About Weakly Bézout Rings

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ABSTRACT: In this paper, we examine the transfer of the proprety weakly Bézout to the trivial ring extensions. These results provide examples of weakly Bézout rings that are not Bézout rings. We show that the proprety weakly Bézout is not stable under finite direct products. Also, the class of 2- Bézout rings and the class of coherent rings are not comparable with the class of weakly Bézout rings.

Key Words: Weakly Bézout ring, Bézout ring, Trivial ring extension, Direct product.

Contents

1. Introduction

All rings considered below are commutative with unit and all modules are unital. A ring R is a Bézout ring if every finitely generated ideal of R is principal. Examples of Bézout rings are valuation rings, elementary divisor rings and Hermite rings. For instance see [6,10,11]. A ring R is called weakly Bézout if every finitely generated ideal of R contained in a principal proper ideal of R is itself principal (see [2, Definition 2]). If R is Bézout, then R is naturally weakly Bézout. Our aim in this paper is to prove that the converse is false in general.

For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence of R-modules:

$$F_n \longrightarrow F_{n-1} \longrightarrow \ldots F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are respectively, finitely generated and finitely presented R-modules.

A ring R is a coherent ring if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and every finitely generated ideals I and J of R [7, Theorem 2.3.2, p.45]. Examples of coherent rings are Noetherian rings, semihereditary rings and Bézout domains (see [7, p.47]).

Given nonnegative integers n and d, a ring R is called an (n, d)-ring if every n-presented R-module has projective dimension $\leqslant d$; and a weak (n, d)-ring if every n-presented cyclic R-module has projective dimension $\leqslant d$ (equivalently, if every $(n-1)$-presented ideal of R has projective dimension $\leqslant d-1$). See for instance [5,9].

A domain R is a Prüfer domain if every finitely generated ideal is projective (see [7, p.26]); equivalently, R is a $(1, 1)$-domain by [5, Theorem 1.3].

A domain R is called a 2-Prüfer domain if every finitely presented submodules of free modules are projective, i.e., $(2, 1)$-domain and weakly 2-Prüfer domain if every finitely presented ideal is projective. Clearly, every 2-Prüfer domain is a weakly 2-Prüfer domain. (For more details, see [5, Section 7]).

We say that R is a 2- Bézout ring if every finitely presented ideal of R is principal see [3]. This led us to consider the relation between the class of weakly Bézout rings and the class of 2-Bézout rings.

Let A be a ring, E be an A-module and $R := A \ltimes E$ be the set of pairs (a, e) with pairwise addition and multiplication given by: $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Considerable work, part of it summarized in Glaz's book [7] and Huckaba's book [8] where R is called the idealization of E by A, has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [7,8,12].

In the context of rings containing regular elements, we show that the notion of weakly Bézout coincides with the definition of Bézout ring. The goal of this work is to exhibit a class of non-Bézout rings which are weakly Bézout rings. We show that the class of weakly Bézout rings is not stable under finite direct products. Also, we show that the class of 2- Bézout rings and the class of coherent rings are not comparable with the class of weakly Bézout rings. For this purpose, we study the transfer of this property to trivial ring extensions.

2. Main results

Definition 2.1. *A ring R is called a weakly Bézout ring if for every finitely generated ideals I and J of R satisfying that $I \subseteq J \subsetneq R$, when J is principal of R, then so is I. (See [2, Definition 2]).*

We begin this section by giving a sufficient condition to have equivalence between Bézout and weakly Bézout properties.

Proposition 2.2. *Let R be a ring.*

1) If R is a Bézout ring, then R is a weakly Bézout ring.
2) Assume that R contains a non-invertible regular element (that is, R is not a total ring of quotients). Then, R is a Bézout ring if and only if R is a weakly Bézout ring.

Proof. 1) Clear.
2) It remains to show that, if R is a weakly Bézout ring and contains a regular element a non-invertible, then R is a Bézout ring. Let I be a finitely generated ideal of R and a a non-invertible regular element of R. Then $aI \subseteq aR$ and so aI is principal since R is weakly Bézout. Thus I is principal since $(I \cong aI)$, as desired. $\square$

Remark 2.3. *By the above result, a non-Bézout ring which is a weakly Bézout ring is necessarily a total ring of quotient.*

In this section, we study the possible transfer of the weakly Bézout property to various trivial extension contexts. First, we examine the context of trivial ring extensions of a local (A, M) by an A-module E such that $ME = 0$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \ltimes E'$ is a ideal of trivial ring extension of A by E.

Theorem 2.4. *Let A be a local ring with maximal ideal M, E be a A-module such that $ME = 0$ and let $R := A \ltimes E$ be the trivial ring extension of A by E.*
1) R is a weakly Bézout ring if and only if so is A.
2) If R is a Bézout ring, then $dim_{(A/M)}E = 1$.

Proof. Assume that R is a weakly Bézout ring. Our aim is to show that A is weakly Bézout. Let $I \subseteq J$ be two ideals of A such that I is finitely generated and J is principal proper. Then, $I \ltimes 0 \subseteq J \ltimes 0$ are two finitely generated proper ideals of R. Moreover, $J \ltimes 0 = Aa \ltimes 0 = R(a, 0)$ for some element a of A and so $I \ltimes 0 \subseteq J \ltimes 0$ is a principal ideal of R since R is a weakly Bézout ring that is, $I \ltimes 0 = R(b, 0) = Ab \ltimes 0$ for some element b of A. Hence $I = Ab$. Conversely, assume that A is a weakly Bézout ring. Our aim is to show that R is a weakly Bézout ring. Let $I := \sum_{i=1}^{i=n} R(a_i, e_i) \subseteq J := R(b, f)$ be two proper ideals of R such that n is a positive integre, $a_i, b \in A$ and $e_i, f \in E$ for each $i \in \{1, \ldots, n\}$, we which to show that

I is principal. Three cases are then possible.

Case 1 If $b = 0$. Then, $a_i = 0$ for all $i = 1, \ldots, n$; $I := 0 \ltimes E_1$ and $J := 0 \ltimes E_2$ where E_1 (resp.,E_2) is a vector subspace of E generated by the vectors $e_1, \ldots, e_n$ (resp., f). Hence, E_1 is a (A/M)-vector space of rank at most 1 (since $E_1 \subseteq E_2 = (A/M)f$) that is, $E_1 = (A/M)h$ where $h \in E_1$. Therefore, $I := 0 \ltimes (A/M)h = R(0, h)$ and so I is a principal ideal of R.

Case 2 If $b \neq 0$ and $a_i = 0$ for all $i \in \{1, \ldots, n\}$. In this case, $I := 0 \ltimes E_1$ and so principal since $J := R(b, f) \subseteq Ab \ltimes (A/M)f$.

Case 3 If $b \neq 0$ and $a_i \neq 0$ for some $i \in \{1, \ldots, n\}$. We assume that $((a_i, e_i)_{i=1}^n)$ is a minimal generating set of I, $I_0 := \sum_{i=1}^{i=n} Aa_i$ and $J_0 := Ab$. Consider the exact sequence of R-modules:

$$0 \longrightarrow Ker(u) \longrightarrow R^n \cong A^n \ltimes E^n \xrightarrow{u} I \longrightarrow 0$$

where $u((c_i, g_i)_{i=1}^n) = \sum_{i=1}^n (c_i, g_i)(a_i, e_i)$. But, $Ker(u) \subseteq (M \ltimes E)^n$ by [13, Lemma 4.43, page 134] since R is local by [1, Theorem 3.2 (1)]. Hence,

$$
\begin{aligned}
Ker(u) &= \{((c_i, g_i))_{i=1}^n \in R^n / \sum_{i=1}^n (c_i, g_i)(a_i, e_i) = 0\} \\
&= \{((c_i, g_i))_{i=1}^n \in R^n / \sum_{i=1}^n c_i a_i = 0\} \quad \text{(since } ME = 0) \\
&= V \ltimes E^n
\end{aligned}
$$

where $V := \{(c)_{i=1}^n \in A^n / \ \sum_{i=1}^n c_i a_i = 0\}$. Also, we have the exact sequence of R-modules:

$$0 \longrightarrow Ker(w) \longrightarrow R^n \xrightarrow{w} I_0 \ltimes 0 \longrightarrow 0$$

where $w((\alpha_i, k_i)) = \sum_{i=n}^n (\alpha_i, k_i)(a_i, 0)$. But, $Ker(w) = \{((\alpha_i, k_i))_{i=1}^n \in R^n / \sum_{i=1}^n \alpha_i a_i = 0\} = V \ltimes E^n$. Therefore, $I \cong I_0 \ltimes 0$ and since $I_0 \subseteq J_0$ (because $I \subseteq J$), then $I_0 = Aa$ for some element $a \in A$ since A is a weakly Bézout ring and so $I_0 \ltimes 0 = Aa \ltimes 0 = R(a, 0)$. Hence, I is a principal ideal of R in all cases. So, R is a weakly Bézout ring.

2) Assume that $\dim_{(A/M)} E \geqslant 2$. Let $e, f \in E$ such that $\{e, f\}$ is a (A/M)-linearly independant set and set $I := R(0, e) + R(0, f)$. We claim that I is not a principal ideal. Deny, there exists $g \in E$ such that $R(0, e) + R(0, f) = R(0, g) = 0 \ltimes Ag$. Hence, $Ae + Af = Ag$ moreover $(A/M)e + (A/M)f = (A/M)g$ (since $ME = 0$) and so $\{e, f\}$ is a (A/M)-linearly dependent set, a contradiction. Therefore, $R(0, e) + R(0, f)$ is not a principal ideal which means that R is not a Bézout ring and this completes the proof of Theorem 2.4.

$\square$

The condition $\dim_{(A/M)} E = 1$ is not sufficient in Theorem 2.4 (2) (see Example 2.8).

Now, we are able to construct a non-Bézout ring which is a weakly Bézout ring.

Example 2.5. *Let K be a field, E be a K-vector space such that $dim_K E \geqslant 2$ and $R := K \ltimes E$. Then:*

1) R is a weakly Bézout ring by Theorem 2.4 (1).

2) R is not a Bézout ring by Theorem 2.4 (2) since $dim_K E \geqslant 2$.

Proposition 2.6. *Let A be a domain which is not a field, E a nonzero A-module and $R := A \ltimes E$ be the trivial ring extension of A by E. If there exists a non-invertible nonzero element a in A such that $aE = 0$. Then, R is not a Bézout ring.*

Proof. Assume that R is a Bézout ring. Let $e \in E - \{0\}$ and set $J := R(a, 0) + R(0, e)$. Then, $J = R(b, x)$ for some $(b, x) \in J$ since R is a Bézout ring. Hence, $Aa = Ab$ and so $b = ua$ for some invertible element u of A (since A is a domain). Therefore, $R(b, x) = R(ua, x) = R(u, 0)(a, u^{-1}x) = R(a, u^{-1}x)$ since $(u, 0)$ invertible in R by [8, Theorem 25.1 (6)]. Then, $J := R(a, u^{-1}x)$. On the other hand $(a, 0) \in J$, so there exists $(c, f) \in R$ such that $(a, 0) = (c, f)(a, u^{-1}x) = (ca, cu^{-1}x)$ since $aE = 0$. Hence, $ca = a$ and $cu^{-1}x = 0$. Thus $c = 1$, $x = 0$; and so $J = Aa \ltimes 0$, which is a contradiction since $e \neq 0$. It follows that R is not a Bézout ring.

$\square$

Now, we give an example of non Bézout ring.

Example 2.7. *Let $A := \mathbb{Z}$ and $E := \mathbb{Z}/6\mathbb{Z}$. Then, $R := A \ltimes E$ is not a Bézout ring by Proposition 2.6.*

Now, we give a second example of a weakly Bézout ring which is not a Bézout ring.

Example 2.8. *Let A be a local Bézout domain which is not a field with maximal ideal M (for instance, $A := K[[X]]$ and $M := (X)$ where X an indeterminate over the field K) and $R := A \ltimes A/M$. Then:*

1) R is a weakly Bézout ring by Theorem 2.4.
2) R is not a Bézout ring by Proposition 2.6.

Next, we explore a different context; namely, the trivial ring extension of a domain A by a K-vector space E, where $K := qf(A)$.

Proposition 2.9. *Let A be a domain which is not a field, $K := qf(A)$, E be a K-vector space and $R := A \ltimes E$ be the trivial ring extension of A by E. If R is a weakly Bézout ring, then so is A and $dim_K E = 1$.*

Before proving Proposition 2.9, we establish the following Lemma.

Lemma 2.10. *. With the notation of Proposition 2.9, let $I := R(a, e)$ be a principal ideal of R, where $a \in A - \{0\}$ and $e \in E$. Then, $I := Aa \ltimes E = R(a, 0)$.*

Proof. Clearly, $I = R(a, e) = \{(b, f)(a, e)/b \in A, f \in E\} = \{(ba, fa + be)/b \in A, f \in E\}$. But $\{af/f \in E\} = E$, hence $I = Aa \ltimes E = R(a, 0)$. $\square$

Proof of Proposition 2.9. Assume that R is a weakly Bézout ring. We claim that A is weakly Bézout. Indeed, Let $I := \sum_{i=1}^{i=n} Aa_i$ for some positive integer n and $J := Ab$ be two proper ideals of A such that $I \subseteq J$. Then, $I \ltimes E := \sum_{i=1}^{i=n} Aa_i \ltimes E = Aa_1 \ltimes E + Aa_2 \ltimes E + \cdots + Aa_n \ltimes E = R(a_1, e_1) + R(a_2, e_2) + \cdots + R(a_n, e_n) = \sum_{i=1}^{i=n} R(a_i, 0)$ contained in $J \ltimes E := Ab \ltimes E = R(b, 0)$. Therefore, $I \ltimes E = R(a, k) = Aa \ltimes E$ for some element (a, k) of R since R is a weakly Bézout ring. Hence, $I = Aa$, and therefore A is a weakly Bézout ring. By way of contradiction, suppose that $\dim_K E \geqslant 2$ and let $\{e, f\}$ be a K-linearly independent set of E. Set $I := R(0, e) + R(0, f) \subseteq Aa \ltimes E$. As in the proof of Theorem 2.4 (2), I is not a principal ideal of R, while I is contained in the principal ideal $Aa \ltimes E$. $\square$

Next, we give an example of non weakly Bézout ring.

Example 2.11. *Let A be a Bézout domain which is not a field, $K := qf(A)$. Then, the trivial ring extension of A by K^2 is not weakly Bézout by Proposition 2.9 .*

It is straightforward to see that if a finite product $R := \prod_{i=1}^{n} R_i$ of commutative rings is a weakly Bézout ring, then R_i is a weakly Bézout for every i, however a finite product of weakly Bézout rings is not necessarily a weakly Bézout ring as shown by the next example.

Example 2.12. *Let A be a local Bézout ring with maximal ideal M and E be a A/M-vector space of finite rank $\geqslant 2$. Let R_1 be a Bézout ring and $R_2 := A \ltimes E$. Then, $R_1 \times R_2$ is not a weakly Bézout ring.*

Proof. It is clear that $R_2 := A \ltimes E$ is a weakly Bézout ring by Theorem 2.4 (1). But, $R_1 \times R_2$ is not a weakly Bézout ring. Indeed, let I be a principal proper ideal of R_1. Then, $I \times (0 \ltimes E)$ is a non-principal finitely generated ideal of $R_1 \times R_2$ contained in the principal ideal $I \times R_2$, which is a contradiction. $\square$

The following two examples show that is the class of weakly Bézout rings and the class of 2-Bézout rings are not comparable.

Example 2.13. *Let A be a discrete valuation domain with maximal ideal M (for instance $A := \mathbb{Z}_{(2)}$ and $M := 2\mathbb{Z}_{(2)}$), E be an (A/M)-vector space of finite rank $\geqslant 2$ and let $R := A \ltimes E$. Then:*
1) R is a weakly Bézout ring.
2) R is not a 2-Bézout ring.

Proof. 1) By Theorem 2.4 (1).

2) Let $(e_i)_{i=1}^{i=n}$ be a basis of the (A/M)-vector space E. Then, $0 \ltimes E$ is a finitely presented ideal of R since R is coherent by [9, Theorem 2.6. (2)]. But $0 \ltimes E$ is not principal. Deny, $0 \ltimes E = R(0,e) = 0 \ltimes Ae$ for some $(0,e) \in 0 \ltimes E$, a contradiction. □

Example 2.14. *Let $T := \mathbb{R}[X]_{(X)} = \mathbb{R} + M$ where X is an indeterminate over $\mathbb{R}$ and $M := XT$ the maximal ideal of T. Set $R := \mathbb{Q} + M$. Then:*
1) R is a 2-Bézout ring.
2) R is not a weakly Bézout ring.

Proof. 1) Let I be a finitely presented proper ideal of R. Then, it is projective since R is a weak $(2,1)$-domain by [5, Corollary 5.2 (i)] and so I is free since R local with maximal ideal M by [4, Theorem 2.1 (d)], that is $I := Ra$ for some reguler element a of R. Hence, R is a 2-Bézout domain.

2) R is not a Bézout domain since R is not a Prüfer domain by [5, Corollary 5.2 (i)] and so R is not a weakly Bézout domain by Proposition 2.2. □

Remark 2.15. *Let R be a Bézout domain. It's clear that R is a coherent domain. But, the reciprocal is not true, in general (see Example 2.17).*

The following two examples show that is the class of weakly Bézout rings and the class of coherent rings are not comparable.

Example 2.16. *Let K be a field, E be a K-vector space of infinite rank and $R := K \ltimes E$. Then :*
1) R is a weakly Bézout ring.
2) R is not a coherent ring.

Proof. 1) By Theorem 2.4 (1) R is a weakly Bézout ring.

2) R is not coherent. By [9, Theorem 2.6 (2)] since E is an K-vector space of infinite rank. □

Example 2.17. *Let K be a field and $R := K[X,Y]$ the polynomial ring, where X and Y are two indeterminate elements. Then :*
1) R is a coherent ring.
2) R is not a weakly Bézout ring.

Proof. 1) R is Noetherian, then R is a coherent ring.

2) (X,Y) is a finitely generated ideal of R which is not principal, then R is not a Bézout domain. Therefore, R is not a weakly Bézout domain, by Proposition 2.2. □

Open Problem. Is the property weakly Bézout stable by homomorphic images and localizations ?

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