On Submanifolds of Sasakian Statistical Manifolds

Mohammad Bagher Kazemi Balgeshir

ABSTRACT: In this paper, invariant and anti-invariant submanifolds of Sasakian statistical manifolds are studied. Necessary and sufficient conditions are given for vanishing the dual connection in the normal bundle. Moreover, existence of a Kaehlerian structure on invariant hypersurfaces of Sasakian statistical manifolds are proved.

Key Words: Information geometry, Sasakian statistical manifold.

Contents

1. Introduction

An important and interesting area in statistical studies is information geometry. Amari's idea for α-connections in this area developed investigating statistical manifolds. In fact a statistical manifold of probability space is a Riemannian manifold $(\bar{M}, g)$ which admits dual connections ∇^α and $\nabla^{(-\alpha)}$ with some conditions. Here, we briefly review the basic definitions in statistical manifold from information geometry point of view. For more details one can refer to [1].

Let $(\chi, \mathcal{B})$ be a measure space such that $\chi \subset \mathbb{R}$ and $\mathcal{B}$ is the σ-algebra of subsets of χ. Let $\mathcal{P}(\chi)$ be the set of probability measures which are defined on χ as following

$$\mathcal{P}(\chi) = \{p(m) : \chi \longrightarrow \mathbb{R} \mid p(m) > 0; \int_\chi p(m)dm = 1\}.$$

Suppose $x = [x^1, \ldots, x^n] \in O \subset \mathbb{R}^n$. Then $\bar{M} = \{p(m, x) \in \mathcal{P}(\chi) \mid m \in \chi, x \in O\}$ is a statistical model (manifold). Let $l(m, x) = \log p(m, x)$ and $\partial_i l = \frac{\partial l}{\partial x_i}, \forall i = 1, \ldots, n$. Define the component of an inner product g

$$g_{ij} = \int \partial_i l(m, x) \partial_j l(m, x) p(m, x) dm.$$

The matrix $g = [g_{ij}]$ is symmetric and positive semi-definite and is called Fisher information metrics of $\bar{M}$.
For $\alpha \in \mathbb{R}$ and taking

$$L_\alpha(p) = \begin{cases} \frac{2}{1-\alpha} p^{\frac{1-\alpha}{2}} & \alpha \neq 1 \\ \log p & \alpha = 1 \end{cases}$$

we put

$$\Gamma^\alpha_{ijk} = \int \partial_i \partial_j L_\alpha(p(m, x)) \partial_k L_{(-\alpha)}(p(m, x)) dm.$$

The functions Γ^α_{ijk} define affine connections ∇^α by the following equations

$$g(\nabla^\alpha_{\partial_i} \partial_j, \partial_k) = \Gamma^\alpha_{ijk}.$$

In information geometry ∇^α's are called α-connections. If g be a Riemannian metric on manifold $\bar{M}$, then ∇^0 ($\alpha = 0$) is the Levi-Civita connection on $(\bar{M}, g)$. It is well known that for any vector fields U, V, W on manifold $\bar{M}$, the connections ∇^α satisfy the following duality condition

$$Ug(V, W) = g(\nabla_U^\alpha V, W) + g(\nabla_U^{(-\alpha)} W, V).$$

In the next section, we recall the statistical manifolds from differential geometry viewpoint and review the concept of almost contact and Sasakian manifolds. In Section 3, we introduce invariant and anti-invariant submanifolds of Sasakian statistical manifolds and give an example. We prove some results about embedding curvature tensors of submanifolds with respect to the dual connections. We give necessary and sufficient conditions for the dual Riemannian curvature on the normal bundle to vanish. Moreover, we obtain a statistical Kaehlerian structure on invariant hypersurfaces of Sasakian statistical manifolds.

2. Statistical manifolds and almost contact manifolds

Let $(\bar{M}, g)$ be a smooth Riemannian manifold with Levi-Civita connection $\widehat{\nabla}$. We denote the set of all vector fields on $\bar{M}$ by $\mathfrak{T}(\bar{M})$.

Definition 2.1. *Let $(\bar{M}, g)$ be a Riemannian manifold which admits an affine connection $\bar{\nabla}$ such that for all $U, V, W \in \mathfrak{T}(\bar{M})$*

i) $\bar{\nabla}_U V - \bar{\nabla}_V U = [U, V]$;

ii) $\bar{\nabla}_V g(U, W) = \bar{\nabla}_U g(V, W)$.

Then $(\bar{M}, g, \bar{\nabla})$ is said to be a statistical manifold.

Moreover, there exists an affine connection $\bar{\nabla}^*$ on $\bar{M}$ which is called a dual connection of $\bar{\nabla}$ with respect to the g, such that

$$Ug(V, W) = g(\bar{\nabla}_U V, W) + g(V, \bar{\nabla}_U^* W). \tag{2.1}$$

It can be verified that $(\bar{\nabla}^*)^* = \bar{\nabla}$ and $\bar{\nabla}^*$ also satisfies conditions (i) and (ii) of the Definition 2.1. From compatibility of $\widehat{\nabla}$ with g and Equation (2.1), we obtain $\bar{\nabla}^* = 2\widehat{\nabla} - \bar{\nabla}$.
For vector fields U and V on $\bar{M}$, $(1, 2)$-tensor field $K_U V$ is defined as following

$$K_U V = \bar{\nabla}_U V - \widehat{\nabla}_U V. \tag{2.2}$$

From (2.2), we find K is symmetric and

$$g(K_U V, W) = g(V, K_U W).$$

Let M be an isometrically immersed m-dimensional submanifold of a statistical manifold $(\bar{M}, g, \bar{\nabla}, \bar{\nabla}^*)$. We also denote the induced metric on M by g. The extension of Gauss formula for affine connections are given [6]

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.3}$$

$$\bar{\nabla}_U^* V = \nabla_U^* V + h^*(U, V), \tag{2.4}$$

for any $U, V \in \mathfrak{T}(M)$. Here, ∇ and ∇^* are induced connections on M and dual with respect to g. Moreover, h and h^* are symmetric and bilinear which is called the embedding curvature tensors of M in $\bar{M}$ for $\bar{\nabla}$ and $\bar{\nabla}^*$, respectively.
We define the curvature $H = \frac{1}{m} trace(h)$. Then we say that M is a totally umbilical submanifold if $h(U, V) = g(U, V)H$ and totally geodesic submanifold if $h(U, V) = 0$ for all $U, V \in \mathfrak{T}(M)$.
For any vector field ζ in normal bundle $\mathfrak{T}^\perp(M)$ and $U, V \in \mathfrak{T}(M)$, Vos [6] proved the Weingarten formula for the statistical structures $(\bar{\nabla}, \bar{\nabla}^*, g)$ as following

$$\bar{\nabla}_U \zeta = -A^*_\zeta U + D_U \zeta, \tag{2.5}$$

$$\bar{\nabla}_U^* \zeta = -A_\zeta U + D^*_U \zeta. \tag{2.6}$$

$A_\zeta U$ and $A^*{}_\zeta U$ are bilinear and related to the embedding curvature tensors by the following equations

$$g(A_\zeta U, V) = g(h(U,V), \zeta), \ g(A^*{}_\zeta U, V) = g(h^*(U,V), \zeta). \tag{2.7}$$

It should be denoted that in the Equations (2.5) and (2.6), D and D^* are Riemannian dual connection on the normal bundle $\mathcal{T}^\perp(M)$.

The Riemannian curvature tensors $\bar{R}$ of $\bar{\nabla}$ and $\bar{R}^*$ of $\bar{\nabla}^*$ are defined

$$\bar{R}(U,V)W = [\bar{\nabla}_U, \bar{\nabla}_V]W - \bar{\nabla}_{[U,V]}W, \ \bar{R}^*(U,V)W = [\bar{\nabla}^*_U, \bar{\nabla}^*_V]W - \bar{\nabla}^*_{[U,V]}W.$$

In addition, they are related by

$$g(\bar{R}(U,V)W, X) = -g(\bar{R}^*(U,V)X, W). \tag{2.8}$$

The statistical manifold $(\bar{M}, \bar{\nabla}^*, g)$ is called of constant curvature c, if the following equation holds [2]

$$R^*(U,V)W = c\{g(V,W)U - g(U,W)V\}.$$

Now, we review some basic properties of Sasakian manifolds.

Definition 2.2. *[7] An odd dimensional Riemannian manifold $(\bar{M}, g)$ is said to be an almost contact metric manifold if it admits a 1-form η, a vector field ξ and $(1,1)$-tensor field ϕ which satisfy*

$$\eta(\xi) = 1, \ \phi^2 U = -U + \eta(U)\xi. \tag{2.9}$$

$$\phi\xi = 0, \ \eta(\phi U) = 0, \tag{2.10}$$

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \tag{2.11}$$

for any $U, V \in \mathcal{T}(M)$.

An almost contact metric structure (g, η, ξ, ϕ) is called a Sasakian manifold [5] if

$$(\widehat{\nabla}_U \phi)V = \eta(V)U - g(U,V)\xi. \tag{2.12}$$

Let M be a submanifold of an almost contact manifold $(\bar{M}, g, \eta, \xi, \phi)$. For all $U \in \mathcal{T}(M)$ and $\zeta \in \mathcal{T}^\perp(M)$, we put $\phi U = TU + NU$ and $\phi\zeta = t\zeta + n\zeta$, where $TU, t\zeta \in \mathcal{T}(M)$ and $NU, n\zeta \in \mathcal{T}^\perp(M)$.

3. Submanifolds of Sasakian statistical manifolds

Definition 3.1. *[4] Let $(\bar{M}, g, \bar{\nabla}, \bar{\nabla}^*)$ be a statistical manifold which admits an almost contact structure (ϕ, ξ, η). Then $(\bar{M}, g, \phi, \xi, \bar{\nabla})$ is said to be a Sasakian statistical manifold if (ϕ, ξ, g) is a Sasakian structure on $\bar{M}$ and $\forall U, V \in \mathcal{T}(\bar{M})$, the $(1,2)$-tensor field K satisfies*

$$K_U \phi V + \phi K_U V = 0. \tag{3.1}$$

Theorem 3.2. *[4] Let $(\bar{M}, g, \bar{\nabla}, \bar{\nabla}^*)$ be a statistical manifold with an almost contact structure (ϕ, ξ, η). $(g, \phi, \xi, \bar{\nabla})$ is an Sasakian statistical structure on $\bar{M}$ if and only if the following two equations hold:*

$$\bar{\nabla}_U(\phi V) - \phi\bar{\nabla}^*_U V = g(V, \xi)U - g(V, U)\xi, \tag{3.2}$$

$$\bar{\nabla}_U \xi = \phi U + g(\bar{\nabla}_U \xi, \xi)\xi. \tag{3.3}$$

Remark that since $(\bar{\nabla}^*)^* = \bar{\nabla}$, the above equations satisfy in dual case.

Definition 3.3. *Let (M, g) be a submanifold of a statistical manifold $(\bar{M}, g, \bar{\nabla}, \bar{\nabla}^*)$ and (ϕ, ξ, η) be an almost contact structure on $\bar{M}$. Then M is called an invariant submanifold if $\forall p \in M$, $\phi(T_pM) \subset T_pM$. Furthermore, if $\forall p \in M$, $\phi(T_pM) \subset T_pM^\perp$, then M is called an anti-invariant submanifold.*

Example 3.4. *Let (ϕ, ξ, η, g) be a Sasakian structure on unit hypersphere S^7. By taking $K(U, V) = \eta(U)\eta(V)\xi$ and $\bar{\nabla} = \widehat{\nabla} + K$, $(S^7, \bar{\nabla}, \phi, \xi, \eta, g)$ is a Sasakian statistical manifold ([4], Ex. 2.15). So, if for $i = 1, 2$, (x^i, y^i, t) be local coordinates of S^7, then we put $\xi = \frac{\partial}{\partial t}$, $\phi(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$, $\phi(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}$, $\phi(\frac{\partial}{\partial t}) = 0$.*

Let $M = (x^1, y^1, 0, 0, 0, 0, t)$ be a 3-dimensional submanifold of S^7. Then M is an invariant submanifold such that ξ is tangent to M.

Remark 3.5. *Let M be a submanifold of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$ and $\xi \in \mathfrak{T}^\perp(M)$. Then from (3.3), for all $U, V \in \mathfrak{T}(M)$, we get*

$$g(\phi U, V) = g(\phi U + g(\bar{\nabla}_U \xi, \xi)\xi, V) = g(\bar{\nabla}_U \xi, V). \tag{3.4}$$

Using (2.6) and (2.7) in Equation (3.4) imply

$$g(\phi U, V) = g(-A^*_\xi U + D_U \xi, V) = g(-A^*_\xi U, V) = -g(h^*(U, V), \xi), \tag{3.5}$$

and since h^* is symmetric, we obtain $g(\phi U, V) = g(\phi V, U) = -g(\phi U, V) = 0$. This means that if the structure vector field ξ is normal to M, then M is an anti-invariant submanifold.

Lemma 3.6. *Let M be an anti-invariant submanifold of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$ and ξ is tangent to M. Then $h^*(U, \xi) = \phi U$ and so $h^*(\xi, \xi) = 0$.*

Proof: From (2.4) and the dual version of (3.3) we have

$$\phi U + g(\bar{\nabla}^*_U \xi, \xi)\xi = \bar{\nabla}^*_U \xi = \nabla^*_U \xi + h^*(U, \xi). \tag{3.6}$$

By taking the tangential and normal components of the above equation we get $\nabla^*_U \xi = g(\bar{\nabla}^*_U \xi, \xi)\xi$ and $h^*(U, \xi) = \phi U$. Using (2.10) implies $h^*(\xi, \xi) = 0$.

$\square$

Remark 3.7. *By considering the dual of equations in the previous lemma we obtain $h(U, \xi) = \phi U$ and $h(\xi, \xi) = 0$.*

Let M be a totally umbilical submanifold. If $h(\xi, \xi) = 0$, we have $h(\xi, \xi) = g(\xi, \xi)H = 0$, thus $H = 0$ and M is a totally geodesic submanifold. So, by using Lemma 3.6, we can state the following corollary.

Corollary 3.8. *Any anti-invariant totally umbilical submanifold M of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$ which is tangent to ξ is a totally geodesic submanifold.*

Lemma 3.9. *Let M be an anti-invariant submanifold of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$ and ξ be normal to M. Then $A^*_\xi = 0$ and $D_U \xi = \phi U + g(\bar{\nabla}_U \xi, \xi)\xi$.*

Proof: Taking account of (2.5) and (3.3) we get

$$\phi U + g(\bar{\nabla}_U \xi, \xi)\xi = \bar{\nabla}_U \xi = D_U \xi - A^*_\xi U. \tag{3.7}$$

So the normal part is $D_U \xi = \phi U + g(\bar{\nabla}_U \xi, \xi)\xi$ and the tangential part is $A^*_\xi = 0$.

$\square$

In the previous lemma, by using $\bar{\nabla}^*$ instead of $\bar{\nabla}$, it can be proved that

$$D^*_U \xi = \phi U + g(\bar{\nabla}^*_U \xi, \xi)\xi, \ A_\xi U = 0. \tag{3.8}$$

Theorem 3.10. *Let M be an anti-invariant submanifold of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$ and ξ be normal to M. Then $R^{*\perp}(U, V)\xi = g(\bar{\nabla}_V \xi, \xi)D^*_U \xi - g(\bar{\nabla}_U \xi, \xi)D^*_V \xi$.*

Proof: Since M is anti-invariant, from (2.4), (2.5) and (3.2), for all $U, V \in \mathfrak{T}(M)$, we have

$$g(U, V)\xi = \bar{\nabla}^*_U \phi V - \phi \bar{\nabla}_U V = -A_{\phi V} U + D^*_U \phi V - \phi \nabla_U V - \phi(h(U, V)). \tag{3.9}$$

Thus

$$D^*_U \phi V = g(U, V)\xi + \phi \nabla_U V + n(h(U, V)). \tag{3.10}$$

Now, for convenience we put $\lambda = g(\bar{\nabla}^*_U \xi, \xi)$ and $\gamma = g(\bar{\nabla}^*_V \xi, \xi)$, so using Equations (3.8), (3.10) and Lemma 3.9 imply

$$D^*_U D^*_V \xi = D^*_U (\phi V + \gamma \xi) = g(U, V)\xi + \phi \nabla_U V + n(h(U, V)) + D^*_U \gamma \xi, \tag{3.11}$$

hence,

$$D^*_V D^*_U \xi = D^*_V (\phi U + \lambda \xi) = g(U, V)\xi + \phi \nabla_V U + n(h(U, V)) + D^*_V \lambda \xi. \tag{3.12}$$

Since D^* is torsion free, we obtain

$$D^*_U D^*_V \xi - D^*_V D^*_U \xi = D^*_U \gamma \xi - D^*_V \lambda \xi + \phi[U, V]. \tag{3.13}$$

So,

$$R^{*\perp}(U, V)\xi = \{U g(\bar{\nabla}_V \xi, \xi) - V g(\bar{\nabla}_U \xi, \xi) - g(\bar{\nabla}_{[U,V]} \xi, \xi))\xi\} + \gamma D^*_U \xi - \lambda D^*_V \xi, \tag{3.14}$$

by applying (2.1) in (3.14) we deduce

$$R^{*\perp}(U, V)\xi = g(\bar{R}^*(U, V)\xi, \xi)\xi + \gamma D^*_U \xi - \lambda D^*_V \xi = \gamma D^*_U \xi - \lambda D^*_V \xi. \tag{3.15}$$

$\square$

Let M be an anti-invariant submanifold normal to ξ. We say that M has minimal codimension if $T^\perp M = \mathcal{D} \oplus < \xi >$, where $\mathcal{D} = \phi(TM)$. In this case, from Lemma 3.9, we have $A^*_\xi U = 0$ and so $g(h^*(U, V), \xi) = g(A^*_\xi U, V) = 0$. Therefore

$$\phi h^*(U, V) \in \mathfrak{T}(M). \tag{3.16}$$

Theorem 3.11. *Let M be an anti-invariant submanifold of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$ and $\xi \in \mathfrak{T}^\perp(M)$. Let M has minimal codimension. Then $\phi R^\perp = 0$ on $\mathcal{D}$ if and only if R^* is of constant curvature 1.*

Proof: From (3.2), Gauss and Weingarten formula, we deduce

$$g(V, \xi)U - g(V, U)\xi = \bar{\nabla}_U \phi V - \phi \bar{\nabla}^*_U V = D_U \phi V - A_{\phi V} X - \phi \nabla^*_U V - \phi h^*(U, V). \tag{3.17}$$

Using (3.16) and taking the normal components of the previous equation we obtain

$$D_U \phi V = -g(V, U)\xi - \phi \nabla^*_U V. \tag{3.18}$$

Thus (3.8) and (3.18) imply

$$D_U D_V \phi W = -U g(V, W)\xi - g(V, W) D_U \xi - g(U, \nabla^*_V W)\xi + \phi \nabla^*_U \nabla^*_V W. \tag{3.19}$$

and

$$D_{[U,V]} \phi W = -g([U, V], W)\xi + \phi \nabla^*_{[U,V]} W = -g(\nabla_U V - \nabla_V U, W)\xi + \phi \nabla^*_{[U,V]} W. \tag{3.20}$$

By Equations (2.1), (3.8), (3.19) and (3.20) we have

$$R^\perp(U, V)\phi W = \phi R^*(U, V)W - g(V, W)(\phi U + g(\bar{\nabla}^*_U \xi, \xi)\xi) +$$
$$g(U, W)(\phi V + g(\bar{\nabla}^*_V \xi, \xi)\xi). \tag{3.21}$$

So, If $\phi R^\perp = 0$ then $R^*(U, V)W = g(V, W)U - g(U, W)V$ and the sectional curvature of R^* is equal to 1.

Conversely, if $R^*(U, V)W = g(V, W)U - g(U, W)V$ then (3.21) implies that

$$\phi R^\perp(U, V)\phi W = 0.$$

Since $\mathcal{D} = \phi(TM)$, $\phi R^\perp = 0$ on $\mathcal{D}$. $\square$

Theorem 3.12. *Let M be an anti-invariant submanifold of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$. Let ξ be in $\mathcal{T}^\perp(M)$ and normal to $\bar{\nabla}^*\xi$. If M is of minimal codimension then $R^\perp = 0$ if and only if R^* is of constant curvature 1.*

Proof: Since ξ is normal to $\bar{\nabla}^*\xi$, from Theorem 3.10, $R^{*\perp}(U,V)\xi = 0$ and therefore from (2.8), $R^\perp(U,V)\xi = 0$. On the other hand, from (3.21) we deduce

$$R^\perp(U,V)\phi W = \phi R^*(U,V)W - g(V,W)\phi U + g(U,W)\phi V \tag{3.22}$$

Thus if R^* is of constant curvature 1 then $\forall \zeta \in \mathcal{T}^\perp(M)$, $R^\perp(U,V)\zeta = 0$ and vice versa. $\square$

Remark 3.13. *An almost Hermitian manifold (M, g, J) with statistical structure (g, ∇, ∇^*) is called a statistical Hermitian manifold. If the 2-form $\omega(U,V) = g(U, JV)$ is parallel with respect to ∇, then (M, g, J) is said to be a holomorphic statistical manifold. In Proposition 2.5. of [3], it has been proved that a statistical Hermitian manifold (M, g, J, ∇) is a holomorphic statistical manifold if and only if (M, g, J) is a Kaehlerian manifold.*

Now, we introduce a Kaehlerian structure on invariant hypersurfaces of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$.

Theorem 3.14. *Let M be an invariant hypersurface of a Sasakian statistical manifold $(\bar{M}, g, \bar{\nabla}, \phi, \xi)$ and $\xi \in \mathcal{T}^\perp(M)$. Then M admits a Kaehlerian structure.*

Proof: For all $U, V \in \mathcal{T}(M)$, we have $g(U, \xi) = 0$. So (2.9) implies

$$\phi^2 U = -U, \tag{3.23}$$

and by defining the complex structure $J = \phi|_M$ on M, from (2.11) we obtain $g(JU, JV) = g(\phi U, \phi V) = g(U, V)$. It follows that (M, g, J) is an almost Hermitian manifold.

On the other hand, $\forall U, V, W \in \mathcal{T}(M)$

$$(\nabla_U \omega)(V, W) = U g(V, JW) - g(\nabla_U V, JW) - g(V, J\nabla_U W). \tag{3.24}$$

Using definition of J and Eq. (2.1) in the second and third terms of the above equation implies

$$(\nabla_U \omega)(V, W) = g(V, \nabla_U^* JW - J\nabla_U W) = g(V, \nabla_U^* \phi W - \phi \nabla_U W). \tag{3.25}$$

Since $\xi \in \mathcal{T}^\perp(M)$, in account of (3.2), we get

$$(\nabla_U \omega)(V, W) = g(V, g(W, \xi)U - g(W, U)\xi) = 0, \tag{3.26}$$

and this means that ω is a parallel 2-form, hence (M, g, J, ∇) is a holomorphic statistical manifold. Furthermore, Remark 3.13 implies (M, g, J) is a Kaehlerian manifold. $\square$

References

1. Amari, S. and Nagaoka, H., *Methods of information geometry.* Transl. Math. Monogr. 191., Amer. Math. Soc. (2000).

2. Aydin, M.E., Mihai, A. and Mihai, I., *Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature.* Bull. Math. Sci. 7, 155—166, (2017).

3. Furuhata, H., *Hypersurfaces in statistical manifolds.* Diff. Geom. Appl. 27, 420–429, (2009).

4. Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K. and Shahid, M. *Sasakian statistical manifolds.* J. Geom. Phys. 117, 179–186, (2017).

5. Majhi, P and Ghosh G., *Certain results on generalized (k,μ)-contact manifolds.* Bol. Soc. Paran. Mat. 37, 131–142, (2019).

6. Vos, P. W., *Fundamental equations for statistical submanifolds with applications to the Bartlett correction.* Ann. Inst. Statist. Math. 41, 429–450, (1989).

7. Yano, K. and Kon, M., *Structures on manifolds.* World Scientific, (1984).

Mohammad Bagher Kazemi Balgeshir,
Department of Mathematics,
University of Zanjan,
Zanjan, Iran.
E-mail address: `mbkazemi@znu.ac.ir`