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Some Results on Generalized Mean Nonexpansive Mapping in Complete Metric Spaces

Akindele Adebayo Mebawondu, Chinedu Izuchukwu, Hammed Anuoluwapo Abass and Oluwatosin Temitope Mewomo

Abstract: In this paper, we obtain sufficient conditions for the existence of a unique fixed point of T- mean nonexpansive mapping and an integral type of T- mean nonexpansive mapping. We also obtain sufficient conditions for the existence of coincidence point and common fixed point for a Jungck-type mean nonexpansive mapping in the frame work of a complete metric space. Some examples of T-mean nonexpansive and Jungck-type mean nonexpansive mappings which are not mean nonexpansive mapping are given. The result obtained generalizes corresponding results in this direction in the literature.

Key Words: T- mean nonexpansive mapping, Fixed point, Jungck-type mean nonexpansive mapping, Integral type T- mean nonexpansive mapping, Metric space.

Contents

1 Introduction and Preliminaries **1**

2 Existence and Uniqueness of Fixed Point of Generalized Mean Nonexpansive Mappings **5**

3 Common Fixed Point Theorem for Jungck-Mean Nonexpansive Mapping **11**

4 Conclusion **14**

1. Introduction and Preliminaries

Let (X, d) denote a metric space and C be a nonempty closed and convex subset of X. A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if

$$Tx = x. \tag{1.1}$$

Definition 1.1. *The mapping $T : C \to C$ is said to be*

1. *an α-contraction if there exists $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall \quad x, y \in C, \tag{1.2}$$

2. *Kannan contraction if there exists $b \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad \forall \quad x, y \in C,$$

3. *Chatterjea contraction if there exists $c \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \quad \forall \quad x, y \in C,$$

4. *Zamfirescu contraction if there exist $a \in [0, 1), b, c \in (0, \frac{1}{2})$ such that for all $x, y \in C$, one has at least one of the following:*
   - *(a)* $d(Tx, Ty) \leq ad(x, y)$;
   - *(b)* $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
   - *(c)* $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

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Banach in 1922, introduced the Banach contraction principle as follows:

Theorem 1.2. *[1] Let (X,d) be a complete metric space and $T : X \to X$ be a self map that satisfies condition (1.2). Then T has a unique fixed point $x^* \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = x^*$.*

Banach contraction principle can be seen as the pivot of the theory of fixed points and its applications. The theory of fixed points plays an important role in nonlinear functional analysis and it is very useful for showing the existence and uniqueness theorems for nonlinear differential and integral equations. The importance of Theorem 1.2 cannot be over emphasized in the study of fixed point theory and its applications. Thereafter, the Banach contraction principle has been extended and generalized by researchers in this area. Researchers in this area generalize the well celebrated Banach contraction principle by considering a class of nonlinear mappings and spaces which are more general than the class of a contraction mappings and metric spaces (see [5,6,7,8,9,15,20,21,23] and the references therein). In [12], Goebel generalized condition (1.2) by introducing a continuous mapping S in place of the identity mapping ($Ix = x$), such that S commutes with T and $T(X) \subset S(X)$. More precisely, he introduced the following definition.

Definition 1.3. *Let $S, T : Y \to X$ be two mappings, T is called a Jungck contraction if there exists a real number $\delta \in [0,1)$ such that*

$$d(Tx,Ty) \leq \delta d(Sx,Sy) \tag{1.3}$$

for all $x,y \in X$.

In 1976, Jungck [10], proved a common fixed point theorem for commuting maps under the condition that $X = Y$. The result is as follows:

Theorem 1.4. *Let (X,d) be a complete metric space. Suppose the mappings $S,T : X \to X$ satisfies condition (1.3) such that (T,S) are commuting pair, $T(X) \subseteq S(X)$ and S is continuous. Then T and S have a unique common fixed point say $p \in X$.*

Remark 1.5. *Clearly, if we take $Sx = x$ in (1.3) for all $x \in X$, we obtain condition (1.2).*

We now recall the following definition of Jungck-type mappings.

Definition 1.6. *Let (X,d) be a metric space and $T,S : X \to X$ be two mappings. A mapping T is said to be*

1. *Jungck Kannan contraction if there exists $b \in (0,\frac{1}{2})$ such that*

$$d(Tx,Ty) \leq b[d(Sx,Tx) + d(Sy,Ty)], \quad \forall \quad x,y \in C,$$

2. *Jungck Chatterjea contraction if there exists $c \in (0,\frac{1}{2})$ such that*

$$d(Tx,Ty) \leq c[d(Sx,Ty) + d(Sy,Tx)], \quad \forall \quad x,y \in C,$$

3. *Jungck Zamfirescu contraction if there exist $a \in [0,1), b,c \in (0,\frac{1}{2})$ such that for all $x,y \in C$, one has at least one of the following:*
   (a) $d(Tx,Ty) \leq ad(Sx,Sy)$;
   (b) $d(Tx,Ty) \leq b[d(Sx,Tx) + d(Sy,Ty)]$;
   (c) $d(Tx,Ty) \leq c[d(Sx,Ty) + d(Sy,Tx)]$.

Remark 1.7. *Clearly, if $Sx = x$, we obtain Definition 1.1.*

Definition 1.8. *[11] Let X be a nonempty set and $S,T : X \to X$ be any two mappings.*

1. *A point $x \in X$ is called:*

   (a) *coincidence point of* S *and* T *if* $Sx = Tx$,
   (b) *common fixed point of* S *and* T *if* $x = Sx = Tx$.

2. *If* $y = Sx = Tx$ *for some* $x \in X$, *then* y *is called the point of coincidence of* S *and* T.
3. *A pair* (S,T) *is said to be:*
   (a) *commuting if* $TSx = STx$ *for all* $x \in X$,
   (b) *weakly compatible if they commute at their coincidence points, that is* $STx = TSx$, *whenever* $Sx = Tx$.

In 2002, Branciari [3] established a fixed point theorem which is a generalization of Theorem 1.2 for a mapping defined as follows:

Definition 1.9. *Let* (X,d) *be a complete metric space. A mapping* $T : X \to X$ *is said to be a* $\int \varphi$*-Banach contraction, if there exists* $a \in [0,1)$ *such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)d(t) \leq a \int_0^{d(x,y)} \varphi(t)d(t) \quad \forall\ x,y \in X, \tag{1.4}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *and* φ *satisfies the following conditions*

1. φ *is Lebesgue integrable mapping which is summable on each compact subset of* $\mathbb{R}^+$,
2. *for each* $\epsilon > 0, \int_0^\epsilon \varphi(t)d(t) > 0$.

He further gave examples of mappings satisfying condition (1.4) that does not satisfy condition (1.2) and established the following result.

Theorem 1.10. *[3] Let* (X,d) *be a complete metric space and let* $T : X \to X$ *be a* $\int \varphi$*-Banach contraction. Then* T *has a unique fixed point* $p \in X$ *such that for all* $x \in X, \lim_{n\to\infty} T^n x = p$.

Remark 1.11. *It is easy to see that condition (1.4) is a generalization of condition (1.2).*

In 2009, Beiranvand et al. [2], introduced the notions of T-Banach contraction and T-contractive mapping, and then they extended the Banach contraction principle and Edelstein's fixed point theorem. In addition, S. Moradi [16] introduced the T-Kannan contractive mapping and extended the Kannan's fixed point theorem [13]. They established the condition for the existence and unique fixed point for mapping in this class as follows:

Theorem 1.12. *[2] Let* (X,d) *be a complete metric space. Suppose the mappings* $S,T : X \to X$ *satisfying*

$$d(TSx,TSy) \leq ad(Tx,Ty)$$

for all $x,y \in X$, *where* $a \in [0,1)$ *such that* T *is one to one, continuous and subsequentially convergent. Then* S *has a unique fixed point* $p \in X$. *Moreover, if* T *is sequentially convergent, then for all* $x \in X$, $\lim_{n\to\infty} S^n x = p$.

Theorem 1.13. *[13] Let* (X,d) *be a complete metric space. Suppose the mappings* $S,T : X \to X$ *satisfying*

$$d(TSx,TSy) \leq b[d(Tx,TSx) + d(Ty,TSy)]$$

for all $x,y \in X$, *where* $b \in (0,\frac{1}{2})$ *such that* T *is one to one, continuous and subsequentially convergent. Then* S *has a unique fixed point* $p \in X$. *Moreover, if* T *is sequentially convergent, then for all* $x \in X$, $\lim_{n\to\infty} S^n x = p$.

Here we recall the definitions of the following classes of T-contraction mappings.

Definition 1.14. *Let* (X,d) *be a metric space and* $T,S : X \to X$ *be two mappings. A mapping* S *is said to be*

1. *T- Banach contraction if there exists $a \in [0,1)$ such that*

$$d(TSx, TSy) \leq ad(Tx, Ty), \quad \forall \quad x, y \in C,$$

2. *T-Kannan contraction if there exists $b \in (0, \frac{1}{2})$ such that*

$$d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)], \quad \forall \quad x, y \in C,$$

3. *T-Chatterjea contraction if there exists $c \in (0, \frac{1}{2})$ such that*

$$d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)], \quad \forall \quad x, y \in C,$$

4. *T-Zamfirescu contraction if there exist $a \in [0,1), b, c \in (0, \frac{1}{2})$ such that for all $x, y \in C$, one has at least one of the following:*
   (a) $d(TSx, TSy) \leq ad(Tx, Ty)$;
   (b) $d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)]$;
   (c) $d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)]$.

Remark 1.15. *Clearly, if $Tx = x$, we obtain Definition 1.1.*

The concept of T-contraction mapping was introduced and studied by Morales and Rojas [17,18] in cone metric spaces. They established the strong convergence and fixed point theorems for T-Kannan, T-Zamfirescu and T-weakly contraction mappings. The existence of fixed point for T-Zamfirescu operators [17] in complete metric spaces was established. Furthermore, they proved a convergence theorem using T-Picard iteration for the class of T-Zamfirescu operators. They obtained the following result:

Theorem 1.16. *[17] Let (X, d) be a complete metric space and $S, T : X \to X$ be two mappings such that T is continuous, one to one and subsequentially convergent. If S is a T-Zamfirescu operator, then S has a unique fixed point. Moreover, if T is sequentially convergent, then for every $x_0 \in X$, the T-Picard iteration associated to S, $TS^n x_0$ converges to Tx, where x is the fixed point of X.*

Definition 1.17. *[2] Let (X, d) be a metric space and $T : X \to X$ be a nonlinear mapping.*

1. *The mapping T is said to be sequentially convergent if for every sequence $\{y_n\} \subset X$, such that Ty_n is convergent, then $\{y_n\}$ is convergent.*
2. *The mapping T is said to be subsequentially convergent if for every $\{y_n\} \subset X$, such that Ty_n is convergent, then $\{y_n\}$ has a convergent subsequence.*

In 1975, Zhang [25] introduced and studied the class of mean nonexpansive mappings in Banach spaces, he proved the unique existence of fixed points for this class of mappings in Banach spaces with normal structure. We recall that, a mapping $S : X \to X$ is said to be mean nonexpasive if there exist $a, b \geq 0$ with $a + b \leq 1$ such that

$$\|Sx - Sy\| \leq a\|x - y\| + b\|x - Sy\|, \tag{1.5}$$

for all $x, y \in X$.
In 2007, Wu [22] proved that if $a + b < 1$, then mean nonexpansive mapping S has a unique fixed point. Zuo in [27] proved that a mean nonexpansive mapping has approximate fixed point sequence and under some suitable conditions he got some existence and uniqueness theorems of fixed point. In 2015, Zhou and Cui in [26] studied the existence of fixed points for mean nonexpansive mappings in CAT(0) spaces. They also obtain the demiclosed principle for mean nonexpansive mappings in CAT(0) spaces. In addition, they proved a Δ-convergence theorem and a strong convergence theorem of the Ishikawa iteration process for mean nonexpansive mappings under the proper restrictions in CAT(0) spaces. In 2017, Chen et al. [4], introduced the concept of a mean nonexpansive set-valued mapping in Banach spaces, and extend Nadler's

fixed point theorem and Lim's fixed point theorem to the case of mean nonexpansive set-valued mappings.

To the best of our knowledge, there is no discussion so far concerning the extension or generalization of the concept of mean nonexpansive mapping using the idea of T mappings, integral type of T mappings and Jungck-type mappings. Motivated by the research work described above and the research work in this direction, our purpose in this paper is to obtain sufficient conditions for the existence of a unique fixed point of T- mean nonexpansive mapping and an integral type of T- mean nonexpansive mapping. We also obtain sufficient conditions for the existence of coincidence point and common fixed point for a Jungck-type mean nonexpansive mapping in the frame work of a complete metric space.

2. Existence and Uniqueness of Fixed Point of Generalized Mean Nonexpansive Mappings

In this section, we introduce the T mean nonexpansive mapping, integral type mapping and generalize the existence and uniqueness result of mean nonexpansive mappings.

Definition 2.1. *Let (X,d) be a metric space and $T,S : X \to X$ be two mappings. A mapping S is said to be T- mean nonexpansive if there exist $a,b \geq 0$ with $a+b \leq 1$ such that*

$$d(TSx,TSy) \leq ad(Tx,Ty) + bd(Tx,TSy), \quad \forall \quad x,y \in X. \tag{2.1}$$

Remark 2.2. *In the above definition, if $Tx = x$, we obtain condition (1.5).*

We now give examples of mappings that satisfy condition (2.1) but do not satisfy condition (1.5).

Example 2.1. *Let $X = [1,\infty)$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Let $S,T : X \to X$ be defined by $Tx = 1 + \ln x$ and*

$$Sx = \begin{cases} \sqrt[10]{x} & if \quad x \neq 2 \\ 1 \quad if & x = 2. \end{cases}$$

Then, S is T-mean nonexpansive, but not mean nonexpansive.
Proof:
To show that S is T-mean nonexpansive, we consider the following cases.
Case 1: *Let $x = y = 2$ and $a = \frac{1}{2}, b = \frac{1}{3}$, we have*

$$d(TSx,TSy) = 0 \leq \frac{1}{2}d(Tx,Ty) + \frac{1}{3}d(Tx,TSy).$$

Case 2: *Let $x,y \neq 2$ and $a = \frac{1}{2}, b = \frac{1}{3}$, we have*

$$\begin{aligned} d(TSx,TSy) &= |1 + \ln(x)^{\frac{1}{10}} - (1 + \ln(y)^{\frac{1}{10}})| = \frac{1}{10}|\ln x - \ln y| \\ &< \frac{1}{2}|\ln x - \ln y| + \frac{1}{3}|1 + \ln x - (1 + \ln(y)^{\frac{1}{10}})| \\ &= ad(Tx,Ty) + bd(Tx,TSy) \end{aligned}$$

Case 3: *Let $x \neq 2, y = 2$ and $a = \frac{1}{2}, b = \frac{1}{3}$, we have*

$$\begin{aligned} d(TSx,TSy) = |1 + \ln(x)^{\frac{1}{10}} - 1| &= \frac{1}{10}\ln(x) \\ &< \frac{1}{3}|\ln x| = \frac{1}{3}|1 + \ln x - 1| \\ &< \frac{1}{2}|\ln x - \ln y| + \frac{1}{3}|1 + \ln x - 1| \\ &= ad(Tx,Ty) + bd(Tx,TSy). \end{aligned}$$

Case 4: *Using similar approach as in case 3, for $y \neq 2, x = 2$ and $a = \frac{1}{2}, b = \frac{1}{3}$, we have*

$$\begin{aligned} d(TSx, TSy) &= \frac{1}{10}\ln y \leq \frac{1}{2}|\ln 2 - \ln y| + \frac{1}{3}|\ln 2 - \ln y^{\frac{1}{10}}| \\ &= ad(Tx, Ty) + bd(Tx, TSy). \end{aligned}$$

To show that S is not mean nonexpansive, we suppose that it is mean nonexpansive and obtain a contradiction. Suppose that there exist nonnegative real numbers a and b such that $a + b \leq 1$ and $d(Sx, Sy) \leq ad(x, y) + bd(x, Sy)$ for all $x, y \in [1, \infty]$. In particular, for $x = 1$ and $y = 2$, we have that

$$\begin{aligned} d(Sx, Sy) = 0 &\leq ad(x, y) + bd(x, Sy) \\ &\leq d(x, y). \end{aligned}$$

Therefore, we have that S is a nonexpansive mapping, but this contradicts the fact that S is not continuous. Since every nonexpansive mapping is continuous. Hence, S is not mean nonexpansive.

Example 2.2. *Let $X = [0, 2]$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let $S, T : X \to X$ be defined by*

$$Tx = \begin{cases} 1 - x & \text{if} \quad x \in [0, 1] \\ 2 - x & \text{if} \quad x \in (1, 2]. \end{cases}$$

and

$$Sx = \begin{cases} 1 & \text{if} \quad x \in [0, 1) \\ 2 & \text{if} \quad x \in [1, 2]. \end{cases}$$

Then S is T- mean nonexpansive mapping, but not mean nonexpansive.

Proof:

For any $a, b \geq 0$ with $a + b \leq 1$ and for all $x, y \in [0, 2]$, we have that $d(TSx, TSy) = 0$. Hence, we have that S is T-mean nonexpansive. To show that S is not mean nonexpansive, we suppose that it is mean nonexpansive and obtain a contradiction. Suppose that there exist nonnegative real numbers a and b such that $a + b \leq 1$ and $d(Sx, Sy) \leq ad(x, y) + bd(x, Sy)$ for all $x, y \in [1, \infty]$. In particular, for $x = 1$ and $y = 2$, we have that

$$\begin{aligned} d(Sx, Sy) = 0 &\leq ad(x, y) + bd(x, Sy) \\ &\leq d(x, y). \end{aligned}$$

Therefore, we have that S is a nonexpansive mapping, but this contradicts the fact that S is not continuous. Since every nonexpansive mapping is continuous. Hence, S is not mean nonexpansive.

Theorem 2.3. *Let (X, d) be a complete metric space and $S : X \to X$ be T-mean nonexpansive mapping. If T is continuous, one to one and subsequentially convergent. Then S has a unique fixed point.*

Proof:

Let $x_0 \in X$ and define the sequence $Tx_{n+1} = TSx_n$. Now observe that

$$\begin{aligned} d(Tx_{n+1}, Tx_n) = d(TSx_n, TSx_{n-1}) &\leq ad(Tx_n, Tx_{n-1}) + bd(Tx_n, TSx_{n-1}) \\ &= ad(Tx_n, Tx_{n-1}) + bd(Tx_n, Tx_n) \\ &= ad(Tx_n, Tx_{n-1}). \end{aligned} \tag{2.2}$$

Also,

$$\begin{aligned} d(Tx_n, Tx_{n-1}) &= d(TSx_{n-1}, TSx_{n-2}) \\ &\leq ad(Tx_{n-1}, Tx_{n-2}) + bd(Tx_{n-1}, TSx_{n-2}) \\ &= ad(Tx_{n-1}, Tx_{n-2}) + bd(Tx_{n-1}, Tx_{n-1}) \\ &= ad(Tx_{n-1}, Tx_{n-2}). \end{aligned} \tag{2.3}$$

Substituting (3.3) into (3.2), we have

$$d(Tx_{n+1}, Tx_n) = d(TSx_n, TSx_{n-1}) \leq a^2 d(Tx_{n-1}, Tx_{n-2})$$

and inductively, we have

$$d(Tx_{n+1}, Tx_n) \leq a^n d(Tx_1, Tx_0).$$

Furthermore, for $n > m$,

$$\begin{aligned} d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_{n-2}) + \cdots + d(Tx_{m+1}, Tx_m) \\ &= d(TSx_{n-1}, TSx_{n-2}) + d(TSx_{n-2}, TSx_{n-3}) + \cdots + d(TSx_m, TSx_{m-1}) \\ &\leq [a^{n-1} + a^{n-2} + \cdots + a^m] d(Tx_1, Tx_0) \\ &= \sum_{i=m}^{n-1} a^i d(Tx_1, Tx_0) \\ &\leq \frac{a^m}{1-a} d(Tx_1, Tx_0) \to 0 \quad \text{as} \quad m, n \to \infty. \end{aligned}$$

$\{Tx_n\}$ is a Cauchy sequence. Since X is complete, then $\{Tx_n\}$ is convergent. Also, since T is subsequentially convergent, it follows that $\{x_n\}$ converges. Suppose that $\{x_n\}$ converges to say x^*. Using the continuity of T, we also have that $\lim_{n\to\infty} Tx_n = Tx^*$. Then

$$\begin{aligned} d(TSx^*, Tx^*) &\leq d(TSx^*, TSx_n) + d(TSx_n, Tx^*) \\ &\leq ad(Tx^*, Tx_n) + bd(Tx^*, TSx_n) + d(TSx_n, Tx^*) \\ &= ad(Tx^*, Tx_n) + (1+b)d(Tx^*, Tx_{n+1}). \end{aligned} \tag{2.4}$$

By taking limit on both sides of (2.4), we have

$$\lim_{n\to\infty} d(TSx^*, Tx^*) = 0.$$

It follows that

$$TSx^* = Tx^*.$$

Since T is one to one, we have that

$$Sx^* = x^*.$$

To show that x^* is the unique fixed point of S, we suppose on the contrary that there exists another fixed point say y^* such that $x^* \neq y^*$. That is $Sx^* = x^*$ and $Sy^* = y^*$

$$\begin{aligned} d(Tx^*, Ty^*) &= d(TSx^*, TSy^*) \\ &\leq ad(Tx^*, Ty^*) + bd(Tx^*, TSy^*) \\ &= (a+b)d(Tx^*, Ty^*) \\ &\leq d(Tx^*, Ty^*). \end{aligned}$$

Clearly, we have that $d(Tx^*, Ty^*) = d(Tx^*, Ty^*)$, if not we get a contradiction $d(Tx^*, Ty^*) < d(Tx^*, Ty^*)$. Since, T is one to one, we have $x^* = y^*$.

Theorem 2.4. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$ satisfy*

$$d(TSx, TSy) \leq ad(Tx, Ty) + 2bd(Tx, TSy)$$

for all $x, y \in X$, where $a, b \geq 0$ and $a + 2b < 1$ such that T is continuous, one to one and subsequentially convergent. Then S has a unique fixed point.

Proof:
The proof is similar to Theorem 2.3. Thus, we omit it.

Theorem 2.5. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$ satisfy*

$$d(TSx, TSy) \leq a[d(Tx, Ty) + d(Tx, TSy)]$$

for all $x, y \in X$, where $a \in [0, 1)$ such that T is continuous, one to one and subsequentially convergent. Then S has a unique fixed point.

Proof:
The proof is similar to Theorem 2.3. Thus, we omit it.

Corollary 2.6. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$ satisfying*

$$d(TSx, TSy) \leq ad(Tx, Ty)$$

for all $x, y \in X$, where $a, b \geq 0$ and $a \in [0, 1)$ such that T is continuous, one to one and subsequentially convergent. Then S has a unique fixed point.

Khan, Swaleh and Sessa in [14] introduced the concept of alternating distance function, which is defined as follows: A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an alternating distance function if the following conditions are satisfied:

1. $\phi(0) = 0$,
2. ϕ is monotonically nondecreasing,
3. ϕ is continuous.

Using this concept of alternating distance function, we introduce a new class of mappings which is a generalization of (2.1) and (1.5).

Definition 2.7. *Let (X, d) be a metric space and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an alternating distance function. A mapping S is said to be (ϕ, T)-mean nonexpansive if there exists $a, b \geq 0$ with $a + b \leq 1$ such that*

$$d(TSx, TSy) \leq ad(Tx, Ty) + b\phi(d(Tx, TSy)), \quad \forall \quad x, y \in X. \tag{2.5}$$

Remark 2.8. *If $\phi(t) = t$, for all $t > 0$, condition (2.5) becomes condition (2.1). Also, if $Tx = x$ and $\phi(t) = t$ for all $t > 0$, we obtain condtion (1.5).*

Theorem 2.9. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$ satisfying condition (2.5) such that T is continuous, one to one and subsequentially convergent and $\phi(0) = 0$. Then S has a unique fixed point.*

Proof:
Let $x_0 \in X$ and define the sequence $Tx_{n+1} = TSx_n$. Using the fact that $\phi(0) = 0$, we have

$$\begin{aligned} d(Tx_{n+1}, Tx_n) = d(TSx_n, TSx_{n-1}) &\leq ad(Tx_n, Tx_{n-1}) + b\phi(d(Tx_n, TSx_{n-1})) \\ &= ad(Tx_n, Tx_{n-1}) + b\phi(d(Tx_n, Tx_n)) \\ &= ad(Tx_n, Tx_{n-1}) + b\phi(0) \\ &= ad(Tx_n, Tx_{n-1}). \end{aligned} \tag{2.6}$$

The existence and uniqueness of the fixed point follows from Theorem 2.3.

We introduce an integral type mapping of T-mean nonexpansive mapping.

Definition 2.10. *Let (X,d) be a metric space and let $T,S:X\to X$ be two mappings. A mapping S is said to be $T-\int\varphi$ if there exist $a,b\geq 0$ with $a+b\leq 1$ such that for all $x,y\in X$,*

$$\int_0^{d(TSx,TSy)}\varphi(t)dt\leq a\int_0^{d(Tx,Ty)}\varphi(t)dt+b\int_0^{d(Tx,TSy)}\varphi(t)dt, \tag{2.7}$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ and φ satisfy

1. *φ is Lebesgue integrable mapping which is summable on each compact subset of $\mathbb{R}^+$,*
2. *for each $\epsilon>0$, $\int_0^\epsilon\varphi(t)d(t)>0$.*

Remark 2.11. *Note that if $\varphi(t)=1$ for all $t>0$, we have that*

$$\begin{aligned}\int_0^{d(TSx,TSy)}\varphi(t)dt&=d(TSx,TSy)\\&\leq ad(Tx,Ty)+bd(Tx,TSy)\\&=\int_0^{d(Tx,Ty)}\varphi(t)dt+b\int_0^{d(Tx,TSy)}\varphi(t)dt.\end{aligned}$$

In addition if $Tx=x$, we obtain condition (1.5).

Theorem 2.12. *Let (X,d) be a complete metric space. Suppose mappings $S,T:X\to X$ satisfying condition (2.7) such that T is continuous, one to one and subsequentially convergent. Then S has a unique fixed point.*

Proof:
Let $x_0\in X$ and define the sequence $Tx_{n+1}=TSx_n$. Now observe that

$$\begin{aligned}\int_0^{d(Tx_{n+1},Tx_n)}\varphi(t)dt&=\int_0^{d(TSx_n,TSx_{n-1})}\varphi(t)dt\\&\leq a\int_0^{d(Tx_n,Tx_{n-1})}\varphi(t)dt+b\int_0^{d(Tx_n,TSx_{n-1})}\varphi(t)dt\\&=a\int_0^{d(Tx_n,Tx_{n-1})}\varphi(t)dt+b\int_0^{d(TSx_{n-1},TSx_{n-1})}\varphi(t)dt\\&=a\int_0^{d(Tx_n,Tx_{n-1})}\varphi(t)dt.\end{aligned} \tag{2.8}$$

Inductively, we have that

$$\int_0^{d(Tx_{n+1},Tx_n)}\varphi(t)dt\leq a^n\int_0^{d(Tx_1,Tx_0)}\varphi(t)dt.$$

taking limit on both sides, we have

$$\lim_{n\to\infty}\int_0^{d(Tx_{n+1},Tx_n)}\varphi(t)dt\leq 0.$$

Using the fact that for each $\epsilon>0$, $\int_0^\epsilon\varphi(t)d(t)>0$, it follows that

$$\lim_{n\to\infty}d(Tx_{n+1},Tx_n)=0. \tag{2.9}$$

We need to show that $\{Tx_n\}$ is Cauchy in X. On the contrary, we suppose that $\{Tx_n\}$ is not Cauchy. Then, there exists and $\epsilon>0$ and subsequences $\{m_k\}$ and $\{n_k\}$ of $\{x_n\}$ such that $\{m_k\}<\{n_k\}<\{m_{k+1}\}$ with

$$d(Tx_{m_k},Tx_{n_k})\geq\epsilon,\qquad d(Tx_{m_k},Tx_{n_{k-1}})<\epsilon. \tag{2.10}$$

Using (2.9) and (2.10), we have

$$\begin{aligned} d(Tx_{m_{k-1}}, Tx_{n_{k-1}}) &\leq d(Tx_{m_{k-1}}, Tx_{m_k}) + d(Tx_{m_k}, Tx_{n_{k-1}}) \\ &\leq \epsilon + d(Tx_{m_{k-1}}, Tx_{m_k}). \end{aligned}$$

It follow that

$$\lim_{k\to\infty} d(Tx_{m_{k-1}}, Tx_{n_{k-1}}) \leq \epsilon = \int_0^{\epsilon} \varphi(t)dt.$$

Using (2.8) and (2.10), we also have that

$$\begin{aligned} \epsilon = \int_0^{\epsilon} \varphi(t)dt \leq \int_0^{d(Tx_{m_k}, Tx_{n_k})} \varphi(t)dt \\ \leq a \int_0^{d(Tx_{m_{k-1}}, Tx_{n_{k-1}})} \varphi(t)dt, \end{aligned}$$

taking limit as $k \to \infty$, we have

$$\int_0^{\epsilon} \varphi(t)dt \leq a \int_0^{\epsilon} \varphi(t)dt,$$

which is a contradiction, since $0 \leq a < 1$. Thus $\{Tx_n\}$ is Cauchy in X. $\{Tx_n\}$ is convergent, also since T is subsequentially convergent, it follows that $\{x_n\}$ converges. Suppose that $\{x_n\}$ converges to say x^*. Using the continuity of T, we also have that $\lim_{n\to\infty} Tx_n = Tx^*$. Then

$$\begin{aligned} \int_0^{d(TSx^*, Tx_{n+1})} \varphi(t)dt &= \int_0^{d(TSx^*, TSx_n)} \varphi(t)dt \\ &\leq a \int_0^{d(Tx^*, Tx_n)} \varphi(t)dt + b \int_0^{d(Tx^*, TSx_n)} \varphi(t)dt \\ &= a \int_0^{d(Tx^*, Tx_n)} \varphi(t)dt + b \int_0^{d(Tx^*, Tx_{n+1})} \varphi(t)dt \end{aligned}$$

taking on both side, we have

$$\int_0^{d(TSx^*, Tx^*)} \varphi(t)dt \leq 0.$$

Using the fact that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)d(t) > 0$, it follows that

$$\lim_{n\to\infty} d(TSx^*, Tx^*) = 0.$$

Therefore, we have

$$TSx^* = Tx^*.$$

Since T is one to one, we have that

$$Sx^* = x^*.$$

To show that x^* is the unique fixed point of S. On the contrary, we suppose that there exists another fixed point say y^* such that $x^* \neq y^*$. That is $Sx^* = x^*$ and $Sy^* = y^*$.

$$\begin{aligned}\int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t) &= \int_0^{d(TSx^*,TSy^*)} \varphi(t)d(t)\\ &\leq a\int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t) + b\int_0^{d(Tx^*,TSy^*)} \varphi(t)d(t)\\ &\leq a\int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t) + b\int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t)\\ &= (a+b)\int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t)\\ &\leq \int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t).\end{aligned}$$

Clearly, we have that $\int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t) = \int_0^{d(Tx^*,Ty^*)} \varphi(t)d(t)$. Thus, we have that $d(Tx^*,Ty^*) = d(Tx^*,Ty^*)$ and since, T is one to one, we have $x^* = y^*$.

3. Common Fixed Point Theorem for Jungck-Mean Nonexpansive Mapping

In this section, we generalize the mean nonexpansive mapping by using the concept of Jungck-type mapping.

Definition 3.1. *Let (X,d) be a metric space and $T,S : X \to X$ be two mappings. T is said to be Jungck mean nonexpansive if there exist $a,b \geq 0$ with $a+b \leq 1$ such that*

$$d(Tx,Ty) \leq ad(Sx,Sy) + bd(Sx,Ty), \quad \forall \quad x,y \in X. \tag{3.1}$$

We give an example of a mapping that satisfy condition (3.1) but does not satisfy condition (1.5).

Example 3.1. *Let $X = [0,2]$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Let $S,T : X \to X$ be defined by*

$$Tx = \begin{cases} 1 & if \quad x \in [0,1) \\ 2 & if \quad x \in [1,2] \end{cases}$$

and

$$Sx = \begin{cases} 0 & if \quad x \in [0,1) \\ 2 & if \quad x \in [1,2]. \end{cases}$$

Then the pair S,T satisfy condition (3.1) but S and T does not satisfy condition (1.5).

Proof:
For $a = \frac{2}{3}, b = \frac{1}{3}$, and for all $x,y \in [0,1)$ and $x,y \in [1,2]$, we have that $d(Tx,Ty) = 0$. Thus, the pair (S,T) satisfy condition (3.1).
Case 1: *When $x \in [0,1)$ and $y \in [1,2]$, we have*

$$d(Tx,Ty) = 1 < 2 = \frac{2}{3}|0-2| + \frac{1}{3}|0-2| = ad(Sx,Sy) + bd(Sx,Ty).$$

Case 2: *When $x \in [1,2]$ and $y \in [0,1)$, we have*

$$d(Tx,Ty) = 1 < 1.5 \leq \frac{2}{3}|2-0| + \frac{1}{3}|2-1| = ad(Sx,Sy) + bd(Sx,Ty).$$

Thus, the pair (S,T) satisfy condition (3.1).

To show that S is not mean nonexpansive. We suppose that S is mean nonexpansive, so therefore, there exists nonnegative real numbers a and b, such that $a+b \leq 1$ and $d(Sx,Sy) \leq ad(x,y)+bd(x,Sy)$ for all $x,y \in [0,2]$. Now suppose that $x=2$ and $y=1$, we have that

$$\begin{aligned} d(Sx,Sy) = 0 &\leq ad(x,y)+bd(x,Sy) \\ &\leq d(x,y). \end{aligned}$$

Therefore, S is a nonexpansive mapping, but this contradicts the fact that S is not continuous. Hence S is not mean nonexpansive.

Example 3.2. *Let $X=[0,2]$ and $d(x,y)=\max\{x,y\}$ for all $x,y \in X$. Clearly, (X,d) is a metric space. Let $S,T : X \to X$ be defined by*

$$Tx = \begin{cases} 0 & if \quad x \in [0,1) \\ \frac{1}{2x} & if \quad x \in [1,3] \end{cases}$$

and

$$Sx = \begin{cases} 3x & if \quad x \in [0,1) \\ 4-x & if \quad x \in [1,3]. \end{cases}$$

Then, the pair (S,T) satisfy condition (3.1).

Proof:
Clearly, $T(0)=S(0)=0$, else, we have $Tx \neq Sx$ for all $x \in X$. For $x,y \in [0,1)$, we have that $d(Tx,Ty)=0$. Thus, we will only consider three cases.
Case 1: *For $x,y \in [1,3]$, we have*

$$\begin{aligned} d(Tx,Ty) &= \max\left\{\frac{1}{2x},\frac{1}{2y}\right\} \\ &\leq \frac{1}{2}\max\{4-x,4-y\}+\frac{1}{3}\max\{4-x,\frac{1}{2y}\}, \end{aligned}$$

which implies $d(Tx,Ty) \leq ad(Sx,Sy)+bd(Sx,Ty)$.
Case 2: *For $x \in [0,1)$ and $y \in [1,3]$, we have*

$$\begin{aligned} d(Tx,Ty) &= \max\left\{0,\frac{1}{2y}\right\} = \frac{1}{2y} \\ &\leq \frac{1}{2}\max\{3x,4-y\}+\frac{1}{3}\max\{3x,\frac{1}{2y}\}, \end{aligned}$$

which implies $d(Tx,Ty) \leq ad(Sx,Sy)+bd(Sx,Ty)$.
Case 3: *For $y \in [0,1)$ and $x \in [1,3]$, we have*

$$\begin{aligned} d(Tx,Ty) &= \max\left\{\frac{1}{2x},0\right\} = \frac{1}{2x} \\ &\leq \frac{1}{2}\max\{4-x,3y\}+\frac{1}{3}\max\{3-x,0\}, \end{aligned}$$

which implies $d(Tx,Ty) \leq ad(Sx,Sy)+bd(Sx,Ty)$.

Theorem 3.2. *Let (X,d) be a complete metric space. Suppose the mappings $S,T : X \to X$ satisfying*

$$d(Tx,Ty) \leq ad(Sx,Sy)+bd(Sx,Ty)$$

for all $x,y \in X$, where $a,b \geq 0$ with $a+b \leq 1$ such that $T(X) \subseteq S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover if T and S are weakly compatible, T and S have a unique common fixed point.

Proof:
For every any $x_0 \in X$, there exists $x_1 \in X$ such that $Sx_1 = Tx_0$, since $T(X) \subseteq S(X)$. Using this fact, for any $x_n \in X$, there exist $x_{n+1} \in X$ such that $Sx_{n+1} = Tx_n$. Now observe that

$$\begin{aligned} d(Tx_n, Tx_{n-1}) &\leq ad(Sx_n, Sx_{n-1}) + bd(Sx_n, Tx_{n-1}) \\ &= ad(Sx_n, Sx_{n-1}) + bd(Sx_n, Sx_n) \\ &= ad(Sx_n, Sx_{n-1}). \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned} d(Sx_n, Sx_{n-1}) &= d(Tx_{n-1}, Tx_{n-2}) \\ &\leq ad(Sx_{n-1}, Sx_{n-2}) + bd(Sx_{n-1}, Tx_{n-2}) \\ &= ad(Sx_{n-1}, Sx_{n-2}) + bd(Sx_{n-1}, Sx_{n-1}) \\ &= ad(Sx_{n-1}, Sx_{n-2}). \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$d(Tx_n, Tx_{n-1}) \leq a^2 d(Sx_{n-1}, Sx_{n-2})$$

and inductively, we have

$$d(Tx_n, Tx_{n-1}) = d(Sx_{n+1}, Sx_n) \leq a^n d(Sx_1, Sx_0).$$

Furthermore, for $n > m$,

$$\begin{aligned} d(Sx_n, Sx_m) &\leq d(Sx_n, Sx_{n-1}) + d(Sx_{n-1}, Sx_{n-2}) + \cdots + d(Sx_{m+1}, Sx_m) \\ &= d(Tx_{n-1}, Tx_{n-2}) + d(Tx_{n-2}, Tx_{n-3}) + \cdots + d(Tx_m, Tx_{m-1}) \\ &\leq [a^{n-1} + a^{n-2} + \cdots + a^m] d(Sx_1, Sx_0) \\ &= \sum_{i=m}^{n-1} a^i d(Sx_1, Sx_0) \\ &\leq \frac{a^m}{1-a} d(Sx_1, Sx_0) \to 0 \quad \text{as} \quad n, m \to \infty. \end{aligned}$$

$\{Sx_n\}$ is a Cauchy sequence. Since $S(X)$ is complete, then there exists $x^* \in S(X)$ such that $\lim_{n\to\infty} Sx_n = x^*$. More so, we can find $y^* \in Y$ such that $Sy^* = x^*$ and

$$\begin{aligned} d(Sx_n, Ty^*) &= d(Ty^*, Tx_{n-1}) \\ &\leq ad(Sy^*, Sx_{n-1}) + bd(Sy^*, Tx_{n-1}) \\ &= ad(Sx_{n-1}, x^*) + bd(Sx_n, x^*), \end{aligned}$$

taking limit on both sides, we have

$$\lim_{n\to\infty} d(Sx_n, Ty^*) = 0.$$

In addition, we have

$$\lim_{n\to\infty} d(Sx_n, Sy^*) = 0.$$

By the uniqueness of limit, we obtain $Ty^* = Sy^* = x^*$. Thus x^* is the coincidence point of T and S. To show that x^* is the unique coincidence point, we suppose that there exists another coincidence point say y^* such that $x^* \neq y^*$. Then, there exist say $x_1, y_1 \in Y$ such that $Sx_1 = Tx_1 = x^*$ and $Sy_1 = Ty_1 = y^*$. Then, we have that

$$\begin{aligned} d(x^*, y^*) &= d(Tx_1, Ty_1) \\ &\leq ad(Sx_1, Sy_1) + bd(Sx_1, Ty_1) \\ &= (a+b)d(x^*, y^*) \\ &\leq d(x^*, y^*). \end{aligned}$$

Clearly, we have that $d(x^*, y^*) = d(x^*, y^*)$, if not we get a contradiction $d(x^*, y^*) < d(x^*, y^*)$. Hence, we have that $x^* = y^*$. Since, S and T are weakly compatible and $x^* = Ty^* = Sy^*$, then $Tx^* = TTy^* = TSy^* = STy^* = Sx^*$. Thus, Tx^* is a point of coincidence of S and T, the point of coincidence is unique, we then have $x^* = Tx^*$. Hence, $Tx^* = Sx^* = x^*$ and therefore x^* is unique common fixed point of S and T.

Theorem 3.3. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$*

$$d(Tx, Ty) \leq ad(Sx, Sy) + 2bd(Sx, Ty)$$

for all $x, y \in X$, where $a, b \geq 0$ with $a + 2b \leq 1$ such that $T(X) \subseteq S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover if T and S are weakly compatible, T and S have a unique common fixed point.

Proof:
The proof is similar to that of Theorem 3.2.

Theorem 3.4. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$ satisfying*

$$d(Tx, Ty) \leq a[d(Sx, Sy) + d(Sx, Ty)]$$

for all $x, y \in X$, where $a \in [0, 1)$, such that $T(X) \subseteq S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover if T and S are weakly compatible, T and S have a unique common fixed point.

Proof:
The proof is similar to that of Theorem 3.2.

Corollary 3.5. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$*

$$d(Tx, Ty) \leq ad(Sx, Sy)$$

for all $x, y \in X$, where $a \in [0, 1)$ such that $T(X) \subseteq S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover if T and S are weakly compatible, T and S have a unique common fixed point.

4. Conclusion

In this work, we have extend and improve various fixed point results in metric spaces. More so, the results obtained in this paper generalize and complement many well-known results in Hilbert spaces and Banach spaces.

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Akindele Adebayo Mebawondu,
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal,
Durban, South Africa.
E-mail address: dele@aims.ac.za, 216028272@ukzn.ac.za

and

Chinedu Izuchukwu,
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal,
Durban, South Africa.
E-mail address: izuchukwuc@ukzn.ac.za, izuchukwu_c@yahoo.com

and

Hammed Anuoluwapo Abass,
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal,
Durban, South Africa.
E-mail address: hammedabass548@gmail.com

and

Oluwatosin Temitope Mewomo,
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal,
Durban, South Africa.
E-mail address: mewomoo@ukzn.ac.za