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Properties of a Certain Class of Multivalent Functions

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ABSTRACT: The object of this paper is to derive some properties for certain class of multivalent functions. Also we obtain some properties of an integral operator for functions in this class.

Key Words: Multivalent functions, p-valent starlike and convex functions of order α, p-valent functions, Univalent functions, Generalized Libera integral operator, Differential subordination.

Contents

1. Introduction

Denote by $\mathcal{A}_p(n)$, with $p,\, n \in \mathbb{N} := \{1, 2, \dots\}$, the class of multivalent analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k,\ z \in \mathbb{U}.$$

For $0 \le \alpha < p - q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $p > q$, we say that the function $f \in \mathcal{A}_p(n)$ is in the class $\mathbb{S}^*_{p,q}(n,\alpha)$ if it satisfies the inequality

$$\operatorname{Re} \frac{z f^{(1+q)}(z)}{f^{(q)}(z)} > \alpha,\ z \in \mathbb{U}.$$

Also, the function $f \in \mathcal{A}_p(n)$ is in the class $\mathbb{K}_{p,q}(n,\alpha)$ if it satisfies

$$\operatorname{Re} \left(1 + \frac{z f^{(2+q)}(z)}{f^{(1+q)}(z)}\right) > \alpha,\ z \in \mathbb{U}.$$

We note that $\mathbb{S}^*_{p,q}(1,\alpha) =: \mathbb{S}^*_{p,q}(\alpha)$ and $\mathbb{K}_{p,q}(1,\alpha) =: \mathbb{K}_{p,q}(\alpha)$ were studied by Aouf in [1,2,3]. Also, the classes $\mathbb{S}^*_{p,0}(n,\alpha) =: \mathbb{S}^*(p,n,\alpha)$ and $\mathbb{K}_{p,0}(n,\alpha) =: \mathbb{K}(p,n,\alpha)$ are the classes of *p-valent starlike and convex functions of order α*, respectively, with $0 \le \alpha < p$, which were introduced and studied by Owa in [10].

For two functions f and g, analytic in $\mathbb{U}$, the function f is said to be *subordinated* to g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, i.e. w is analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if g is univalent in $\mathbb{U}$, then (see [4] and [7]).

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathbb{B}_{p,q}(n,\alpha)$ denote the class of functions $f \in \mathcal{A}_p(n)$ satisfying the subordination

$$\frac{f^{(q)}(z)}{z^{p-q}} \prec \delta(p,q) + [\delta(p,q) - \alpha]\, z, \tag{1.1}$$

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where $\delta(p,q)=\dfrac{p!}{(p-q)!}$, $p>q$, and $0\le\alpha<\delta(p,q)$. The subordination (1.1) is equivalent to

$$\left|\frac{f^{(q)}(z)}{z^{p-q}}-\delta(p,q)\right|<\delta(p,q)-\alpha,\ z\in\mathbb{U},$$

and we note that $\mathbb{B}_{p,0}(n,\alpha)=:\mathbb{A}_p(n,\alpha)$ (see Saitoh [17]).

For $f\in\mathcal{A}_p(n)$ we define the *generalized Libera integral operator* $J_{c,p}$ by

$$J_{c,p}^{q;n}f^{(q)}(z):=\frac{c+p-q}{z^c}\int_0^z t^{c-1}f^{(q)}(t)dt$$
$$=\delta(p,q)z^{p-q}+\sum_{k=p+n}^{\infty}\frac{c+p-q}{c+k-q}\delta(k,q)a_kz^{k-q},\ z\in\mathbb{U},\tag{1.2}$$

with

$$c+p-q>0.$$

We note that the operator $J_{c,p}^{0,1}=:J_{c,p}$ was introduced and studied by Saitoh in [18], and $J_{c,p}^{0,n}=:J_{c,p}^{n}$ was also studied by Saitoh [17].

In order to drive our results we have to recall the following lemmas.

Lemma 1.1. *[5,6] Let $\omega(z)=b_nz^n+b_{n+1}z^{n+1}+\dots$ be analytic in $\mathbb{U}$ such that $\omega(z)\not\equiv0$. If $z_0=r_0e^{i\theta}$ $(r_0<1)$ and $\max\limits_{|z|\le r_0}|\omega(z)|=|\omega(z_0)|$, then $z_0\omega'(z_0)=k\omega(z_0)$, where $k\ge n\ge1$.*

The following lemma is a special case of [8, Theorem 8]:

Lemma 1.2. *Let $f\in\mathcal{A}(p):=\mathcal{A}_p(1)$, and if there exists a $(p-k+1)$-valent starlike function $g(z)=z^{p-k+1}+b_{p-k+2}z^{p-k+2}+\dots$ satisfying*

$$\operatorname{Re}\frac{zf^{(k)}(z)}{g(z)}>0,\ z\in\mathbb{U},$$

then f is p-valent in $\mathbb{U}$.

Lemma 1.3. *[15, Theorem 2] If $f\in\mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left[\left(\frac{f(z)}{z}\right)^2\frac{1}{f'(z)}\right]\ge\frac{1}{2},\ z\in\mathbb{U},$$

then f is univalent in $\mathbb{U}$.

2. Properties of the class $\mathbb{B}_{p,q}(n,\alpha)$

In the reminder of this paper we assume that $0\le\alpha<\delta(p,q)$, β, $\gamma\ge0$, p, $n\in\mathbb{N}$, $q\in\mathbb{N}_0$ and $p>q$.

Theorem 2.1. *If $f\in\mathcal{A}_p(n)$ satisfies*

$$\left|\beta\frac{f^{(q)}(z)}{z^{p-q}}+\gamma\frac{f^{(1+q)}(z)}{z^{p-q-1}}-[\beta+\gamma(p-q)]\delta(p,q)\right|$$
$$<[\delta(p,q)-\alpha][\beta+\gamma(p-q+n)],\ z\in\mathbb{U},\tag{2.1}$$

then $f\in\mathbb{B}_{p,q}(n,\alpha)$.

Proof. If is easy to check that the function $f_0(z) = z^p$ satisfies the inequality (2.1) and $f_0 \in \mathbb{B}_{p,q}(n, \alpha)$. For $f \in \mathcal{A}_p(n)$, with $f(z) \not\equiv z^p$, let define the function ω by

$$\frac{f^{(q)}(z)}{z^{p-q}} = \delta(p,q) + [\delta(p,q) - \alpha]\omega(z),\ z \in \mathbb{U}. \tag{2.2}$$

Then, $\omega(z) = \omega_n z^n + \omega_{n+1} z^{n+1} + \dots$ is analytic in $\mathbb{U}$, with $\omega(z) \not\equiv 0$, and differentiating (2.2) we have

$$\frac{f^{(1+q)}(z)}{z^{p-q-1}} = (p-q)\delta(p,q) + [\delta(p,q) - \alpha][(p-q)\omega(z) + z\omega'(z)],\ z \in \mathbb{U}. \tag{2.3}$$

From (2.2) and (2.3) we obtain that

$$\beta \frac{f^{(q)}(z)}{z^{p-q}} + \gamma \frac{f^{(1+q)}(z)}{z^{p-q-1}} - [\beta + \gamma(p-q)]\delta(p,q)$$
$$= [\delta(p,q) - \alpha]\Big\{[\beta + \gamma(p-q)]\omega(z) + \gamma z\omega'(z)\Big\},\ z \in \mathbb{U}.$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\max_{|z| \le |z_0|} |\omega(z)| = |\omega(z_0)| = 1$. Then, applying Lemma 1.1 we obtain that $z_0\omega'(z_0) = m\omega(z_0)$, with $m \ge n \ge 1$, and letting $\omega(z_0) = e^{i\theta}$, $\theta \in [0, 2\pi)$, we have

$$\left| \beta \frac{f^{(q)}(z_0)}{z_0^{p-q}} + \gamma \frac{f^{(1+q)}(z_0)}{z_0^{p-q-1}} - [\beta + \gamma(p-q)]\delta(p,q) \right|$$
$$= [\delta(p,q) - \alpha]\Big|[\beta + \gamma(p-q)]e^{i\theta} + \gamma m e^{i\theta}\Big| \ge [\delta(p,q) - \alpha](\beta + \gamma(p-q) + \gamma n)$$
$$= [\delta(p,q) - \alpha][\beta + \gamma(p-q+n)],$$

which contradicts the assumption (2.1). Therefore, $|\omega(z)| < 1$, $z \in \mathbb{U}$, that is

$$\left| \frac{f^{(q)}(z)}{z^{p-q}} - \delta(p,q) \right| < \delta(p,q) - \alpha,\ z \in \mathbb{U},$$

hence $f \in \mathbb{B}_{p,q}(n, \alpha)$. □

Putting $\gamma = 1 - \beta$, with $0 \le \beta \le 1$ in Theorem 2.1 we obtain:

Corollary 2.2. *If $f \in \mathcal{A}_p(n)$ satisfies*

$$\left| \beta \left(\frac{f^{(q)}(z)}{z^{p-q}} - \delta(p,q) \right) + (1-\beta) \left(\frac{f^{(1+q)}(z)}{z^{p-q-1}} - \delta(p,q+1) \right) \right|$$
$$< [\delta(p,q) - \alpha][\beta + (1-\beta)(p-q+n)],\ z \in \mathbb{U},$$

for some β, $0 \le \beta \le 1$, then, $f \in \mathbb{B}_{p,q}(n, \alpha)$.

For $\beta = 0$, Corollary 2.2 reduces to the following special case:

Example 2.3. *If $f \in \mathcal{A}_p(n)$ satisfies*

$$\left| \frac{f^{(1+q)}(z)}{z^{p-q-1}} - \delta(p,q+1) \right| < [\delta(p,q) - \alpha](p-q+n),\ z \in \mathbb{U},$$

then $f \in \mathbb{B}_{p,q}(n, \alpha)$.

Putting $\beta = 1/2$ in Corollary 2.2 or $\beta = \gamma \ge 0$ in Theorem 2.1 we obtain the next result:

Example 2.4. *If $f \in \mathcal{A}_p(n)$ satisfies*

$$\left|\frac{f^{(q)}(z)}{z^{p-q}} + \frac{f^{(1+q)}(z)}{z^{p-q-1}} - (p-q+1)\delta(p,q)\right| < [\delta(p,q)-\alpha](p-q+n+1),\ z \in \mathbb{U},$$

then, $f \in \mathbb{B}_{p,q}(n,\alpha)$.

Considering $\beta = 1-(p-q+n)\gamma \geq 0$ in Theorem 2.1 we deduce the following implication:

Example 2.5. *If $f \in \mathcal{A}_p(n)$ satisfies*

$$\left|[1-(p-q+n)\gamma]\frac{f^{(q)}(z)}{z^{p-q}} + \gamma\frac{f^{(1+q)}(z)}{z^{p-q-1}} - (1-n\gamma)\delta(p,q)\right| < \delta(p,q)-\alpha,\ z \in \mathbb{U},$$

for some γ, with $0 \leq \gamma \leq \dfrac{1}{p-q+n}$, then $f \in \mathbb{B}_{p,q}(n,\alpha)$.

Remark 2.1. *Putting $q = j-1$, with $1 \leq j \leq p$, $n = 1$ and $\alpha = 0$ in Theorem 2.1 we obtain the result of Patel and Mohanty [16, Theorem 3].*

Corollary 2.6. *If $f \in \mathcal{A}(p)$ satisfies the condition (2.1) for $n = 1$, then f is a p-valent function in $\mathbb{U}$.*

Proof. If $f \in \mathcal{A}(p)$ satisfies the condition (2.1) for $n = 1$, from Theorem 2.1 it follows that $f \in \mathbb{B}_{p,q}(1,\alpha)$, that is

$$\left|\frac{f^{(q)}(z)}{z^{p-q}} - \delta(p,q)\right| < \delta(p,q)-\alpha,\ z \in \mathbb{U}. \tag{2.4}$$

Therefore, we obtain that

$$\operatorname{Re}\frac{zf^{(q)}(z)}{z^{p-q+1}} > \alpha \geq 0,\ z \in \mathbb{U}.$$

Since the function $g(z) := z^{p-q+1}$ is $(p-q+1)$-valent starlike in $\mathbb{U}$, in view of Lemma 1.2 the function f is p-valent in $\mathbb{U}$. □

Remark 2.2. *If we take $q = j-1$, with $2 \leq j \leq p$, and $\alpha = 0$ in Corollary 2.6 we obtain the result of Patel and Mohanty [16, Corollary 5].*

If we take $\alpha = \beta = 0$ and $\gamma = n = 1$ in Theorem 2.1, using from the above proof that the inequality (2.4) implies that f is a p-valent function, we deduce the next result:

Corollary 2.7. *If $f \in A(p)$ satisfies*

$$\left|\frac{f^{(1+q)}(z)}{z^{p-q-1}} - \delta(p,q+1)\right| < \delta(p,q)(p-q+1),\ z \in \mathbb{U},$$

then

$$\left|\frac{f^{(q)}(z)}{z^{p-q}} - \delta(p,q)\right| < \delta(p,q),\ z \in \mathbb{U},$$

that is f is a p-valent function in $\mathbb{U}$.

Remark 2.3. *Putting $q = p-1$, with $p \geq 2$, in Corollary 2.7 we obtain [16, Corollary 6] (see also [9, Theorem 2]):*

If $f \in \mathcal{A}(p)$ satisfies

$$\left|f^{(p)}(z) - p\,!\right| < 2(p\,!),\ z \in \mathbb{U},$$

then f is p-valent in $\mathbb{U}$.

Theorem 2.8. *If $f \in \mathcal{A}_p(n)$ satisfies $f^{(q)}(z)f^{(1+q)}(z) \neq 0$ for $0 < |z| < 1$, and*

$$\left|\alpha\left(\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - (p-q)\right) - \left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (p-q)\right)\right| < \frac{n}{1+\beta},\ z \in \mathbb{U}, \tag{2.5}$$

for some $\alpha \geq 0$ and $\beta > 0$, then

$$\operatorname{Re}\left[\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha} \frac{\delta(p,q+1)z^{p-q-1}}{f^{(1+q)}(z)}\right] > 1 - \frac{1}{\beta},\ z \in \mathbb{U}. \tag{2.6}$$

Proof. If is easy to check that the function $f_0(z) = z^p$ satisfies the inequalities (2.5) and (2.6). For $f \in \mathcal{A}_p(n)$, with $f(z) \not\equiv z^p$, let define the function ω by

$$\omega(z) = \beta\left[\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha} \frac{\delta(p,q+1)z^{p-q-1}}{f^{(1+q)}(z)} - 1\right],\ z \in \mathbb{U}. \tag{2.7}$$

Since $f \in \mathcal{A}_p(n)$ and $f^{(q)}(z)f^{(1+q)}(z) \neq 0$ for $0 < |z| < 1$, it follows that ω is analytic in $\mathbb{U}$ with $\omega(z) = \omega_n z^n + \omega_{n+1}z^{n+1} + \dots$, and $\omega(z) \not\equiv 0$. Differentiating (2.7) we have

$$\alpha\left(\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - (p-q)\right) - \left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} - (p-q)\right) = \frac{z\omega'(z)}{\beta + \omega(z)},\ z \in \mathbb{U}. \tag{2.8}$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\max\limits_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$. Then, by using Lemma 1.1 we have $z_0\omega'(z_0) = m\omega(z_0)$, with $m \geq n \geq 1$, and letting $\omega(z_0) = e^{i\theta}$, $\theta \in [0, 2\pi)$ we have

$$\begin{aligned}
&\left|\alpha\left(\frac{z_0 f^{(1+q)}(z_0)}{f^{(q)}(z_0)} - (p-q)\right) - \left(1 + \frac{z_0 f^{(2+q)}(z_0)}{f^{(1+q)}(z_0)} - (p-q)\right)\right| \\
&= \left|\frac{z_0\omega'(z_0)}{\beta + \omega(z_0)}\right| = \left|\frac{m}{\beta + e^{i\theta}}\right| = \frac{m}{\left(1 + 2\beta\cos\theta + \beta^2\right)^{1/2}} \geq \frac{n}{1+\beta},
\end{aligned}$$

which contradicts the assumption (2.5). Therefore, $|\omega(z)| < 1$, $z \in \mathbb{U}$, that is

$$\left|\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha} \frac{\delta(p,q+1)z^{p-q-1}}{f^{(1+q)}(z)} - 1\right| < \beta^{-1},\ z \in \mathbb{U},$$

which implies (2.6). □

If we take $q = 0$ in Theorem 2.8 obtain the following corollary which was obtained by Owa [12, Theorem 2] for $p = 1$:

Corollary 2.9. *If $f \in \mathcal{A}_p(n)$ satisfies $f(z)f'(z) \neq 0$ for $0 < |z| < 1$, and*

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - p\right) - \left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right| < \frac{n}{1+\beta},\ z \in \mathbb{U},$$

for some $\alpha \geq 0$ and $\beta > 0$, then

$$\operatorname{Re}\left[\left(\frac{f(z)}{z^p}\right)^{\alpha} \frac{pz^{p-1}}{f'(z)}\right] > 1 - \frac{1}{\beta},\ z \in \mathbb{U}.$$

Putting $p = 1$ and $\alpha = \beta = 2$ in Corollary 2.9 and using Lemma 1.3 we deduce the following corollary which improve the result obtained by Owa [12, Corollary 4]:

Corollary 2.10. *If $f \in \mathcal{A}_1(n)$ satisfies $f(z)f'(z) \neq 0$ for $0 < |z| < 1$, and*

$$\left| \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{2f'(z)} - 1 \right| < \frac{n}{6},\ z \in \mathbb{U},$$

then

$$\operatorname{Re}\left[\left(\frac{f(z)}{z} \right)^2 \frac{1}{f'(z)} \right] > \frac{1}{2},\ z \in \mathbb{U},$$

that is f is univalent in $\mathbb{U}$.

For $\alpha = 1$ and $\alpha = 0$, respectively, in Corollary 2.9 reduces to the next sufficient conditions of p-valent starlikeness and p-valent close-to-convexity, respectively:

Corollary 2.11. *If $f \in \mathcal{A}_p(n)$ satisfies $f(z)f'(z) \neq 0$, $0 < |z| < 1$, and*

$$\left| \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{n}{2},\ z \in \mathbb{U},$$

then f is p-valently starlike in $\mathbb{U}$.

Corollary 2.12. *If $f \in \mathcal{A}_p(n)$ satisfies $f(z)f'(z) \neq 0$, $0 < |z| < 1$, and*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| < \frac{n}{2},\ z \in \mathbb{U},$$

then f is p-valently close-to-convex in $\mathbb{U}$.

3. Some properties for the operator $\mathbf{J}_{c,p}^{q,n}$

Theorem 3.1. *If $f \in \mathbb{B}_{p,q}(n, \alpha)$, then*

$$\frac{J_{c,p}^{q,n} f^{(q)}(z)}{z^{p-q}} \prec \delta(p,q) + \frac{[\delta(p,q) - \alpha]\delta(p+n,q)(c+p-q)}{c+p+n-q} z. \tag{3.1}$$

Proof. If is clear to check that the function $f_0(z) = z^p \in \mathbb{B}_{p,q}(n, \alpha)$ and (3.1) holds. Assuming that If $f \in \mathbb{B}_{p,q}(n, \alpha)$ and $f(z) \not\equiv z^p$, let define the function ω by

$$\frac{J_{c,p}^{q,n} f^{(q)}(z)}{z^{p-q}} = \delta(p,q) + \frac{[\delta(p,q) - \alpha]\delta(p+n,q)(c+p-q)}{c+p+n-q} \omega(z),\ z \in \mathbb{U}. \tag{3.2}$$

Then, ω is analytic in $\mathbb{U}$ with $\omega(z) = \omega_n z^n + \omega_{n+1} z^{n+1} + \ldots$, and $\omega(z) \not\equiv 0$. Using (1.2) we get

$$\left(J_{c,p}^{q,n} f^{(q)}(z) \right)' = (c+p-q) \frac{f^{(q)}(z)}{z} - c \frac{J_{c,p}^{q,n} f^{(q)}(z)}{z},\ z \in \mathbb{U}, \tag{3.3}$$

and from (3.2) and (3.3) we obtain

$$\frac{f^{(q)}(z)}{z^{p-q}} - \delta(p,q) = \frac{[\delta(p,q) - \alpha]\delta(p+n,q)(c+p-q)}{c+p+n-q} \left(\omega(z) + \frac{z\omega'(z)}{c+p-q} \right),\ z \in \mathbb{U}.$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\max\limits_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$. Then, by using Lemma 1.1 we have we have $z_0 \omega'(z_0) = m\omega(z_0)$, with $m \geq n \geq 1$, hence

$$\begin{aligned} \left| \frac{f^{(q)}(z_0)}{z_0^{p-q}} - \delta(p,q) \right| &= \frac{[\delta(p,q) - \alpha]\delta(p+n,q)(c+p-q)}{c+p+n-q} \left| \omega(z_0) + \frac{z_0\omega'(z_0)}{c+p-q} \right| \\ &\geq \frac{[\delta(p,q) - \alpha]\delta(p+n,q)(c+p-q)}{c+p+n-q} \left(1 + \frac{m}{c+p-q} \right) \\ &= \frac{[\delta(p,q) - \alpha]\delta(p+n,q)(c+p-q)}{c+p+n-q} \cdot \frac{c+p-q+m}{c+p-q} \geq \delta(p,q) - \alpha, \end{aligned}$$

which contradicts that $f \in \mathbb{B}_{p,q}(n, \alpha)$. Therefore, $|\omega(z)| < 1$, $z \in \mathbb{U}$, that is (3.1) holds. $\square$

Taking $c=0$ in Theorem 3.1 we obtain the following corollary:

Corollary 3.2. *If $f\in\mathbb{B}_{p,q}(n,\alpha)$, then*

$$\frac{(p-q)\int_0^z \frac{f^{(q)}(t)}{t}dt}{z^{p-q}} \prec \delta(p,q)+\frac{[\delta(p,q)-\alpha]\delta(p+n,q)(p-q)}{p+n-q}z.$$

Putting $q=0$ in Corollary 3.2 we obtain the result of Saitoh [17, Corollary 5].

Theorem 3.3. *If $f\in\mathbb{B}_{p,q}(n,\alpha)$, and*

$$\frac{[\delta(p,q)-\alpha]\delta(p+n,q)(c+p-q)}{\delta(p,q)(c+p+n-q)}\le 1,$$

then

$$\operatorname{Re}\left(e^{i\eta}\frac{J_{c,p}^{q,n}f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)>0,\ z\in\mathbb{U},$$

where

$$|\eta|\le\frac{\pi}{2}-\sin^{-1}\left(\frac{[\delta(p,q)-\alpha]\delta(p+n,q)(c+p-q)}{\delta(p,q)(c+p+n-q)}\right).$$

The bound of $|\eta|$ is the best possible for the function given by

$$f^{(q)}(z)=\delta(p,q)z^{p-q}+[\delta(p,q)-\alpha]\delta(p+n,q)z^{p+n-q}. \tag{3.4}$$

Proof. If $f\in\mathbb{B}_{p,q}(n,\alpha)$, from Theorem 3.1 we have

$$\left|\frac{J_{c,p}^{q,n}f^{(q)}(z)}{\delta(p,q)z^{p-q}}-1\right|<\frac{[\delta(p,q)-\alpha]\delta(p+n,q)(c+p-q)}{\delta(p,q)(c+p+n-q)},\ z\in\mathbb{U},$$

and from here we deduce that

$$\operatorname{Re}\left(e^{i\eta}\frac{J_{c,p}^{q,n}f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)>0,\ z\in\mathbb{U},$$

for

$$|\eta|\le\frac{\pi}{2}-\sin^{-1}\left(\frac{[\delta(p,q)-\alpha]\delta(p+n,q)(c+p-q)}{\delta(p,q)(c+p+n-q)}\right).$$

Further, the bound of $|\eta|$ is obtained for the function $f\in\mathbb{B}_{p,q}(n,\alpha)$ defined by

$$\frac{J_{c,p}^{q,n}f^{(q)}(z)}{\delta(p,q)z^{p-q}}=1+\frac{(\delta(p,q)-\alpha)\delta(p+n,q)(c+p-q)}{\delta(p,q)(c+p+n-q)}z^n,\ z\in\mathbb{U},$$

which is equivalent to (3.4). □

Putting $c=0$ in Theorem 3.3 we obtain:

Corollary 3.4. *If $f\in\mathbb{B}_{p,q}(n,\alpha)$, and*

$$\frac{[\delta(p,q)-\alpha]\delta(p+n,q)(p-q)}{\delta(p,q)(p+n-q)}\le 1,$$

then

$$\operatorname{Re}\left[\frac{(p-q)e^{i\eta}}{\delta(p,q)z^{p-q}}\int_0^z\frac{f^{(q)}(t)}{t}dt\right]>0,\ z\in\mathbb{U},$$

where

$$|\eta|\le\frac{\pi}{2}-\sin^{-1}\left(\frac{[\delta(p,q)-\alpha]\delta(p+n,q)(p-q)}{\delta(p,q)(p+n-q)}\right).$$

The bound of $|\eta|$ is the best possible for the function given by (3.4).

Remark 3.1. *(i) Putting $q = 0$ in Theorem 2.1 and Examples 2.4 and 2.5 we obtain the results due to Sekine and Owa [19];*

(ii) Taking $q = 0$ in Corollary 2.2, Examples 2.3 and 2.4 we obtain the results of Saitoh [17, Theorem 1 and Corollaries 2 and 3];

(iii) Putting $n = 1$ and $q = j$, with $1 \leq j \leq p$, $p \in \mathbb{N}$, in Corollary 2.2, Examples 2.3 and 2.4 we get the results obtained by Owa [11, Theorem 2, Corollaries 3 and 4];

(iv) For $q = 0$, Theorem 3.1 reduces to the result of Saitoh [17, Theorem 2], and corrects the result obtained by Owa and Ma [14, Theorem 1];

(v) Taking $p = 1$ and $q = 0$ in Theorem 3.1 we obtain the result of Saitoh [17, Corollary 4], and corrects the result obtained by Owa and Hu [13, Theorem 1];

(vi) Putting $p = 1$ and $q = 0$ in Theorem 3.3 we get the result due to Saitoh [17, Corollary 6], and corrects the result of Owa and Hu [13, Theorem 2];

(vii) For $q = 0$, Corollary 3.4 reduces to the result obtained by Saitoh [17, Corollary 7].

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