \mu h(u(1)), \end{cases} \tag{3.16}$$

where $T(u) := M\left(\int_0^1 \frac{1}{p(x)}(|u'|^{p(x)} + \alpha(x)|u|^{p(x)})dx\right)[-(|u'|^{p(x)-2}u')' + \alpha(x)|u|^{p(x)-2}u]$, $\theta : (0,1) \to \mathbb{R}$ is a non-negative and non-zero function, $\theta \in L^1(]0,1[)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g, h : \mathbb{R} \to \mathbb{R}$ are as introduced in the problem (1.1) in Introduction.

Set $F(x,t) = \theta(x)F(t)$ for every $(x,t) \in (0,1) \times \mathbb{R}$, where $F(t) = \int_0^t f(\xi)\, d\xi$ for all $t \in \mathbb{R}$. The following existence results are consequences of Theorem 3.3

Theorem 3.5. *Suppose that*

(f_6) *there exists a positive constant* ϵ *such that*

$$\sup_{x\in(0,1)} \theta(x) \ \max\left\{\limsup_{u\to 0}\frac{F(u)}{|u|^{p^-}}, \limsup_{|u|\to+\infty}\frac{F(u)}{|u|^{p^-}}\right\} < \epsilon;$$

(f_7) *there exists a positive constant* d *such that*

$$\tilde{M}\left([d]_p\vartheta_1 + [d]_p\vartheta_2\right) \neq 0 \quad \textit{and} \quad \epsilon < \frac{m_0\int_0^1 F(x,w(x))\,dx}{p^+ 2^{p^+}\tilde{M}\left([d]^p\vartheta_1 + [d]^p\vartheta_2\right)},$$

where w *is given by* (3.13).

Then, for each compact interval $[c,d] \subset (\lambda_3, \lambda_4)$, *(where* λ_3 *and* λ_4 *are the same as* λ_1 *and* λ_2 *but* $\int_0^1 F(x,u(x))\,dx$ *is replaced by* $\int_0^1 \theta(x)F(u(x))\,dx$, *respectively), there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every two non-negative continuous functions* $g, h : \mathbb{R} \to \mathbb{R}$, *there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the problem* (3.16) *has at least three weak solutions whose norms in* X *are less than* R.

Theorem 3.6. *Assume that there exists a positive constant* d *such that*

$$\tilde{M}\left([d]_p\vartheta_1 + [d]_p\vartheta_2\right) > 0 \quad \textit{and} \quad F(d) > 0. \tag{3.17}$$

Moreover, suppose that

$$\limsup_{u\to 0}\frac{f(u)}{|u|^{p^- -1}} = \limsup_{|u|\to+\infty}\frac{f(u)}{|u|^{p^- -1}} = 0. \tag{3.18}$$

Then, for each compact interval $[c,d] \subset (\lambda_3, +\infty)$, *where* λ_3 *is the same as* λ_1 *but* $\int_0^1 F(x,u(x))\,dx$ *is replaced by* $\int_0^1 \theta(x)F(u(x))\,dx$, *there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every two non-negative continuous functions* $g, h : \mathbb{R} \to \mathbb{R}$, *there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the problem* (3.16) *has at least three weak solutions whose norms in* X *are less than* R.

Proof. We easily see that from (3.18), assumption (f_6) is satisfied for every $\epsilon > 0$. Moreover, using (3.17), by choosing $\epsilon > 0$ small enough, one can arrive to assumption (f_7). Hence, the conclusion follows from Theorem 3.5. $\square$

Remark 3.7. From cited results, we realize that nowhere in theorems asymptotic conditions on the functions f, g and h are required, and only the algebraic conditions on f are supposed to guarantee the existence of solutions.

We consider the following example in which the nonlinearity f verifies the hypotheses of Theorem 3.6 and the constructions of the nonlinear functions are partly motivated by [15, Example 3.1].

Example 3.8. *Let* $p(x) = x^2 + 6$, $\alpha(x) = 1$, *for each* $x \in (0,1)$,

$$M(t) = 1 + \frac{1}{\cosh t} \quad \textit{for all } t \geq 0,$$

$\theta(x) = 1$ *for all* $x \in (0,1)$ *and*

$$f(t) = \begin{cases} 2(t+\sin t)^2, & \textit{if } \ t < \pi, \\ 2\pi^2 + \tanh(t-\pi), & \textit{if } \ t \geq \pi. \end{cases}$$

Thus, $m_0 = 1$, $m_1 = 2, p^- = 6$, $p^+ = 7$ *and* f *is a non-negative and continuous function by which choosing* $d = 1$, *we have*

$$F(d) = F(1) = 2\int_0^1 (t+\sin t)^2\,dt > 0.$$

On the other hand, since $\tilde{M}(t) \geq m_0 t$ for each $t \geq 0$, one has

$$\begin{aligned}\tilde{M}(\vartheta_1+\vartheta_2) \geq \vartheta_1+\vartheta_2 &\geq \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\alpha(x)}{p(x)}\,dx \\ &\geq \frac{1}{7}\int_{\frac{1}{3}}^{\frac{2}{3}} dx = \frac{1}{21} > 0.\end{aligned}$$

Moreover, we have

$$\limsup_{u\to 0}\frac{f(u)}{|u|^{p^- -1}} = \lim_{u\to 0}\frac{2(u+\sin u)^2}{|u|^5} = 0$$

and

$$\lim_{|u|\to+\infty}\frac{f(u)}{|u|^{p^- -1}} = \lim_{|u|\to+\infty}\frac{2\pi^2+\tanh(u-\pi)}{|u|^5} = 0.$$

Hence, by applying Theorem 3.6 for each compact interval $[c,d] \subset (0,+\infty)$, there exists $R>0$ with the following property: for every $\lambda \in [c,d]$ and every two non-negative continuous functions $g,h:\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, the problem

$$\begin{cases} \Upsilon(u) = \lambda f(u), & \text{in } (0,1),\\ |u'(0)|^4 u'(0) = -\mu g(u(0)), & \\ |u'(1)|^5 u'(1) = \mu h(u(1)), & \end{cases}$$

where

$$\Upsilon(u) := \left(1+\frac{1}{\cosh\left(\int_0^1 \frac{1}{x^2+6}\left[|u'|^{x^2+6}+|u|^{x^2+6}\right]dx\right)}\right)\left[-\left(|u'|^{x^2+4}u'\right)' + |u|^{x^2+4}u\right],$$

has at least three weak solutions whose norms in X (by $p(x)=x^2+6$) are less than R.

Remark 3.9. We point out that the same statements of the above given results can be obtained by considering the special case $M(t)=b_1+b_2t$ for $t\in[\iota,\kappa]$, where b_1,b_2,ι and κ are positive numbers. In fact, we have

$$\tilde{M}(t)=\int_0^t M(\xi)\,d\xi = \int_0^t (b_1+b_2\xi)\,d\xi = b_1t+\frac{b_2}{2}t^2 = \frac{(b_1+b_2t)^2}{2b_2}-\frac{b_1^2}{2b_2} \quad \text{for } t\in[\iota,\kappa],$$

$$m_0 = b_1+b_2\iota \quad \text{and} \quad m_1 = b_1+b_2\kappa.$$

Arguing as in the proof of Theorem 3.1, three weak solutions can be obtained.

References

1. C.O. Alves, F.J.S.A. Corrêa, T.F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type,* Comput. Math. Appl. **49** (2005), 85-93.
2. G. Bonanno, A. Chinnì, *Discontinuous elliptic problems involving the $p(x)-$Laplacian*, Math. Nachr. **284** (2011), 639-652
3. F. Cammaroto, A. Chinnì, B. Di Bella, *Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian,* Nonlinear Anal. **71** (2009), 4486-4492.
4. G. Bonanno, B. Di Bella, *A boundary value problem for fourth-order elastic beam equations*, J. Math. Anal. Appl. 343(2008), 1166-1176.
5. G. Bonanno, B. Di Bella and D. O. Regan, *Non-trivial solutions for nonlinear fourth-order elastic beam equations*, Comput. Math. Appl. 62 (2011), 1862- 1869.
6. G. Bonanno, G. D'Aguì and A. Sciammetta, *One-dimensional non- linear boundary value problems with variable exponent,* Discrete and continuous Dynamical Systems, Series S, 11(2) (2018), 179-191.
7. G. Bonanno, B. Di Bella and D. O. Regan, *Non-trivial solutions for nonlinear fourth-order elastic beam equations*, Comput. Math. Appl. 62 (2011), 1862-1869.

8. A. Cabada, J.A. Cid and L. Sanchez, *Positivity and lower and upper solutions for for fourth-order boundary value problems*, Nonlinear Anal. 67 (2007), 1599-1612.

9. G. D'Aguì, *Second-order boundary-value problems with variable exponent*, Electron. J. Differential Equations, Vol. **2014** (2014), No. 68, 1-10.

10. L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev spaces with variable exponents,* Lecture Notes in Mathematics, **2017** Springer-Verlag, Heidelberg, 2011.

11. X.L. Fan, X.Y. Han, *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$,* Nonlinear Anal. **59** (2004), 173-188.

12. X.L. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$,* J. Math. Anal. Appl. **263** (2001), 424-446.

13. S. Heidarkhani, S. Moradi and D. Barill, *Existence results for second- order boundary-value problems with variable exponents,* Nonlinear Anal. (RWA), 44 (2018), 40-53.

14. S. Heidarkhani, S. Moradi and S. A. Tersian, *Three solutions for second-order boundary-value problems with variable exponents* Electronic Journal of Qualitative Theory of Differential Equations **13** 2018, 1-19.

15. S. Heidarkhani, A. Salari, G. Caristi, D. Barilla, *Perturbed nonlocal fourth order equations of Kirchhoff type with Navier boundary conditions,* Bound. Value Probl. **2017** (2017), 86.

16. G. Kirchhoff, *Vorlesungen uber mathematische Physik*, Mechanik. Teubner, Leipzig (1883).

17. O. Kováčik, J. Rákosník, *On the spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. **41** (1991), 592-618.

18. L. Li, C-L. Tang, *Existence of three solutions for (p, q)-biharmonic systems,* Nonlinear Anal. TMA **73** (2010), 796-805.

19. L. Li, C-L. Tang, *Three solutions for a Navier boundary value problem involving the p-biharmonic,* Nonlinear Anal. TMA **72** (2010), 1339-1347.

20. M. Mihăilescu, *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplacian operator,* Nonlinear Anal. **67** (2007), 1419-1425.

21. G. Molica Bisci, D. Repovš, *Multiple solutions of p-biharmonic equations with Navier boundary conditions,* Complex Var. Elliptic Equ. **59** (2014), 271-284.

22. J. Musielak, *Orlicz spaces and modular spaces,* Lecture Notes in Mathematics 1034, Springer, Berlin, 1983.

23. C. Qian, Z. Shen, M. Yang, *Existence of solutions for $p(x)$-Laplacian nonhomogeneous Neumann problems with indefinite weight,* Nonlinear Anal. **11** (2010), 446-458.

24. B. Ricceri, *A further three critical points theorem,* Nonlinear Anal. TMA **71** (2009), 4151-4157.

25. X.J. Wang, R. Yuan, *Existence of periodic solutions for $p(t)$-Laplacian systems,* Nonlinear Anal. **70** (2009), 866-880.

26. V. V. Zhikov, *Averaging of functionals in the calculus of variations and elasticity,* Math. USSR Izv. **29** (1987), 33-66.

Iman E. Eskandarkolaei,
Department of Mathematics,
Faculty of Mathematical Sciences, University of Mazandaran, Babolsar,
Iran.
E-mail address: imanen64@yahoo.com

and

Somayeh Khademloo,
Department of Mathematics,
Faculty of Basic Sciences, Babol (Noushirvani) University of Technology, Babol,
Iran.
E-mail address: s.khademloo@nit.ac.ir

and

Ghasem A. Afrouzi,
Department of Mathematics,
Faculty of Mathematical Sciences, University of Mazandaran, Babolsar
Iran.
E-mail address: afrouzi@umz.ac.ir