



## Common Fixed Point for Multivalued $(\psi-G)$ -Contraction Mappings in Partial Metric Spaces with a Graph Structure\*

S. Benchabane, S. Djebali

ABSTRACT: In the present work, we first discuss the definition of a multivalued  $(\psi-G)$ -contraction mapping in a metric space endowed with a graph as introduced in [13] and we suggest a generalization. Then, we prove a common fixed point theorem for multivalued  $(\psi-G)$ -contraction mappings in partial metric spaces endowed with a graph. An example of application illustrates the main existence result and some known existence results are derived.

Key Words: Common fixed point, Multi-valued mappings,  $(\psi-G)$ -contraction, Partial metric.

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### 1. Introduction and preliminaries

The notion of partial metric is an important generalization of the classical concept of metric. It was introduced in 1994 by Matthews [20] who extended the Banach contraction principle in the setting of partial metric spaces. His result has been generalized in several directions by many authors (we refer the reader to [4], [8], [12], [18], [19], [22] and references therein).

In 2008, Jachymski [17] introduced the concept of  $G$ -contraction, that is a single-valued contraction mapping in a metric space with a graph structure (it preserves the edges and decreases weights of edges of the graph). Then Banach's contraction principle in ordered metric spaces was generalized.

Following J.R. Nadler [21], some authors have considered in a natural way the fixed point theory for multi-valued mappings in partial metric spaces endowed with a graph (see, e.g., [1], [3], [7])

More recently, Dehkordi and Ghods [13] defined a multi-valued  $(\psi, G)$ -contraction (Definition 1.9) in some metric spaces and obtained a fixed point result.

Our contribution, in this paper, is part of these extensions. More precisely, we have introduced a refined version of a  $(\psi, G)$ -contraction (Definition 1.11) and have obtained a new common fixed point theorem for a pair of multivalued mappings in a partial metric space endowed with a graph (Theorem 2.1). Then some consequences have been derived.

It is well known now that some fixed point results on partial metric spaces are not real generalizations (see, e.g., [11,14,15,24]). In this work this new version of  $(\psi, G)$ -contraction is still interesting in the setting of metric spaces.

Before coming to the main result of this paper, we first recall some basic definitions and auxiliary results on partial metric spaces that are well developed in the recent literature (see, e.g., [9], [16], [22], [23]).

**Definition 1.1.** [20] *A partial metric on a nonempty set  $X$  is a function  $p : X \times X \rightarrow \mathbb{R}^+$  such that for all  $x, y, z \in X$ :*

$$(P_1) \quad x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y),$$

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- (P<sub>2</sub>)  $p(x, x) \leq p(x, y)$ ,  
(P<sub>3</sub>)  $p(x, y) = p(y, x)$ ,  
(P<sub>4</sub>)  $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ .

A partial metric space is a pair  $(X, p)$  such that  $X$  is a nonempty set and  $p$  is a partial metric on  $X$ . Clearly a metric space is a partial metric space.

If  $p(x, y) = 0$ , then (P<sub>1</sub>) and (P<sub>2</sub>) imply that  $x = y$ . But the converse does not in general hold true. A trivial example of a partial metric space is the pair  $(\mathbb{R}^+, p)$ , where  $p : X \times X \rightarrow \mathbb{R}^+$  is defined as  $p(x, y) = \max\{x, y\}$ .

Each partial metric  $p$  on  $X$  generates a  $T_0$  topology  $\tau_p$  on  $X$  which has as a base the family of open  $p$ -balls  $\{B_p(x, \varepsilon), x \in X, \varepsilon > 0\}$ , where  $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$  for all  $x \in X$  and  $\varepsilon > 0$ .

If  $p$  is a partial metric on  $X$ , then the functions  $p^s, p^w : X \times X \rightarrow \mathbb{R}$  given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad \text{and} \quad p^w(x, y) = p(x, y) - \min\{p(x, x), p(y, y)\}$$

are equivalent metrics on  $X$ .

In the topology  $\tau_p$ , the classical notions of convergence are recovered.

**Definition 1.2.** Let  $(X, p)$  be a partial metric space. Then

- (a) A sequence  $(x_n)_n$  in  $(X, p)$  is said to be convergent to a point  $x \in X$  with respect to  $\tau_p$  if  $p(x, x) = \lim_{n \rightarrow +\infty} p(x, x_n)$ .  
(b) A sequence  $(x_n)_n$  in  $X$  will be a Cauchy sequence if  $\lim_{m, n \rightarrow +\infty} p(x_n, x_m)$  exists and is finite.  
(c) A partial metric space  $(X, p)$  is called a complete partial metric space if every Cauchy sequence  $(x_n)_n$  in  $X$  converges with respect to  $\tau_p$  to a point  $x \in X$ .

**Lemma 1.3.** [20] Let  $(X, p)$  be a partial metric space.

- (a) A sequence  $(x_n)_n$  in  $(X, p^s)$  is said to be convergent to a point  $x \in X$  if and only if

$$p(x, x) = \lim_{n \rightarrow +\infty} p(x, x_n) = \lim_{m, n \rightarrow +\infty} p(x_n, x_m).$$

- (b) A sequence  $(x_n)_n$  in  $X$  is Cauchy with respect to  $p$  if and only if it is Cauchy with respect to  $p^s$ .  
(c) A partial metric space  $(X, p)$  is complete if and only if the metric space  $(X, p^s)$  is complete.

A subset  $A$  of  $X$  is called closed in  $(X, p)$  if it is closed with respect to  $\tau_p$ . Let  $C_p(X)$  be the collection of all nonempty and closed subsets of  $X$  with respect to the partial metric  $p$ . For  $A \in C_p(X)$ , define  $p(x, A) = \inf_{y \in A} p(x, y)$ . Note that  $p(x, A) = 0 \Rightarrow p^s(x, A) = 0$ , where  $p^s(x, A) = \inf_{y \in A} p^s(x, y)$ .

**Lemma 1.4.** [9] Let  $(X, p)$  be a partial metric space and  $A$  any nonempty set in  $X$ . Then

$$(a) \quad a \in \overline{A} \text{ if and only if } p(a, A) = p(a, a),$$

where  $\overline{A}$  denotes the closure of  $A$  with respect to the partial metric  $p$ .

- (b) If  $A$  is closed in  $(X, p)$ , then  $A$  is closed in  $(X, p^s)$ . Notice that as in metric spaces,  $A$  is closed in  $(X, p)$  if and only if  $\overline{A} = A$ .

Some fixed point theorems in partial metric spaces can be found, e.g., in [22]. Next, we present some basic definitions from graph theory needed in the sequel. A graph  $G$  is an ordered pair  $(V, E)$ , where  $V$  is a set and  $E \subset V \times V$  is a binary relation on  $V$ . Elements of  $E$  are called edges and are denoted by  $E(G)$  while elements of  $V$ , denoted  $V(G)$ , are called vertices. If the direction is imposed in  $E$ , that is the edges are directed, then we obtain a digraph (directed graph). Hereafter, we assume that  $G$  has no parallel edges, i.e. two vertices cannot be connected by more than one edge. Doing this,  $G$  can be identified with the pair  $(V(G), E(G))$ . If  $x$  and  $y$  are vertices of  $G$ , then a path in  $G$  from  $x$  to  $y$  of length  $k \in \mathbb{N}$  is a finite sequence  $(x_n)_n, n \in \{0, 1, 2, \dots, k\}$  of vertices such that  $x = x_0, \dots, x_k = y$  and  $(x_{n-1}, x_n) \in E(G)$  for  $n \in \{1, 2, \dots, k\}$ . A graph  $G$  is connected if there is a path between any two

vertices and it is weakly connected if  $\tilde{G}$  is connected, where  $\tilde{G}$  denotes the undirected graph obtained from  $G$  by ignoring the direction of edges. Let  $G^{-1}$  be the graph obtained from  $G$  by reversing the direction of edges (the conversion of the graph  $G$ ). We have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat  $\tilde{G}$  as a directed graph for which the set of edges is symmetric; then we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Let  $G_x$  be the component of  $G$  consisting of all the edges and vertices which are contained in some path in  $G$  beginning at  $x$ . If  $G$  is such that  $E(G)$  is symmetric, then for  $x \in V(G)$ , we may define the equivalence class  $[x]_G$  on  $V(G)$  by  $R: xRy$  if there is a path in  $G$  from  $x$  to  $y$ . Then  $V(G_x) = [x]_G$ .

Throughout this paper,  $(X, p)$  denotes a partial metric space,  $G = (V(G), E(G))$  is a directed graph without parallel edges such that  $V(G) = X$ , and  $(x, x) \notin E(G)$  (the graph does not contain loops).

**Remark 1.5.** *By contrast with most of recent papers, you need not assume here  $E(G)$  to be symmetric.*

Consider the following class of functions:

**Definition 1.6.**  $\Psi = \{\psi : [0, +\infty) \rightarrow [0, +\infty) \text{ is nondecreasing}\}$  which satisfies the following conditions:

- (i) for every  $(t_n) \subset \mathbb{R}^+$ ,  $\psi(t_n) \rightarrow 0$  if and only if  $t_n \rightarrow 0$ ;
- (ii) for every  $t_1, t_2 \in \mathbb{R}^+$ ,  $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$ ;
- (iii) for any  $t > 0$  we have  $\psi(t) \leq t$ .

The following Lemma will be useful in explaining Definition 1.11. It is the analogous of the one proved by Nadler [Remark, p. 480] [21] when dealing with multi-valued contraction mappings.

**Lemma 1.7.** [10] *Let  $A, B \in CB^p(X)$ ,  $a \in A$ . Then for each  $\varepsilon > 0$ , there exists  $b \in B$  such that*

$$p(a, b) \leq H_p(A, B) + \varepsilon.$$

Then we can deduce

**Lemma 1.8.** *Let  $A, B \in CB^p(X)$ ,  $a \in A$  and  $\psi \in \Psi$ . Then for each  $\varepsilon > 0$ , there exists  $b \in B$  such that*

$$\psi(p(a, b)) \leq \psi(H_p(A, B)) + \varepsilon.$$

In [Definition 2.1] [13], the authors introduced the following:

**Definition 1.9.** *Let  $(X, d)$  be a complete metric space and  $G$  be a directed graph with no-parallel edges,  $E(G)$  is symmetric and  $(x, x) \notin E(G)$ . Two mappings  $F, T : X \rightarrow C(X)$  are said to be a common  $(G-\psi)$  contraction if there exists  $k \in (0, 1)$  such that*

- (i)  $\psi(H(F(x), T(y))) \leq k\psi((d(x, y)))$ , for all  $(x, y) \in E(G)$ ,  $(x \neq y)$
- (ii) for all  $(x, y) \in E(G)$  if  $u \in F(x)$  and  $v \in T(y)$  are such that

$$\psi(d(u, v)) \leq k\psi((d(x, y))) + \varepsilon, \text{ for each } \varepsilon > 0 \text{ then } (u, v) \in E(G)$$

Arguing as [5,6], we observe that Definition 1.9 is not appropriate because of condition (ii). Here is a counter-example.

**Example 1.10.** *Consider the space  $\mathbb{R}^2$  endowed with the Euclidean distance  $d$  and let the graph  $G$  be defined by*

$$((x_1, x_2), (y_1, y_2)) \in E(G) \Leftrightarrow x_1 + y_1 < x_2 + y_2 \text{ with } x_1 \neq y_1 \text{ and } x_2 \neq y_2.$$

Let  $A$  be the unit ball of  $\mathbb{R}^2$ , that is

$$A = \{x = (x_1, x_2) \in \mathbb{R}^2 : d_2(x, 0) = x_1^2 + x_2^2 \leq 1 \text{ and } x_1 \leq x_2\}$$

Define the multivalued maps  $F, T : \mathbb{R}^2 \rightarrow CB(\mathbb{R}^2)$  by  $F(x) = T(x) = A$ . Then  $H(F(x), T(y)) = 0$ , for all  $x, y \in \mathbb{R}^2$  and  $\psi(t) = t$ . Moreover there exists  $k \in (0, 1)$  such that

$$H(F(x), T(y)) \leq kd(x, y), \text{ for all } (x, y) \in E(G), (x \neq y),$$

However, if  $(x, y) = ((0, 3), (3, 3))$ , then  $(x, y) \in E(G)$  and  $d_2(x, y) = 3$ . Since  $d_2(x, y) = 3$ , then condition (ii) holds if and only if for any  $u, v \in A$  with  $d(u, v) \leq 3k + \varepsilon$  for all  $\varepsilon > 0$ , we must have  $(u, v) \in E(G)$ . But this is not the case, if for instance we take  $(u, v) = ((k, k), (\frac{k}{2}, \frac{k}{2}))$ , for  $d(u, v) = \frac{k}{\sqrt{2}} \leq 3k + \varepsilon$ , for all  $\varepsilon > 0$  while  $(u, v) \notin E(G)$ .

The following definition is suggested as alternative for Definition 1.9. The introduction of function  $M_p$  inspires the concept of graphic  $F$ -contraction used in [Definition 1.2] [2].

**Definition 1.11.** Let  $(X, p)$  be a partial metric space endowed with a graph  $G$ . Two mappings  $F, T : X \rightarrow C_p(X)$  are said to be a common  $(\psi$ - $G$ ) contraction if for all  $x, y \in X$  such that  $(x, y) \in E(G)$ , we have

(1)  $a \in F(x)$  implies that there exists  $b \in T(y)$  such that  $(a, b) \in E(G)$  and

$$\psi(p(a, b)) \leq k(p(x, y))\psi((M_p(Fx, Ty)) + L\varphi(N_p(Fx, Ty))),$$

(2)  $a \in T(x)$  implies that there exists  $b \in F(y)$  such that  $(a, b) \in E(G)$  and

$$\psi(p(a, b)) \leq k(p(x, y))\psi((M_p(Tx, Fy)) + L\varphi(N_p(Tx, Fy))),$$

where

$$N_p(Fx, Ty) = \min\{P^w(x, F(x)), P^w(y, T(y)), P^w(y, F(x)), P^w(x, T(y))\},$$

$$P^w(x, F(x)) = \inf\{p^w(x, y) : y \in F(x)\},$$

$$M_p(Fx, Ty) = \max\left\{p(x, y), p(x, F(x)), p(y, T(y)), \frac{p(y, F(x)) + p(x, T(y))}{2}\right\},$$

$k : (0, +\infty) \rightarrow [0, 1)$  satisfies  $\limsup_{s \rightarrow t^+} k(s) < 1$ , for all  $t \in [0, +\infty)$ ,  $L \geq 0$ ,  $\psi \in \Psi$ , and  $\varphi$  is nondecreasing and satisfies conditions (i) in Definition 1.6.

**Remark 1.12.** In Example 1.10 and according to Definition 1.11,  $F$  and  $T$  have a common  $(\psi$ - $G$ ) contraction. Indeed, pick  $p = d$  and  $L = 0$ . For  $(x, y) \in E(G)$ , put  $d_1 = kM(x, y)$  and let  $a \in A$ . Then it is sufficient to take  $b \in B(a, \frac{d_1}{2})$ .

The following condition first appeared in [17].

**Property (A):** for any sequence  $(x_n)_n$  in  $X$ , if  $x_n \rightarrow x$  and  $(x_n, x_{n+1}) \in E(G)$ , for all  $n \in \mathbb{N}$ , then  $(x_n, x) \in E(G)$  for  $n \in \mathbb{N}$ .

With this condition, Jachymski showed that in a complete metric space, a  $G$ -contraction has a fixed if and only if  $X_f \neq \emptyset$ , where

$$X_f = \{x \in X : (x, y) \in E(G) \text{ for some } y \in F(x)\}. \quad (1.1)$$

$\mathbb{N} = \{1, 2, \dots\}$  refers to the set of natural numbers.

## 2. Main result

Our existence results for common fixed points are collected in the following:

**Theorem 2.1.** Let  $(X, p)$  be a complete partial metric space endowed with a directed graph  $G$  and suppose that the triple  $(X, d, G)$  has Property (A). Let  $F, T : X \rightarrow C_p(X)$  be a common  $(G$ - $\psi)$  contraction. Then the following statements hold.

- (1) For every  $x \in X_F$ , mappings  $F$  and  $T|_{[x]_{\tilde{G}}}$  have a common fixed point.
- (2) If  $X_F \neq \emptyset$  and  $G$  is weakly connected, then  $F$  and  $T$  have a common fixed point in  $X$ .
- (3) If  $X' = \cup\{[x]_{\tilde{G}} : x \in X_F\}$ , then  $F$  and  $T|_{X'}$  have a common fixed point.
- (4) If  $F \subseteq E(G)$ , then  $F$  and  $T$  have a common fixed point.

*Proof. Claim 1. (a) We first construct a Cauchy sequence.* Let  $x_0 \in X_F$ , then there is an  $x_1 \in F(x_0)$  such that  $(x_0, x_1) \in E(G)$ . Since  $F$  and  $T$  are a common  $(\psi-G)$  contraction, then there exists  $x_2 \in T(x_1)$  such that  $(x_1, x_2) \in E(G)$  and

$$\begin{aligned} \psi(p(x_1, x_2)) &\leq k(p(x_0, x_1))\psi(M_p(Fx_0, Tx_1)) + L\varphi(N_p(Fx_0, Tx_1)) \\ &\leq k(p(x_0, x_1))\psi(M_p(Fx_0, Tx_1)) + L\varphi(P^w(x_1, F(x_0))) \\ &\leq k(p(x_0, x_1))\psi(M_p(Fx_0, Tx_1)) + L\varphi(p^w(x_1, x_1)) \\ &= k(p(x_0, x_1))\psi(M_p(Fx_0, Tx_1)). \end{aligned}$$

Since  $(x_1, x_2) \in E(G)$  and  $F, T$  are common  $(G-\psi)$  contraction, then there exists  $x_3 \in F(x_2)$  such that  $(x_2, x_3) \in E(G)$  and

$$\begin{aligned} \psi(p(x_2, x_3)) &\leq k(p(x_1, x_2))\psi(M_p(Tx_1, Fx_2)) + L\varphi(N_p(Tx_1, Fx_2)) \\ &\leq k(p(x_1, x_2))\psi(M_p(Tx_1, Fx_2)) + L\varphi(P^w(x_2, T(x_1))) \\ &\leq k(p(x_1, x_2))\psi(M_p(Tx_1, Fx_2)) + L\varphi(p^w(x_2, x_2)) \\ &= k(p(x_1, x_2))\psi(M_p(Tx_1, Fx_2)). \end{aligned}$$

By induction, we construct a sequence  $(x_n)_n$  such that  $x_{2n+1} \in F(x_{2n})$ ,  $x_{2n+2} \in T(x_{2n+1})$ ,  $(x_n, x_{n+1}) \in E(G)$ , and

$$\psi(p(x_n, x_{n+1})) \leq \begin{cases} k(p(x_{n-1}, x_n))\psi(M_p(Fx_{n-1}, Tx_n)), & \text{for odd } n \\ k(p(x_{n-1}, x_n))\psi(M_p(Tx_{n-1}, Fx_n)), & \text{for even } n. \end{cases}$$

Let us distinguish between two cases:

*Case 1:  $n$  is odd.* We have

$$\begin{aligned} M_p(Fx_{2k}, Tx_{2k+1}) &= \max \left\{ p(x_{2k}, x_{2k+1}), p(x_{2k}, F(x_{2k})), p(x_{2k+1}, T(x_{2k+1})), \right. \\ &\quad \left. \frac{p(x_{2k+1}, F(x_{2k})) + p(x_{2k}, T(x_{2k+1}))}{2} \right\} \\ &\leq \max \left\{ p(x_{2k}, x_{2k+1}), p(x_{2k}, x_{2k+1}), p(x_{2k+1}, x_{2k+2}), \right. \\ &\quad \left. \frac{p(x_{2k+1}, x_{2k+1}) + p(x_{2k}, x_{2k+2})}{2} \right\} \\ &\leq \max \left\{ p(x_{2k}, x_{2k+1}), p(x_{2k+1}, x_{2k+2}), \right. \\ &\quad \left. \frac{p(x_{2k}, x_{2k+1}) + p(x_{2k+1}, x_{2k+2})}{2} \right\} \\ &= \max \left\{ p(x_{2k}, x_{2k+1}), p(x_{2k+1}, x_{2k+2}) \right\}. \end{aligned}$$

If  $M_p(Fx_{2k}, Tx_{2k+1}) = p(x_{2k+1}, x_{2k+2})$ , then

$$\psi(p(x_{2k+1}, x_{2k+2})) \leq k(p(x_{2k}, x_{2k+1}))\psi(p(x_{2k+1}, x_{2k+2})) < \psi(p(x_{2k+1}, x_{2k+2})),$$

which is a contradiction. Therefore  $M_p(Fx_{2k}, Tx_{2k+1}) = p(x_{2k}, x_{2k+1})$ .

*Case 2:  $n$  is even.* In an analogous manner, we can show that

$$M_p(Tx_{2k+2}, Fx_{2k+1}) = p(x_{2k+1}, x_{2k+2}).$$

Hence for all  $n \in \mathbb{N}$ , we have

$$\psi(p(x_n, x_{n+1})) \leq k(p(x_{n-1}, x_n))\psi(p(x_{n-1}, x_n)). \quad (2.1)$$

Since  $0 < k(p(x_{n-1}, x_n)) < 1$  for all  $n \in \mathbb{N}$ , then

$$\psi(p(x_n, x_{n+1})) < \psi(p(x_{n-1}, x_n)),$$

that is  $(\psi(p(x_n, x_{n+1})))_n$  is a decreasing sequence of positive numbers. Let

$$l = \lim_{n \rightarrow \infty} \psi(p(x_n, x_{n+1})) \geq 0.$$

Taking the limit in (2.1) yields

$$l \leq l \limsup_{n \rightarrow \infty} k(p(x_{n-1}, x_n)).$$

Since  $\limsup_{n \rightarrow \infty} k(p(x_{n-1}, x_n)) < 1$ , then  $l = 0$ . Hence  $\lim_{n \rightarrow \infty} \psi(p(x_n, x_{n+1})) = 0$ . By definition of  $\psi$ ,  $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0$ .

(b)  $(x_n)_n$  is a Cauchy sequence in  $(X, p)$ . Since  $\limsup_{s \rightarrow t^+} k(s) < 1$ , for every nonnegative  $t$ , then there exist  $\delta > 0$  and  $a \in (0, 1)$  such that

$$k(t) < a, \forall t \in (0, \delta).$$

Since  $\lim_{n \rightarrow \infty} p(x_{n-1}, x_n) = 0$ , then there exists  $n_0 \in \mathbb{N}$  such that  $p(x_{n-1}, x_n) < \delta$  for all  $n \geq n_0$ . From the inequality in (2.1) we obtain that for  $n \geq n_0$

$$\psi(p(x_n, x_{n+1})) \leq a\psi(p(x_{n-1}, x_n)) \leq \dots \leq a^{n-N+1}\psi(p(x_N, x_{N+1})).$$

Hence for all  $m, n \in \mathbb{N}$  with  $m > n \geq n_0$ , we have

$$\begin{aligned} \psi(p(x_n, x_m)) &\leq \psi(\sum_{i=n}^{m-1} p(x_i, x_{i+1}) - \sum_{i=n+1}^{m-1} p(x_i, x_i)) \\ &\leq \sum_{i=n}^{m-1} \psi(p(x_i, x_{i+1})) \\ &\leq \sum_{i=n}^{m-1} a^{i-N+1} \psi(p(x_N, x_{N+1})). \end{aligned}$$

Consequently

$$\begin{aligned} \psi(p^s(x_n, x_m)) &\leq \psi(2p(x_n, x_m)) \\ &\leq 2\psi(p(x_n, x_m)) \\ &\leq 2\psi(p(x_N, x_{N+1})) \sum_{i=n}^{m-1} a^{i-N+1}. \end{aligned}$$

Taking the limit as  $n, m \rightarrow \infty$ , we get  $\psi(p^s(x_n, x_m)) \rightarrow 0$ . By definition of  $\psi$ ,  $p^s(x_n, x_m) \rightarrow 0$  as  $n, m \rightarrow \infty$ . This shows that  $(x_n)_n$  is a Cauchy sequence in  $(X, p^s)$ . Since  $(X, p)$  is complete,  $(X, p^s)$  is also complete by Lemma 1.3. Then there exists  $x \in X$  such that  $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$ . Appealing again to Lemma 1.3, we get

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_n, x_{n+m}) = 0.$$

By Property (A),  $(x_n, x) \in E(G)$ , for all  $n \in \mathbb{N}$ . Again two cases are discussed separately.

*Case 1:  $n = 2k$  is even.* Since  $F$  and  $T$  are common  $(\psi-G)$  contraction, then there exists  $y_k \in T(x)$  such that for all  $k \in \mathbb{N}$

$$\begin{aligned} \psi(p(x_{2k+1}, y_k)) &\leq k(p(x_{2k}, x))\psi(M_p(Fx_{2k}, Tx)) + L\varphi(N_p(Fx_{2k}, Tx)) \\ &\leq k(p(x_{2k}, x))\psi(M_p(Fx_{2k}, Tx)) + L\varphi(P^w(x, F(x_{2k}))) \\ &\leq k(p(x_{2k}, x))\psi(M_p(Fx_{2k}, Tx)) + L\varphi(p(x, x_{2k+1})), \end{aligned}$$

where

$$\begin{aligned} M_p(Fx_{2k}, Tx) &= \max \left\{ p(x_{2k}, x), p(x_{2k}, F(x_{2k})), p(x, T(x)), \frac{p(x, F(x_{2k})) + p(x_{2k}, T(x))}{2} \right\} \\ &\leq \max \left\{ p(x_{2k}, x), p(x_{2k}, x_{2k+1}), p(x, T(x)), \frac{p(x, F(x_{2k})) + p(x_{2k}, T(x))}{2} \right\}. \end{aligned}$$

Then we can choose  $n_1 \in \mathbb{N}$  such that  $M_p(Fx_{2k}, Tx) = p(x, T(x))$  for all  $k \geq n_1$ . Since  $y_k \in T(x)$ , we have for all  $k \geq n_1$

$$\begin{aligned} \psi(p(x_{2k+1}, y_k)) &\leq k(p(x_{2k}, x))\psi(p(x, T(x))) + L\varphi(p(x, x_{2k+1})) \\ &\leq k(p(x_{2k}, x))\psi(p(x, y_k)) + L\varphi(p(x, x_{2k+1})) \\ &\leq \psi(p(x, x_{2k+1})) + k(p(x_{2k}, x))\psi(p(x_{2k+1}, y_k)) \\ &\quad + L\varphi(p(x, x_{2k+1})). \end{aligned}$$

Hence

$$(1 - k(p(x_{2k}, x)))\psi(p(x_{2k+1}, y_k)) \leq \psi(p(x, x_{2k+1})) + L\varphi(p(x, x_{2k+1})).$$

Taking the limit as  $k \rightarrow \infty$  and using properties of functions  $\psi$ ,  $\varphi$ , and  $k$ , we obtain that

$$\lim_{k \rightarrow \infty} \psi(p(x_{2k+1}, y_k)) = 0.$$

For all  $k \geq n_1$ , we have

$$\begin{aligned} \psi(p(x, T(x))) &\leq \psi(p(x, x_{2k+1}) + p(x_{2k+1}, T(x)) - p(x_{2k+1}, x_{2k+1})) \\ &\leq \psi(p(x, x_{2k+1})) + \psi(p(x_{2k+1}, y_k)). \end{aligned}$$

Passing to the limit as  $k \rightarrow \infty$ , we obtain  $\psi(p(x, T(x))) = 0$  which implies that  $p(x, T(x)) = 0$ . Hence  $p(x, x) = p(x, T(x)) = 0$ . By Lemma 1.4, we conclude that  $x \in T(x)$ .

*Case 2:  $n = 2k + 1$  is odd.* Arguing as in Case 1, we find that  $x \in F(x)$ .

Since  $(x_n, x_{n+1}) \in E(G)$  and  $(x_n, x) \in E(G)$ , for  $n \in \mathbb{N}$ , we conclude that  $(x_0, x_1, x_2, \dots, x_n, x)$  is a path in  $G$  and so  $x \in [x_0]_G$ .

**Claim 2.** Since  $X_F \neq \emptyset$ , then there exists an  $x_0 \in X_F$ . In addition  $G$  is weakly connected, then  $[x_0]_G = X$  and by Claim 1,  $F$  and  $T$  have a common fixed point.

**Claim 3.** The result follows from Claim 1 and Claim 2.

**Claim 4.**  $F \subseteq E(G)$  implies that all  $x \in X$  are such that there exists some  $u \in F(x)$  with  $(x, u) \in E(G)$ , so  $X_F = X$  which implies that  $F$  and  $T$  have a common fixed point, which completes the proof of Theorem 2.1.  $\square$

### 3. Example of application

To support our result, an example of application is developed. We have modified [Example 2.3] [13] so that  $E(G)$  is no longer symmetric. We have also introduced the function  $\psi$  both with a partial metric  $p$ .

Let  $X = \{\frac{1}{2^n}, n \in \mathbb{N}\} \cup \{0, 1\}$  and  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Let

$$E(G) = \{(\frac{1}{2^n}, 0), (\frac{1}{2^n}, \frac{1}{2^{n+1}}), n \in \mathbb{N}\} \cup \{(1, 0)\},$$

$\psi(t) = \frac{t}{t+1}$ ,  $L > 0$ , and  $k \in [\frac{2}{3}, 1)$ . Notice that  $E(G)$  is not symmetric. Let  $F$  and  $T : X \rightarrow C_p(X)$  be defined by

$$F(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{\frac{1}{2^2}\}, & \text{if } x = 1, \\ \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \dots\}, & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}. \end{cases}$$

$$T(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{\frac{1}{2^4}\}, & \text{if } x = 1, \\ \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \dots\}, & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}. \end{cases}$$

Then  $F$  and  $T$  are a common  $(\psi-G)$  contraction and  $0 \in F(0) \cap T(0)$ . To check this, let  $x, y \in X$  be such that  $(x, y) \in E(G)$  and consider three cases:

*Case 1.* If  $(x, y) = (\frac{1}{2^n}, 0)$ , then

(i)  $F(\frac{1}{2^n}) = \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \dots\}$  and  $T(0) = \{0\}$ . For  $a = \frac{1}{2^{n+s}}$  where  $s \in \{2, 3, \dots\}$ , let  $b = 0$  and

$$\begin{aligned} k\psi(M_p(Fx, Ty)) + L\varphi(N_p(Fx, Ty)) &= k\frac{1}{1+2^n} + L\varphi(N_p(Fx, Ty)) \\ &\geq \frac{1}{1+2^{n+s}} = \psi(p(a, b)). \end{aligned}$$

(ii)  $T\left(\frac{1}{2^n}\right) = \left\{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \dots\right\}$  and  $F(0) = \{0\}$ . For  $a = 0$ , let  $b = \frac{1}{2^{n+2}}$  and

$$\begin{aligned} k\psi(M_p(Tx, Fy)) + L\varphi(N_p(Tx, Fy)) &= k\frac{1}{1+2^n} + L\varphi(N_p(Tx, Fy)) \\ &\geq \frac{1}{1+2^{n+2}} = \psi(p(a, b)). \end{aligned}$$

*Case 2.* If  $(x, y) = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right)$ , then

(i)  $F\left(\frac{1}{2^n}\right) = \left\{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \dots\right\}$  and  $T\left(\frac{1}{2^{n+1}}\right) = \left\{\frac{1}{2^{n+3}}, \frac{1}{2^{n+4}}, \dots\right\}$ . For  $a = \frac{1}{2^{n+s}}$  where  $s \in \{2, 3, \dots\}$ , let  $b = \frac{1}{2^{n+s+1}}$  and

$$\begin{aligned} k\psi(M_p(Fx, Ty)) + L\varphi(N_p(Fx, Ty)) &= k\frac{1}{1+2^n} + L\varphi(N_p(Fx, Ty)) \\ &\geq \frac{1}{1+2^{n+s}} = \psi(p(a, b)). \end{aligned}$$

(ii)  $T\left(\frac{1}{2^n}\right) = \left\{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \dots\right\}$  and  $F\left(\frac{1}{2^{n+1}}\right) = \left\{\frac{1}{2^{n+3}}, \frac{1}{2^{n+4}}, \dots\right\}$ . For  $a = \frac{1}{2^{n+s+1}}$  where  $s \in \{2, 3, \dots\}$ , let  $b = \frac{1}{2^{n+s}}$  and

$$\begin{aligned} k\psi(M_p(Tx, Fy)) + L\varphi(N_p(Tx, Fy)) &= k\frac{1}{1+2^n} + L\varphi(N_p(Tx, Fy)) \\ &\geq \frac{1}{1+2^{n+s}} = \psi(p(a, b)). \end{aligned}$$

*Case 3.* If  $(x, y) = (1, 0)$ , then

(i)  $F(1) = \left\{\frac{1}{2^2}\right\}$  and  $T(0) = \{0\}$ . Hence  $\left(\frac{1}{2^2}, 0\right) \in E(G)$  and

$$\begin{aligned} k\psi(M_p(Fx, Ty)) + L\varphi(N_p(Fx, Ty)) &= k\frac{1}{2} + L\varphi(N_p(Fx, Ty)) \\ &\geq \frac{1}{1+2^2} = \psi(p(a, b)). \end{aligned}$$

(ii)  $T(1) = \left\{\frac{1}{2^4}\right\}$  and  $F(0) = \{0\}$ . Hence  $\left(\frac{1}{2^4}, 0\right) \in E(G)$  and

$$\begin{aligned} k\psi(M_p(Tx, Fy)) + L\varphi(N_p(Tx, Fy)) &= k\frac{1}{2} + L\varphi(N_p(Tx, Fy)) \\ &\geq \frac{1}{1+2^4} = \psi(p(a, b)). \end{aligned}$$

**Remark 3.1.** In the previous example, if we take  $p(x, y) = d(x, y) = |x - y|$  and

$$E(G) = \left\{\left(\frac{1}{2^n}, 0\right), \left(0, \frac{1}{2^n}\right), \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right), \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right), n \in \mathbb{N}\right\} \cup \{(1, 0), (0, 1)\},$$

i.e.,  $E(G)$  is symmetric, then

(1)  $F$  and  $T$  are a common  $(\psi-G)$  contraction.

(2)  $F$  and  $T$  are not a common  $(\psi-G)$  contraction (with respect to Definition 1.9). Indeed if  $(x, y) = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right)$ , then  $F\left(\frac{1}{2^n}\right) = \left\{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \dots\right\}$  and  $T\left(\frac{1}{2^{n+1}}\right) = \left\{\frac{1}{2^{n+3}}, \frac{1}{2^{n+4}}, \dots\right\}$ . Let  $a = \frac{1}{2^{n+2}}$  and  $b = \frac{1}{2^{n+4}}$ . Then there exists  $k \in (0, 1)$  such that

$$\begin{aligned} k\psi(d(x, y)) &= k\frac{1}{1+2^{n+1}} + \varepsilon \\ &\geq \frac{1}{3+2^{n+4}} = \psi(d(a, b)), \end{aligned}$$

for all  $\varepsilon > 0$ , but  $\left(\frac{1}{2^{n+2}}, \frac{1}{2^{n+4}}\right) \notin E(G)$ .

#### 4. Common fixed point theorems

We end this paper with some consequences.

**Corollary 4.1.** Let  $(X, p)$  be a complete partial metric space endowed with a directed graph  $G$  and suppose that the triplet  $(X, p, G)$  has Property (A). Let  $F, T : X \rightarrow C_p(X)$  be mappings with the property that for any  $x, y \in X$  such that  $(x, y) \in E(G)$ , we have the conditions:

(1)  $a \in F(x)$  implies that there exists  $b \in T(y)$  with  $(a, b) \in E(G)$  and

$$\begin{aligned} \psi(p(a, b)) &\leq a\psi(p(x, y)) + b\psi(p(x, F(x))) + c\psi(p(y, T(y))) \\ &\quad + d\psi\left(\frac{p(y, F(x)) + p(x, T(y))}{2}\right) + L\varphi(N_p(Fx, Ty)), \end{aligned}$$



(2)  $a \in T(x)$  implies that there exists  $b \in F(y)$  with  $(a, b) \in E(G)$  and

$$\begin{aligned} \psi(p(a, b)) &\leq a\psi(p(x, y)) + b\psi(p(x, T(x))) + c\psi(p(y, F(y))) \\ &\quad + d\psi\left(\frac{p(y, T(x)) + p(x, F(y))}{2}\right) + L\varphi(N_p(Tx, Fy)), \end{aligned}$$

where

$$N_p(Fx, Ty) = \min\{P^w(x, F(x)), P^w(y, T(y)), P^w(y, F(x)), P^w(x, T(y))\},$$

$L \geq 0$ ,  $\psi \in \Psi$  and  $\varphi$  is nondecreasing and satisfies conditions (i) of Definition 1.6 and  $a + b + c + d < 1$ . If further there exist  $x_0 \in X$  and  $x_1 \in F(x_0)$  such that  $(x_0, x_1) \in E(G)$ , then  $F$  and  $T$  have a common fixed point.

**Corollary 4.2.** Let  $(X, d)$  be a complete metric space endowed with a directed graph  $G$  and suppose that the triplet  $(X, d, G)$  satisfies Property (A). Let  $F, T : X \rightarrow C(X)$  be mappings with the property that for any  $x, y \in X$  with  $(x, y) \in E(G)$ , we have the conditions:

(1)  $a \in F(x)$  implies that there exists  $b \in T(y)$  such that  $(a, b) \in E(G)$  and

$$\psi(d(a, b)) \leq k(d(x, y))\psi((M(Fx, Ty)) + L\varphi(N(Fx, Ty))),$$

(2)  $a \in T(x)$  implies that there exists  $b \in F(y)$  such that  $(a, b) \in E(G)$  and

$$\psi(d(a, b)) \leq k(d(x, y))\psi((M(Fx, Ty)) + L\varphi(N(Fx, Ty))),$$

where

$$\begin{aligned} M(Fx, Ty) &= \max\left\{d(x, y), d(x, F(x)), d(y, T(y)), \frac{d(y, F(x)) + d(x, T(y))}{2}\right\}, \\ N(Fx, Ty) &= \min\{d(x, F(x)), d(y, T(y)), d(y, F(x)), d(x, T(y))\}, \end{aligned}$$

$k : (0, +\infty) \rightarrow [0, 1)$  satisfies  $\limsup_{s \rightarrow t^+} k(s) < 1$ , for every nonnegative  $t$ ,  $L \geq 0$ ,  $\psi \in \Psi$  and  $\varphi$  is nondecreasing and satisfies conditions (i) of Definition 1.6. If  $E(G) \cap \text{Graph}(F) \neq \emptyset$ , then  $F$  and  $T$  have a common fixed point in  $X$ .

To prove Corollary 4.2, it suffices to pick  $p = d$  in Theorem 2.1.

Finally, we recapture [13, Theorem 2.2]. Recall that  $X_F$  is defined by (1.1).

**Corollary 4.3.** Let  $(X, d)$  be a complete metric space and suppose that the triple  $(X, d, G)$  satisfies Property (A). Let  $F, T : X \rightarrow CB(X)$  be a  $(\psi-G)$  contraction. Then the following statements hold.

- (1) For any  $x \in X_F$ ,  $F$  and  $T|_{[x]_G}$  have a common fixed point,
- (2) If  $X_F \neq \emptyset$  and  $G$  is weakly connected, then  $F$  and  $T$  have a common fixed point in  $X$ .
- (3) If  $X' = \cup\{x]_G : x \in X_F\}$ , then  $F$  and  $T|_{X'}$  have a common fixed point.
- (4) If  $F \subseteq E(G)$ , then  $F$  and  $T$  have a common fixed point.

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*S. Benchabane,*  
*Laboratoire "Théorie du Point Fixe et Applications",*  
*ENS, BP 92 Kouba. Algiers, 16006. Algeria.*  
*E-mail address: benchabane.saadia@ymail.com*

and

*S. Djebali,*  
*Department of Mathematics & Statistics, College of Sciences,*  
*Imam Mohammad Ibn Saud Islamic University (IMSIU)*  
*PB 90950. Riyadh 11623, Saudi Arabia.*  
*E-mail address: djebali@hotmail.com*