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Existence and Uniqueness of Renormalized Solution for Nonlinear Parabolic Equations in Musielak Orlicz Spaces

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ABSTRACT: This paper is devoted to the study of a class of parabolic equation of type

$$\frac{\partial u}{\partial t} - \operatorname{div}\Big(A(x,t,u,\nabla u) + B(x,t,u)\Big) = f \quad \text{in} \quad Q_T,$$

where $\operatorname{div}(A(x,t,u,\nabla u)$ is a Leray-Lions type operator, $B(x,t,u)$ is a nonlinear lower order term and $f \in L^1(Q_T)$. We show the existence and the uniqueness of renormalized solution in the framework of Musielak-Orlicz spaces.

Key Words: Musielak-Orlicz space, Nonlinear parabolic problems, Renormalized solution, Existence and uniqueness, Lower order.

Contents

1. Introduction

Let Ω be a bounded open set of $I\!R^N$ $(N \geq 2)$, T is a positive real number, and $Q_T = \Omega \times (0,T)$. We consider the Dirichlet problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\Big(A(x,t,u,\nabla u) + B(x,t,u)\Big) = f & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,t=0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $A : Q_T \times I\!R \times I\!R^N \to I\!R^N$ is a Leray-Lions operator defined on the inhomogeneous Musielak-Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$, M is a Musielak-Orlicz-function related to the growth of A. $B : Q_T \times I\!R \to I\!R^N$ is a Carathéodory function satisfy only a growth condition (see (3.4)), $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$.

In the case where $M(x,t) = t^p$ (Classical Lebesgue's spaces), many works that show the existence and uniqueness result with $B(x,t,u) = B(u) \in \mathcal{C}^\infty(I\!R^N)$, the control of this term is by using Stokes formula, (see [7]) and by using Gagliardo-Nirenberg inequality type when B depend on variables x,t and u (see [11]).

In the anisotropic case $M(x,t) = t^{p(x)}$ (Lebesgue with variable exponent) we refer to ([5], [9], [10], [18]).

For more general anisotropic N-function, where the operator $A + B$ has exponential or logarithmic growth with respect to ∇u, we refer to [15] and [16].

The study of the problem in the framework of renormalized solutions is motivated by the luck of regularity of the distributional formulation. It's not strong to provide the uniqueness (for more detail see the counterexample in [19]).

For the applied motivation: we refer to Chen, Levine and Rao [9], the authors propose a framework for image restoration based on a variable exponent Laplacian, a second application is modeling the electrorheological fluids [10], [18], the constitutive equation is given by

$$u_t + \operatorname{div}(S(u)) + (u\nabla)u + \nabla\pi = f$$

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where u the velocity, π the pressure, f the external forces and $S(u) = \mu(x)(1 + |\nabla u(x)|^2)^{\frac{p(x)-2}{2}} \nabla u(x)$.

Our novelty in the present paper is to give the existence and uniqueness result of renormalized solution of (1.1) in the general framework inhomogeneous Musielak-Orlicz spaces with a lower order term B which depends on x, t and u, namely with $A(x, t, u, \nabla u)$ is replaced by $A(x, t, u, \nabla u)+B(x, t, u)$, in order to study the behavior of the approximate solutions we call upon compactness tools. The difficulties encountered during the proof of the existence and uniqueness of the solution is that the term B does not satisfy the coercivity condition and nonlinearities are characterized by N-functions $M(x, t)$, for which Δ_2-conditions not imposed, will lose the reflexivity of the space $L_M(Q_T)$ and $W_0^1 L_M(Q_T)$. In the literature, in our knowledge, there is no result of the uniqueness of the operator $A(x, t, u, \nabla u)+B(x, t, u)$ in the framework of Musielak- Orlicz spaces.

This paper is organized as follows. In section 2, we recall some well-known preliminaries, properties of inhomogeneous Musielak-Orlicz spaces. In section 3, we give the definition of a renormalized solution of problem (1.1) and the existence theorem of such a solution. Finally, in section 4, we establish the uniqueness result.

2. Inhomogeous Musielak-Orlicz space- Notation and properties

Let M be a real-valued function defined in $\Omega \times I\!R_+$ and satisfying conditions:

- $M(x, .)$ is a N-function for all $x \in \Omega$, (i.e. convex, non-decreasing, continuous, $M(x, 0) = 0$, $M(x, 0) > 0$ for $t > 0$, $\lim_{t\to 0} \sup_{x\in\Omega} \frac{M(x,t)}{t} = 0$ and $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{M(x,t)}{t} = \infty$).
- $M(., t)$ is a measurable function for all $t \geq 0$.

A function M which satisfies the above conditions is called a Musielak-Orlicz function.

Let $M_x(t) = M(x, t)$, we associate its non-negative reciprocal function M_x^{-1}, with respect to t, that is $M_x^{-1}(M(x, t)) = M(x, M_x^{-1}(t)) = t$.

Let M and P be two Musielak-Orlicz functions, we say that P grows essentially less rapidly than M at 0 (resp. near infinity), and we write $P \ll M$, for every positive constant c, we have $\lim_{t\to 0} \big(\sup_{x\in\Omega} \frac{P(x, ct)}{M(x, t)} \big) = 0$ (resp. $\lim_{t\to\infty} \big(\sup_{x\in\Omega} \frac{P(x, ct)}{M(x, t)} \big) = 0 \big)$.

Proposition 2.1. *([13]) Let $P \ll M$ near infinity and $\forall t > 0$, $\sup_{x\in\Omega} P(x, t) < \infty$, then $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that*

$$P(x, t) \leq M(x, \epsilon t) + C_\epsilon, \forall t > 0. \tag{2.1}$$

The Musielak-Orlicz space $L_M(\Omega)$ is define as

$$L_M(\Omega) = \{u : \Omega \to I\!R \quad \text{mesurable :} \quad \varrho_{M,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

where $\varrho_{M,\Omega}(u) = \int_\Omega M(x, |u(x)|)dx$, equipped with the Luxemburg norm

$$\|u\|_M = \inf \big\{\lambda > 0 : \varrho_{M,\Omega}(\frac{u}{\lambda}) \leq 1\big\}.$$

Denote $\overline{M}(x, s) = \sup_{t\geq 0}(st - M(x, s))$ the conjugate Musielak-Orlicz function of M.

We define $E_M(\Omega)$ as the subset of $L_M(\Omega)$ of all measurable functions $u : \Omega \mapsto I\!R$ such that $\varrho_{M,\Omega}(\frac{u}{\lambda}) < \infty$ for all $\lambda > 0$. It is a separable space and $(E_M(\Omega))^* = L_{\overline{M}}(\Omega)$.

We define the Musielak-Orlicz-Sobolev space as

$$W^1 L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \quad \forall |\alpha| \leq 1\},$$

endowed with the norm

$$\|u\|^1_{M,\Omega} = \inf\{\lambda > 0 : \sum_{|\alpha|\leq 1} \varrho_{M,\Omega}(\frac{D^\alpha u}{\lambda}) \leq 1\}.$$

Lemma 2.2. *([3])(Approximation theorem) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let M and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

1. *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} M(x,1) > c$,*
2. *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$, we have*
$$\frac{M(x,t)}{M(y,t)} \leq |t|^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \geq 1,$$
3. $\displaystyle\int_K M(y,\lambda)dx < \infty, \quad \forall \lambda > 0$ *and for every compact* $K \subset \Omega$,
4. *There exists a constant $C > 0$ such that $\overline{M}(y,t) \leq C$ a.e. in Ω.*

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence.

Example 2.3. *We give some example for a Musielak-Orlicz functions of approximation theorem*

- $M_1(x,t) = |t|^{p(x)}$ *with $p : \Omega \to [1,\infty)$ a measurable function with Log-Hölder continuite*
$$\frac{M_1(x,t)}{M_1(y,t)} = |t|^{p(x)-p(y)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \geq 1.$$
- $M_2(x,t) = \alpha(x)(\exp(|t|) - 1 + |t|)$, $0 < \alpha(x) \in L^\infty(\Omega)$.

Remark that $M_1 \in \triangle_2$ if $p^+ := ess\sup_{x\in\Omega} p(x) < \infty$ while $M_2 \notin \triangle_2$.

Lemma 2.4. *([1])(Modular Poincaré inequality) Under the assumptions of lemma 2.2, and by assuming that $M(x,.)$ decreases with respect to one of coordinate of x, there exists a constant $\delta > 0$ which depends only on Ω such that*
$$\int_\Omega M(x,|u|)dx \leq \int_\Omega M(x,\delta|\nabla u|)dx \quad \text{for all} \quad u \in W_0^1 L_M(\Omega). \tag{2.2}$$

Inhomogeneous Musielak-Orlicz-Sobolev spaces :

Let M be an Musielak-Orlicz function, for each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,
$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\},$$
$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : \nabla_x^\alpha u \in E_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\}.$$
The last space is a subspace of the first one, and both are Banach spaces under the norm
$$\|u\| = \sum_{|\alpha|\leq 1} \|\nabla_x^\alpha u\|_{M,Q_T}.$$

The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show as in [6], that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W_0^{1,x}E_M(Q_T)$, of some subsequence in $\mathcal{D}(Q_T)$ for the modular convergence. This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M$, and the dual space of $W_0^{1,x}E_M(Q_T)$ will be denoted by
$$W^{-1,x}L_{\overline{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M},Q_T}$.

Lemma 2.5. *[13] Let $a < b \in I\!R$ and Ω be a bounded open subset of $I\!R^N$ with the segment property, then* $\{u \in W_0^{1,x}L_M(\Omega\times(a,b)) \cap L^1(\Omega\times(a,b)) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega\times(a,b)) + L^1(\Omega\times(a,b))\} \subset \mathcal{C}([a,b], L^1(\Omega))$.

Lemma 2.6. *([12])*
Under assumptions (3.1)-(3.6), and let (z_n) be a sequence in $W_0^{1,x}L_M(Q_T)$ such that:

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$
$$(A(x, z_n, \nabla z_n))_n \quad is\ bounded\ in\ (L_{\overline{M}}(Q_T))^N,$$
$$\int_{Q_T} [A(x, z_n, \nabla z_n) - A(x, z_n, \nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dxdt \to 0$$

as n and s tend to $+\infty$, and where χ_s is the characteristic function of $Q^s = \{x \in Q_T; |\nabla z| \le s\}$. Then,

$$\nabla z_n \to \nabla z \quad a.e.\ in\ Q_T,$$
$$\lim_{n\to+\infty} \int_{Q_T} A(x, z_n, \nabla z_n)\nabla z_n dxdt = \int_{Q_T} A(x, z, \nabla z)\nabla z dxdt,$$
$$M(x, |\nabla z_n|) \to M(x, |\nabla z|) \quad in\ L^1(Q_T).$$

Finally, T_k, $k > 0$, denotes the truncation function at level k defined on $I\!R$ by

$$T_k(r) = \max(-k, \min(k, r)) \quad \text{for all} \quad r \in I\!R,$$

and $\widetilde{T}_k(s) = \int_0^s T_k(t)dt = \begin{cases} \frac{s^2}{2} & \text{if} \quad |s| \le k \\ k|s| - \frac{k^2}{2} & \text{if} \quad |s| \ge k \end{cases}$.

3. Formulation of the problem and existence of solution

Let Ω be a bounded open subset of $I\!R^N$ $(N \ge 2)$ satisfying the segment property,and let M and P be two Musielak-Orlicz functions such that M and its complementary $\overline{M}$ satisfies conditions of Lemma 2.2, assuming that M decreases with respect to one of coordinate of x and $P \ll M$.
$A : Q_T \times I\!R \times I\!R^N \to I\!R^N$ is Carathéodory function such that there exist a two strict positive constants $\alpha > 0$, $\upsilon > 0$, for a.e. $(x,t) \in Q_T$ and for all $s \in I\!R$, $\xi, \xi^* \in I\!R^N$, $\xi \neq \xi^*$,

$$|A(x, t, s, \xi)| \le \nu(a_0(x, t) + \overline{M}_x^{-1} P(x, |s|)) \quad \text{with} \quad a_0 \in E_{\overline{M}}(Q_T), \tag{3.1}$$

$$(A(x, t, s, \xi) - A(x, t, s, \xi^*))(\xi - \xi^*) > 0, \tag{3.2}$$

$$A(x, t, s, \xi).\xi \ge \alpha M(x, |\xi|). \tag{3.3}$$

$B : Q_T \times I\!R \to I\!R^N$ is a Carathéodory function such that

$$|B(x, t, s)| \le q(x, t)\overline{M}_x^{-1} M(x, \frac{\alpha_0}{\delta}|s|), \tag{3.4}$$

where $\quad 0 < \alpha_0 < 1 \quad$ and $\quad \|q(x,t)\|_{L^\infty(Q_T)} < \frac{\alpha}{\alpha_0+1}$,

$$f \in L^1(Q_T), \tag{3.5}$$

and

$$u_0 \in L^1(\Omega). \tag{3.6}$$

Following [7] and [8] we recall the definition of a renormalized solution to Problem (1.1).

Definition 3.1. *A measurable function u defined on Q_T is a renormalized solution of problem (1.1), if it satisfies the following conditions:*

$$T_k(u) \in W_0^{1,x}L_M(Q_T), \quad \forall k > 0, \tag{3.7}$$

$$\lim_{m\to+\infty} \int_{\{m\le|u|\le m+1\}} A(x, t, u, \nabla u)\nabla u dxdt = 0, \tag{3.8}$$

and if, for every function $S \in W^{2,\infty}(I\!R)$ which is piecewise C^1 and such that S' has a compact support, we have in the sense of distributions

$$\frac{\partial S(u)}{\partial t} - div\Big(S'(u)(A(x,t,u,\nabla u) + B(x,t,u))\Big) \quad (3.9)$$
$$+S''(u)\Big(A(x,t,u,\nabla u) + B(x,t,u)\Big) = fS'(u) \quad in \quad \mathcal{D}(Q_T), \quad (3.10)$$

$$S(u)(t=0) = S(u_0) \qquad in \quad \Omega. \quad (3.11)$$

Theorem 3.2. *Assume that (3.1)-(3.6) hold true. Then there exists at least one renormalized solution u of the problem (1.1) in the sense of the definition 3.1.*

Proof of the existence theorem 3.2
The proof will be divided into several steps.
Truncated problem .
For each $n > 0$, we define the following approximations:

$$A_n(x,t,s,\xi) = A(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ s \in I\!R,\ \forall\ \xi \in I\!R^N, \quad (3.12)$$

$$B_n(x,t,s) = B(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ s \in I\!R, \quad (3.13)$$

$$f_n \in \mathcal{C}^\infty(Q_T) \text{ such that } f_n \to f \text{ strongly in } L^1(Q_T), \quad (3.14)$$

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega). \quad (3.15)$$

And consider the approximate problem:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \text{div}\Big(A_n(x,t,u_n,\nabla u_n) + B_n(x,t,u_n)\Big) = f_n & in \quad Q_T, \\ u_n(x,t) = 0 & on \quad \partial\Omega \times (0,T), \\ u_n(x,t=0) = u_{0n}(x) & in \quad \Omega. \end{cases} \quad (3.16)$$

Let show that the problem (3.16) admits at least one solution. It is easy to see that the operator $A_n(x,t,u_n,\nabla u_n) + B_n(x,t,u_n)$ satisfies the assumptions (A_1), (A_2) and (A_3) (see section conditions on mapping T in J.P. Gossez and V. Mustonen [14]). It remains to shown (A_4).
Indeed, for any fixed $n > 0$, let $u_n \in W_0^{1,x}L_M(Q_T)$ and using (3.4) we get

$$|B_n(x,t,u_n)\nabla u_n| \le \|q(.,.)\|_{L^\infty(Q_T)}\Big(\overline{M}(x, \frac{1}{\epsilon}\overline{M}_x^{-1}M(x, \frac{\alpha_0}{\delta}|T_n(u_n)|)) + \epsilon M(x, |\nabla u_n|)\Big).$$

Then
$$|B_n(x,t,u_n)\nabla u_n| \le d_{n,\epsilon}(x,t) + \epsilon\|q(.,.)\|_{L^\infty(Q_T)}M(x, |\nabla u_n|)$$

where $d_{n,\epsilon} \in L^1(Q_T)$.
Finally

$$(A_n(x,t,u_n,\nabla u_n) + B_n(x,t,u_n))\nabla u_n \ge [\alpha - \epsilon\|q(.,.)\|_{L^\infty(Q_T)}]M(x, |\nabla u_n|) - d_{n,\epsilon}(x,t)$$

we can choose ϵ such that $\epsilon \le \frac{\alpha}{2\|q(.,.)\|_{L^\infty(Q_T)}}$, we obtain

$$(A_n(x,t,u_n,\nabla u_n) + B_n(x,t,n))\nabla u_n \ge \frac{\alpha}{2}M(x, |\nabla u_n|) - d_{n,\epsilon}(x,t).$$

Then the operator $(A_n(x,t,u_n,\nabla u_n) + B_n(x,t,u_n))$ satisfies the coercivity condition and we have the conditions to apply the Proposition 5 of [14] and there exists at least one solution $u_n \in W_0^{1,x}L_M(Q_T)$ of (3.16).

Remark 3.3. *the explicit dependence in x and t of the functions A and B will be omitted so that $A(x,t,u,\nabla u) = A(u,\nabla u)$ and $B(x,t,u) = B(u)$.*

Step 1: A priori estimates.

Lemma 3.4.
Let $\{u_n\}_n$ be a solution of the approximate problem (3.16), then for all $k>0$, there exists a constant C such that

$$\int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \leq kC, \tag{3.17}$$

$$u_n \to u \quad a.e \text{ in } \quad Q_T, \tag{3.18}$$

$$A_n(T_k(u_n),\nabla T_k(u_n)) \quad \text{is bounded in} (L_{\overline{M}}(Q_T))^N. \tag{3.19}$$

Proof. Fixed $k>0$ and $\tau\in(0,T)$. Let $T_k(u_n)\chi_{(0,\tau)}$ as a test function in problem (3.16) and using the Young Inequality we get

$$\int_\Omega \widetilde{T}_k(u_n(\tau))dx + \int_{Q_\tau} A_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt + \int_{Q_\tau} B_n(u_n)\nabla T_k(u_n)dxdt$$
$$= \int_{Q_\tau} f_n T_k(u_n)dxdt + \int_\Omega \widetilde{T}_k(u_{0n})dx. \tag{3.20}$$

By definition of $\widetilde{T}_k$, we deduce $\int_\Omega \widetilde{T}_k(u_n(\tau))dx \geq 0$ and $\int_\Omega \widetilde{T}_k(u_{0n})dx \leq k\|u_0)\|_{L^1(\Omega)}$.
and by (3.4) and Young Inequality we have

$$\int_{Q_\tau} B_n(u_n)\nabla T_k(u_n)dxdt \leq \|q(.,.)\|_{L^\infty(Q_T)}\Big[\alpha_0\int_{Q_\tau} M(x,\frac{|T_k(u_n)|}{\delta})dxdt$$
$$+\int_{Q_\tau} M(x,|\nabla T_k(u_n)|)dxdt\Big],$$

thanks to Lemma 2.4, we obtain

$$\int_{Q_\tau} B_n(u_n)\nabla T_k(u_n)dxdt \leq \|q(.,.)\|_{L^\infty(Q_T)}(\alpha_0+1)\int_{Q_\tau} M(x,|\nabla T_k(u_n)|)dxdt.$$

Returning to (3.20) and using (3.3) we get

$$\int_{Q_\tau} A_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq \|q(.,.)\|_{L^\infty(Q_T)}\frac{(\alpha_0+1)}{\alpha}\int_{Q_\tau} A_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt$$
$$+k\Big[\|f_n\|_{L^1(Q_T)}+\|u_0\|_{L^1(Q_T)}\Big],$$

thus

$$\Big[1-\frac{(\alpha_0+1)}{\alpha}\|q(.,.)\|_{L^\infty(Q_T)}\Big]\int_{Q_T} A_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1.$$

Taking $\frac{1}{c_2}=\Big[1-\frac{(\alpha_0+1)}{\alpha}\|q(.,.)\|_{L^\infty(Q_T)}\Big]>0$, from (3.4), we obtain

$$\int_{Q_\tau} A(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kC,$$

where $C=c_1c_2$. So by (3.3) we get (3.17).
Hence $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$ independently of n and for any $k>0$, so there exists a subsequence still denoted by u_n such that $T_k(u_n)\rightharpoonup \xi_k$ weakly in $W_0^{1,x}L_M(Q_T)$.

On the other hand, using Lemma 2.4 and (3.17), we have

$$\inf_{x\in\Omega} M(x,\frac{k}{\delta})meas\{|u_n|>k\} \leq \int_{|u_n|>k} M(x,\frac{|T_k(u_n)|}{\delta})dxdt$$
$$\leq \int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \leq kC.$$

Then

$$meas\{|u_n| > k\} \leq \frac{kC_2}{\inf_{x\in\Omega} M(x, \frac{k}{\delta})},$$

for all n and for all k.

Assuming that there exists a positive function ψ such that $\lim_{t\to\infty} \frac{\psi(t)}{t} = +\infty$ and $\psi(t) \leq ess \inf_{x\in\Omega} M(x,t)$, $\forall t \geq 0$. Thus, we get

$$\lim_{k\to\infty} meas\{|u_n| > k\} = 0. \tag{3.21}$$

For every $\lambda > 0$, we have

$$\begin{aligned} meas\{|u_n - u_m| > \lambda\} &\leq meas\{|u_n| > k\} + meas\{|u_m| > k\} \\ &\quad + meas\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \end{aligned} \tag{3.22}$$

We can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Q_T. Let $\epsilon > 0$, then by (3.21) and (3.22) there exists $k(\epsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} \leq \epsilon \quad \text{for all} \quad n, m > h(k(\epsilon), \lambda).$$

This proves that $(u_n)_n$ is a Cauchy sequence in measure in Q_T, thus it converges almost everywhere to some measurable function u. Then $T_k(u_n) \to T_k(u)$ weakly in $W_0^{1,x} L_M(Q_T)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$, strongly in $E_M(Q_T)$ and a.e. in Q_T.

Proof of (3.19) : The same way in [2], we deduce that $A_n(x,t,T_k(u_n), \nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q_T))^N$ and we obtain (3.19).

□

Step 2:Almost everywhere convergence of the gradients.

To have that the gradient converges almost everywhere, we need to prove this proposition

Proposition 3.5. *Let $\{u_n\}_n$ be a solution of the approximate problem 3.16, then*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} A(u_n, \nabla u_n)\nabla u_n dxdt = 0, \tag{3.23}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} B(u_n)\nabla u_n dxdt = 0, \tag{3.24}$$

and

$$\nabla u_n \to \nabla u \quad a.e. \ in \quad Q_T. \tag{3.25}$$

Proof.

Taking the function

$$Z_m(u_n) = T_1(u_n - T_m(u_n))$$

then

$$\nabla Z_m(u_n) = \nabla u_n \chi_{\{m\leq|u_n|\leq m+1\}}.$$

Multiplying the approximating equation (3.16) by the test function $Z_m(u_n)$ and using the same argument in step 2, we get

$$\int_{\{m\leq|u_n|\leq m+1\}} A_n(u_n, \nabla u_n)\nabla u_n dxdt \leq C\Big[\int_{Q_T} f_n Z_m(u_n)\, dxdt + \int_{\{|u_{0n}|>m\}} |u_{0n}| dxdt\Big],$$

where $\frac{1}{C} = \Big[1 - \frac{(\alpha_0+1)}{\alpha}\|q(.,.)\|_{L^\infty(Q_T)}\Big] > 0$.

Passing to limit as $n \to +\infty$, since the pointwise convergence of u_n and strongly convergence in $L^1(Q_T)$ of f_n we get

$$\lim_{n\to+\infty} \int_{\{m\leq|u_n|\leq m+1\}} A_n(u_n, \nabla u_n)\nabla u_n dxdt \leq C\Big[\int_{Q_T} f Z_m(u)\, dxdt + \int_{\{|u_0|>m\}} |u_0| dxdt\Big].$$

By using Lebesgue's Theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get (3.23).From (3.3), we also deduce

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} M(x,|\nabla Z_m(u_n)|)dxdt = 0 \tag{3.26}$$

On the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} B_n(u_n)\nabla Z_m(u_n)dxdt \le \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt,$$

$$+\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} \overline{M}(x,|B_n(u_n)|)dxdt.$$

Using the pointwise convergence of u_n and by Lebegue's theorem, in the second term of the right side, we get

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} \overline{M}(x,|B_n(u_n)|)dxdt = \int_{\{m\le|u|\le m+1\}} \overline{M}(x,|B(u)|)dxdt,$$

and also, by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} \overline{M}(x,|B(u)|)dxdt = 0, \tag{3.27}$$

Thus with (3.26) and (3.27), we get the (3.24).

Now let $\upsilon_j \in \mathcal{D}(Q_T)$ be a sequence such that $\upsilon_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence. This specific time regularization of $T_k(\upsilon_j)$ (for fixed $k \ge 0$) is defined as follows.

Let $(\alpha_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_0^\mu \in L^\infty(\Omega)\cap W_0^1L_M(\Omega) \quad \text{for all} \quad \mu > 0, \tag{3.28}$$

$$\|\alpha_0^\mu\|_{L^\infty(\Omega)} \le k, \text{ for all } \quad \mu > 0,$$

and α_0^μ converges to $T_k(u_0)$ a.e. in Ω and $\frac{1}{\mu}\|\alpha_0^\mu\|_{M,\Omega}$ converges to 0 as $\mu \to +\infty$.
For $k \ge 0$ and $\mu > 0$, let us consider the unique solution $(T_k(\upsilon_j))_\mu \in L^\infty(Q)\cap W_0^{1,x}L_M(Q)$ of the monotone problem:

$$\frac{\partial (T_k(\upsilon_j))_\mu}{\partial t} + \mu((T_k(\upsilon_j))_\mu - T_k(\upsilon_j)) = 0 \text{ in } D'(\Omega),$$

$$(T_k(\upsilon_j))_\mu(t=0) = \alpha_0^\mu \text{ in } \Omega.$$

Remark that due to

$$\frac{\partial (T_k(\upsilon_j))_\mu}{\partial t} \in W_0^{1,x}L_M(Q_T).$$

We just recall that,

$$(T_k(\upsilon_j))_\mu \to T_k(u) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly} - * \quad \text{in} \quad L^\infty(Q_T)$$

$$(T_k(\upsilon_j))_\mu \to (T_k(u))_\mu \quad \text{in} \quad W_0^{1,x}L_M(Q_T),$$

for the modular convergence as $j \to +\infty$. Also,

$$(T_k(u))_\mu \to T_k(u) \quad \text{in} \quad W_0^{1,x}L_M(Q_T),$$

for the modular convergence as $\mu \to +\infty$ and

$$||(T_k(\upsilon_j))_\mu||_{L^\infty(Q_T)} \le max(||(T_k(u))||_{L^\infty(Q_T)}, \quad ||\alpha_0^\mu||_{L^\infty(\Omega)}) \le k, \text{ for all } \mu > 0, \text{ and } k > 0.$$

We introduce a sequence of increasing $\mathbf{C}^1(I\!R)$-functions S_m such that

$$S_m(r) = 1 \text{ for } |r| \le m,\ \ S_m(r) = m+1-|r|, \text{ for } m \le |r| \le m+1,\ \ S_m(r) = 0 \text{ for } |r| \ge m+1$$

for any $m \ge 1$. And we denote by $\epsilon(n,\mu,\eta,j,m)$ all quantities (possibly different) such that

$$\lim_{m\to+\infty}\lim_{j\to+\infty}\lim_{\eta\to+\infty}\lim_{\mu\to+\infty}\lim_{n\to+\infty}\epsilon(n,\mu,\eta,j,m) = 0.$$

For fixed $k \ge 0$, let $W^{n,j}_{\mu,\eta} = (T_\eta(T_k(u_n) - T_k(\upsilon_j)_\mu))^+$ and $W^{j}_{\mu,\eta} = (T_\eta(T_k(u) - T_k(\upsilon_j)_\mu))^+$. Multiplying the approximating equation by $\exp(G(u_n)))W^{n,j}_{\mu,\eta}S_m(u_n)$, we obtain:

$$\begin{cases}\displaystyle\int_{Q_T} < \frac{\partial u_n}{\partial t}\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt + \int_{Q_T} a_n(u_n,\nabla u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})S_m(u_n)\,dx\,dt\\ \displaystyle+\int_{Q_T} a_n(u_n,\nabla u_n)\nabla u_n\exp(G(u_n))W^{n,j}_{\mu,\eta}S'_m(u_n)\,dx\,dt - \int_{Q_T} B_n(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})S_m(u_n)\,dx\,dt\\ \displaystyle-\int_{Q_T} B_n(u_n)\nabla u_n\exp(G(u_n))W^{n,j}_{\mu,\eta}S'_m(u_n)\,dx\,dt \le \int_{Q_T} f_n\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt.\end{cases} \tag{3.29}$$

Now we pass to the limit in (3.29) for k real number fixed. In order to perform this task we prove below the following results for any fixed $k \ge 0$:

$$\int_{Q_T}\frac{\partial u_n}{\partial t}\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt \ge \epsilon(n,\mu,\eta,j) \qquad \text{for any } m \ge 1, \tag{3.30}$$

$$\int_{Q_T} B_n(u_n)S_m(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})\,dx\,dt = \epsilon(n,j,\mu) \quad \text{for any } m \ge 1, \tag{3.31}$$

$$\int_{Q_T} B_n(u_n)\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt = \epsilon(n,j,\mu) \qquad \text{for any } m \ge 1, \tag{3.32}$$

$$\int_{Q_T} a_n(u_n,\nabla u_n)\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt \le \epsilon(n,m), \tag{3.33}$$

$$\int_{Q_T} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})\,dx\,dt \le C\eta + \epsilon(n,j,\mu,m), \tag{3.34}$$

$$\int_{Q_T} f_n S_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt \le \epsilon(n,\eta), \tag{3.35}$$

$$\int_{Q_T}\Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]dx\,dt \to 0. \tag{3.36}$$

Proof of (3.30):

Lemma 3.6.

$$\int_{Q_T}\frac{\partial u_n}{\partial t}\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)dxdt \ge \epsilon(n,\mu,\eta,\eta,j) \quad m \ge 1. \tag{3.37}$$

Proof. Is a particular case of the proof in [7], with $b(x,u) = u$. □

Proof of (3.31): If we take $n > m+1$, we get

$$B_n(u_n)\exp(G(u_n))S_m(u_n) = B(T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))S_m(T_{m+1}(u_n)),$$

then $B_n(u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_{\overline{M}}(Q_M)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain

$$B_n(u_n)\exp(G(u_n))S_m(u_n) \to B(u)\exp(G(u))S_m(u),$$

with the modular convergence as $n \to +\infty$, then $B_n(u_n)\exp(G(u_n))S_m(u_n) \to B(u)\exp(G(u))S_m(u)$ for $\sigma(\prod L_{\overline{M}}, \prod L_M)$. In the other hand $\nabla W^{n,j}_{\mu,\eta} = \nabla T_k(u_n) - \nabla(T_k(\upsilon_j))_\mu$ for $|T_k(u_n) - (T_k(\upsilon_j))_\mu| \le \eta$ converge to $\nabla T_k(u) - \nabla(T_k(\upsilon_j))_\mu$ weakly in $(L_M(Q_T))^N$, then

$$\int_{Q_T} B_n(u_n)\exp(G(u_n))S_m(u_n)\nabla W^{n,j}_{\mu,\eta}\,dx\,dt \to \int_{Q_T} B(u)S_m(u)\exp(G(u))\nabla W^{j}_{\mu,\eta}\,dx\,dt,$$

as $n \to +\infty$.

By using the modular convergence of $W^j_{\mu,\eta}$ as $j \to +\infty$ and letting μ tends to infinity, we get (3.31).

Proof of (3.32):

For $n > m+1 > k$, we have $\nabla u_n S'_m(u_n) = \nabla T_{m+1}(u_n)$, a.e. in Q_T. By the almost every where convergence of u_n we have $\exp(G(u_n))W^{n,j}_{\mu,\eta} \to \exp(G(u))W^{j}_{\mu,\eta}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(B_n(T_{m+1}(u_n)))_n$ converge strongly in $E_{\overline{M}}(Q_T)$, then

$$B_n(T_{m+1}(u_n))\exp(G(u_n))\; W^{n,j}_{\mu,\eta} \to B(T_{m+1}(u))\;\exp(G(u))W^{j}_{\mu,\eta},$$

converge strongly in $E_{\overline{M}}(Q_T)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_M(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{m\le|u_n|\le m+1} B_n(T_{m+1}(u_n))\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$\to \int_{m\le|u|\le m+1} B(u)\nabla u\exp(G(u))W^{j}_{\mu,\eta}\,dx\,dt$$

as $n \to +\infty$ with the modular convergence of $W^j_{\mu,\eta}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get 3.32.

Proof of (3.33):

We have

$$\begin{aligned}&\int_{Q_T} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n\exp(G(u_n))\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt\\ &= \int_{m\le|u_n|\le m+1} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt\\ &\le \eta C\int_{m\le|u_n|\le m+1} a_n(u_n,\nabla u_n)\nabla u_n\,dx\,dt.\end{aligned}$$

Using (3.23), we get

$$\int_{Q_T} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,ds \le \epsilon(n,\mu,m).$$

Proof of (3.35):

Since $S_m(r) \le 1$ and $W^{n,j}_{\mu,\eta} \le \eta$ we get

$$\int_{Q_T} f_n S_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt \le \epsilon(n,\eta).$$

Proof of (3.34):

$$\int_{Q_T} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$\begin{aligned}&= \int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)_\mu)\,dx\,dt\\ &\quad - \int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j)_\mu\,dx\,dt. \qquad (3.38)\end{aligned}$$

Since $a_n(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, there exist some $\varpi_{k+\eta} \in (L_{\overline{M}}(Q_T))^N$ such that $a_n(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n)) \to \varpi_{k+\eta}$ weakly in $(L_{\overline{M}}(Q_T))^N$. Consequently,

$$\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j)_\mu)\le\eta\}} a_n(u_n, \nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla T_k(v_j)_\mu \, dx\, dt$$

$$= \int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(v_j)_\mu \varpi_{k+\eta} \, dx\, dt + \epsilon(n), \tag{3.39}$$

where we have used the fact that

$$S_m(u_n) \exp(G(u_n)) \nabla T_k(v_j)_\mu) \chi_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j)_\mu)\le\eta\}}$$

$$\to S_m(u) \exp(G(u)) \nabla T_k(v_j)_\mu) \chi_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j)_\mu)\le\eta\}},$$

strongly in $(E_M(Q_T))^N$.

Letting $j \to +\infty$, we obtain

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(v_j)_\mu \varpi_{k+\eta} \, dx\, dt$$

$$= \int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(u)_\mu \varpi_{k+\eta} \, dx\, dt + \epsilon(n, j).$$

One easily has,

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(u)_\mu \varpi_{k+\eta} \, dx\, dt = \epsilon(n, j, \mu).$$

By (3.29)-(3.35), (3.38) and (3.39) we obtain

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j)_\mu)|\le\eta\}} a_n(T_k(u_n), \nabla T_k(u_n)) S_m(u_n) \exp(G(u_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)_\mu) \, dx\, dt$$

$$\le C\eta + \epsilon(n, j, \mu, m),$$

we know that $\exp(G(u_n)) \ge 1$ and $S_m(u_n) = 1$ for $|u_n| \le k$ then

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j)_\mu)|\le\eta\}} a_n(T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)_\mu) \, dx\, dt \le C\eta + \epsilon(n, j, \mu, m). \tag{3.40}$$

Proof of (3.36):
Setting for $s > 0$, $Q^s = \{(x,t) \in Q_T : |\nabla T_k(u)| \le s\}$ and $Q^s_j = \{(x,t) \in Q_T : |\nabla T_k(v_j)| \le s\}$ and denoting by χ^s and χ^s_j the characteristic functions of Q^s and Q^s_j respectively. Let $0 < \delta < 1$, and define

$$\Theta_{n,k} = (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u)).$$

For $s > 0$, we have

$$0 \le \int_{Q^s} \Theta^\delta_{n,k} \, dx\, dt = \int_{Q^s} \Theta^\delta_{n,k} \chi_{|T_k(u_n)-T_k(v_j)_\mu|\le\eta)} \, dx\, dt + \int_{Q^s} \Theta^\delta_{n,k} \chi_{|T_k(u_n)-T_k(v_j)_\mu|>\eta)} \, dx\, dt.$$

With the Hölder inequality, the first and the second term of the right-side hand can written as

$$\int_{Q^s} \Theta^\delta_{n,k} \chi_{|T_k(u_n)-T_k(v_j)_\mu|\le\eta)} \, dx\, dt \le (\int_{Q^s} \Theta_{n,k} \chi_{|T_k(u_n)-T_k(v_j)_\mu|\le\eta)} \, dx\, dt)^\delta (\int_{Q^s} dx\, dt)^{1-\delta}$$

$$\leq C_1(\int_{Q^s} \Theta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dx\,dt)^\delta.$$

and

$$\int_{Q^s} \Theta^\delta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta)}\,dx\,dt \leq (\int_{Q^s} \Theta_{n,k}\,dx\,dt)^\delta(\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta)} dx\,dt)^{1-\delta}.$$

Since $a(T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, while $\nabla T_k(u_n)$ is bounded in $(L_M(Q_T))^N$ we have

$$\int_{Q^s} \Theta^\delta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta)}\,dx\,dt \leq C_2 meas\{(x,t)\in Q_T : |T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}^{1-\delta}.$$

We obtain,

$$\int_{Q^s} \Theta^\delta_{n,k}\,dx\,dt \leq C_1(\int_{Q^s} \Theta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dx\,dt)^\delta$$
$$+C_2 meas\{(x,t)\in Q_T : |T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}^{1-\delta}.$$

On the other hand,

$$\int_{Q^s} \Theta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dx\,dt$$
$$\leq \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi_s))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx\,dt.$$

For each $s>r, r>0,$ one has

$$0 \leq \int_{Q^r\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt$$
$$\leq \int_{Q^s\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt$$
$$= \int_{Q^s\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi_s))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx\,dt$$
$$\leq \int_{Q\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(u)\chi^s)\,dx\,dt$$
$$= \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(\upsilon_j)\chi^s_j))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi^s_j)\,dx\,dt$$
$$+\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi^s_j - \nabla T_k(u)\chi^s)\,dx\,dt$$
$$+\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} (a(T_k(u_n),\nabla T_k(\upsilon_j)\chi^s_j) - a(T_k(u_n),\nabla T_k(u)\chi^s))\nabla T_k(u_n)\,dx\,dt$$
$$-\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(T_k(u_n),\nabla T_k(\upsilon_j)\chi^s_j)\nabla T_k(\upsilon_j)\chi^s_j)\,dx\,dt$$

$$+\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta} a(T_k(u_n),\nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s)\,dx\,dt$$
$$= I_1(n,j,s)+I_2(n,j)+I_3(n,j)+I_4(n,j,\mu)+I_5(n,\mu).$$

We go to the limit as n, j, μ, and $s\to+\infty$

$$\begin{aligned} I_1 =& \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt \\ &-\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt \\ &-\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta} a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s))\,dx\,dt. \end{aligned}$$

Using (3.40), the first term of the right-hand side, we get

$$\begin{aligned} &\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt \\ &\le C\eta+\epsilon(n,m,j,s)-\int_{|u|>k\cap|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta} a(T_k(u),0)\nabla T_k(\upsilon_j)_\mu\,dx\,dt \\ &\le C\eta+\epsilon(n,m,j,\mu). \end{aligned}$$

The second term of the right-hand side tends to

$$\int_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt,$$

since $a(T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, there exist some $\varpi_k\in(L_{\overline{M}}(Q_T))^N$ such that (for a subsequence still denoted by u_n

$$a(T_k(u_n),\nabla T_k(u_n))\to\varpi_k \quad\text{in}\quad (L_M(Q_T))^N \quad\text{for}\quad \sigma(\Pi L_{\overline{M}},\Pi E_M).$$

In view of the fact that

$$(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta}\to(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta},$$

strongly in $(E_M(Q_T))^N$ as $n\to+\infty$.
The third term of the right-hand side tends to

$$\int_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta} a(T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u)-\nabla T_k(\upsilon_j)\chi_j^s))\,dx\,dt.$$

Since

$$a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta}\to a(T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta},$$

in $(E_{\overline{M}}(Q_T))^N$ while

$$(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s))\to(\nabla T_k(u)-\nabla T_k(\upsilon_j)\chi_j^s)),$$

in $(L_M(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$.

Passing to limit as $j\to+\infty$ and $\mu\to+\infty$ and using Lebesgue's theorem, we have

$$I_1\le C\eta+\epsilon(n,j,s,\mu).$$

For what concerns I_2, by letting $n\to+\infty$, we have

$$I_2\to\int_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta)} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\,dx\,dt.$$

Since $a(T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k$ in $(L_{\overline{M}}(Q_T))^N$, for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$,
and

$$(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} \to (\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\leq\eta},$$

strongly in $(E_M(Q_T))^N$.

Passing to limit $j \to +\infty$, and using Lebesgue's theorem, we have

$$I_2 = \epsilon(n, j).$$

Similar ways as above give

$$I_3 = \epsilon(n, j).$$

$$I_4 = \int_{|T_k(u)-T_k(u)_\mu|\leq\eta)} a(T_k(u), \nabla T_k(u))\nabla T_k(u)\, dx\, dt + \epsilon(n, j, \mu, s, m).$$

$$I_5 = \int_{|T_k(u)-T_k(u)_\mu|\leq\eta)} a(T_k(u), \nabla T_k(u))\nabla T_k(u)\, dx\, dt + \epsilon(n, j, \mu, s, m).$$

Finally, we obtain,

$$\int_{Q^s} \Theta_{n,k}\, dx\, dt \leq C_1(C\eta + \epsilon(n, \mu, \eta, m))^\delta + C_2(\epsilon(n, \mu,))^{1-\delta}.$$

Which yields, by passing to the limit sup over n, j, μ , s and η

$$\int_{\{T_\eta(T_k(u_n)-T_k(\upsilon_j))\geq 0\}\cap Q^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))$$
$$(\nabla T_k(u_n) - \nabla T_k(u))dxdt = \epsilon(n), \tag{3.41}$$

Taking on the hand the function $W_\eta^{n,j} = T_\eta(T_k(u_n) - T_k(\upsilon_j))^-$ and $W_\eta^j = T_\eta(T_k(u) - T_k(\upsilon_j))^-$.

Multiplying the approximating equation by $\exp(G(u_n))W_\eta^{n,j} S_m(u_n)$, we obtain

$$\int_{\{T_\eta(T_k(u_n)-T_k(\upsilon_j))\leq 0\}\cap Q^r} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))$$
$$\times (\nabla T_k(u_n) - \nabla T_k(u))dxdt = \epsilon(n), \tag{3.42}$$

by (3.41) and (3.42) we get

$$\int_{Q^r} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))dxdt = \epsilon(n)$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Q^r, and since r is arbitrary,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q_T.$$

Step 3: We show that u satisfies the Definition 3.1

For this, let show that (3.8) holds. We have for any $m > 0$,

$$\begin{aligned}\int_{\{m\leq|u_n|\leq m+1\}} A(u_n, \nabla u_n)\nabla u_n dxdt &= \int_{Q_T} A(u_n, \nabla u_n)[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)]dxdt \\ &= \int_{Q_T} A(T_{m+1}(u_n), \nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)dxdt \\ &- \int_{Q_T} A(T_m(u_n), \nabla T_m(u_n))\nabla T_m(u_n)dxdt.\end{aligned}$$

According to Lemma 2.6, we pass to the limit as n tends to $+\infty$ for fixed $m > 0$ and we obtain

$$\begin{aligned}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} A(u_n,\nabla u_n)\nabla u_n dxdt &= \int_{Q_T} A(T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)dxdt\\ &\quad-\int_{Q_T} A(T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt\\ &=\int_{\{m\le|u|\le m+1\}} A(u,\nabla u)\nabla u dxdt,\end{aligned}$$

with (3.23), we obtain easily (3.8).
Note that, similarly we deduce

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} B(u)\nabla u dxdt = 0. \tag{3.43}$$

Let $S\in W^{2,\infty}(I\!R)$ which is piecewise C^1 and such that S' has a compact support, Let $K>0$ such that $supp(S')\subset[-K,K]$. Pointwise multiplication of the approximate problem (3.16) by $S'(u_n)$, we get

$$\begin{cases}\dfrac{\partial S(b(u_n))}{\partial t}+\operatorname{div}\Big(S'(u_n)(A(u_n,\nabla u_n)-B(u_n))\Big)\\ +S''(u_n)\Big(A(u_n,\nabla u_n)-B(u_n)\Big)\nabla u_n = fS'(u_n).\end{cases} \tag{3.44}$$

Now we will pass to the limit as $n\to+\infty$ of each term of (3.44),

Limit of $\frac{\partial S(u_n)}{\partial t}$: since S is bounded, and $S(u_n)$ converges to $S(u)$ a.e. in Q_T and weakly in $L^\infty(Q_T)$, then $\frac{\partial S(u_n)}{\partial t}$ converges to $\frac{\partial S(u)}{\partial t}$ in $\mathcal{D}'(Q_T)$.

Limit of $S'(u_n)A(u_n,\nabla u_n)$: since $supp(S')\subset[-K,K]$ we have

$$S'(u_n)A(u_n,\nabla u_n)=S'(u_n)A(T_k(u_n),\nabla T_k(u_n)) \text{ a.e. in } Q_T.$$

The pointwise convergence of u_n to u, the bounded character of S', and by Lemma 2.6 and Proposition 3.5, we conclude $A(T_k(u_n),\nabla T_k(u_n))$ converges to $A(T_k(u_n),\nabla T_k(u_n))$ weakly in $(L_{\overline{M}}(Q_T))^N$ allows us to obtain $S'(u_n)A(T_k(u_n),\nabla T_k(u_n))$ converges to $S'(u)A(T_k(u),\nabla T_k(u))$ weakly for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, and $S'(u)A(T_k(u),\nabla T_k(u)=S'(u)A(u,\nabla u)$ a.e. in Q_T.

Limit of $S''(u_n)A(u_n,\nabla u_n)\nabla u_n$: since $supp(S')\subset[-K,K]$, we get

$$S''(u_n)A(u_n,\nabla u_n)\nabla u_n=S''(u_n)A(T_k(u_n),\nabla T_k(u_n))\nabla u_n \text{ a.e. in } Q_T.$$

The pointwise convergence of $S''(u_n)$ to $S''(u)$ as n tends to $+\infty$, the bounded character of S'' and by Lemma 2.6 and Proposition 3.5, we conclude

$$S''(u_n)A(T_k(u_n),\nabla T_k(u_n))\nabla u_n \rightharpoonup S''(u)A(T_k(u),\nabla T_k(u))\nabla u \text{ weakly in } L^1(Q_T)$$

as $n\to+\infty$, and

$$S''(u)A(T_k(u),\nabla T_k(u))\nabla u=S''(u)A(u,\nabla u)\nabla u \text{ a.e. in } Q_T.$$

Limit of $S'(u_n)B(u_n)$: Since $supp(S')\subset[-K,K]$ we have

$$S'(u_n)B(u_n)=S'(u_n)B(T_k(u_n)) \text{ a.e. in } Q_T.$$

In a similar way, we obtain

$$S'(u_n)B(u_n)\to S'(u)B(u) \text{ weakly for } \sigma(\Pi L_{\overline{M}},\Pi E_M).$$

Limit of $S''(u_n)B(u_n)\nabla u_n$: Also we have

$$S''(u_n)B(u_n)\nabla u_n = S''(u_n)B(T_k(u_n))\nabla T_k(u_n).$$

Using the weakly convergence of truncation, it is possible to prove that,

$$S''(u_n)B(u_n)\nabla u_n \to S''(u)B(u)\nabla u \text{ strongly in } L^1(Q_T).$$

Limit of $f_n S'(u_n)$: we have $u_n \to u$ a.e. in Q_T, S' is piecewise C^1. It is enough to use (3.14) to get that $f_n S'(u_n) \to f S'(u)$ strongly in $L^1(Q_T)$.

Finally, to show (3.11), remark that S being bounded, $S(u_n)$ is bounded in $L^\infty(Q_T)$. the equation (3.44) allows to show that $\frac{\partial S(u_n)}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$. By Lemma 2.5 implies that $S(u_n)$ lies in a compact set of $C^0([0,T]; L^\infty(\Omega))$. It follows that, on one hand, $S(u_n(t=0))$ converges to $S(u(t=0))$ strongly in $L^1(Q_T)$. On the other hand, the smoothness of S imply that $S(u(t=0) = S(u_0)$ in Ω. This complete the existence result. □

4. Uniqueness result

Before showing the uniqueness of the solution of the problem (1.1), we will give the following technical lemma.
Let u and v be two renormalized solutions of the problem (1.1) and let us define for any $0 < k < s$,

$$\Gamma(u,v,s,k) = \int_{\{s-k<|u|<s+k\}} \Big(A(u,\nabla u)\nabla u + |B(u)||\nabla u| \Big) dxdt + \int_{\{s-k<|v|<s+k\}} \Big(A(v,\nabla v)\nabla v + |B(v)||\nabla v| \Big) dxdt. \tag{4.1}$$

Lemma 4.1. *Assume that (3.1)-(3.6) hold, then*

$$\liminf_{s\to+\infty} \limsup_{k\to 0} \frac{1}{k}\Gamma(u,v,s,k) = 0. \tag{4.2}$$

Proof.
Define the two functions,

$$L_1(s) = \int_{\{0<u<s\}} \Big(A(u,\nabla u)\nabla u + |B(u)||\nabla u| \Big) dxdt + \int_{\{0<v<s\}} \Big(A(v,\nabla v)\nabla v + |B(v)||\nabla v| \Big) dxdt, \tag{4.3}$$

and

$$L_2(s) = \int_{\{-s<u<0\}} \Big(A(u,\nabla u)\nabla u + |B(u)||\nabla u| \Big) dxdt + \int_{\{-s<v<0\}} \Big(A(v,\nabla v)\nabla v + |B(v)||\nabla v| \Big) dxdt. \tag{4.4}$$

Due to (3.2) the function L_1 and L_2 are monotone increasing. L_1 and L_2 are derivable almost everywhere see [17], with L_1' and L_2' measurable and that we have for any $s > \eta > 0$

$$L_1(s) - L_1(\eta) \geq \int_\eta^s L_1'(\xi)d\xi \quad \text{and} \quad L_2(s) - L_2(\eta) \geq \int_\eta^s L_2'(\xi)d\xi, \tag{4.5}$$

and for almost any $s > 0$

$$L_1'(s) = \frac{1}{2}\limsup_{k\to 0} \frac{1}{k}\Big[\int_{\{s-k<u<s+k\}} \Big(A(u,\nabla u)\nabla u + |B(u)||\nabla u| \Big) dxdt$$

$$+\int_{\{s-k<v<s+k\}}\Big(A(v,\nabla v)\nabla v+|B(v)||\nabla v|\Big)dxdt\Big], \tag{4.6}$$

and

$$L_2'(s)=\frac{1}{2}\limsup_{k\to 0}\frac{1}{k}\Big[\int_{\{-s-k<u<-s+k\}}\Big(A(u,\nabla u)\nabla u+|B(u)||\nabla u|\Big)dxdt$$

$$+\int_{\{-s-k<v<-s+k\}}\Big(A(v,\nabla v)\nabla v+|B(v)||\nabla v|\Big)dxdt\Big]. \tag{4.7}$$

If the thesis of the lemma is not true, let $\epsilon_0>0$ and let $n_0>0$ be a real number such that for every real number $s\geq n_0$ we have

$$\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k)\geq\epsilon_0. \tag{4.8}$$

On the other hand, we have for almost $s\geq n_0$,

$$\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k)=2(L_1'(s)+L_2'(s)),$$

then, from (4.6), (4.7) and (4.8) it follows that $L_1'(\xi)+L_2'(\xi)\geq\frac{\epsilon_0}{2}$.
In view of (4.5), we deduce that for any $s>\eta>n_0$ we have

$$L_1(s)-L_1(\eta)+L_2(s)-L_2(\eta)\geq\frac{\epsilon_0}{2}(s-\eta). \tag{4.9}$$

Taking $s=n+1$ and $\eta=n$ with $n>n_0$ we have

$$\int_{\{n\leq|u|\leq n+1\}}\Big(A(u,\nabla u)\nabla u+|B(u)||\nabla u|\Big)dxdt$$

$$+\int_{\{n\leq|v|\leq n+1\}}\Big(A(v,\nabla v)\nabla v+|B(v)||\nabla v|\Big)dxdt\geq\frac{\epsilon_0}{2}.$$

The last inequality contradicts (3.8) and (3.43). □

Theorem 4.2. *Assume that assumptions (3.1)-(3.6) hold true and moreover that for any compact set $D\subset I\!R$, there exists $L_D\in E_{\overline{M}}(Q_T)$ and $\rho_D>0$ such that $\forall s,\overline{s}\in D$,*

$$|A(x,t,s,\xi)-A(x,t,\overline{s},\xi)|\leq\Big(L_D(x,t)+\rho_D\overline{P}^{-1}P(|\xi|)\Big)|s-\overline{s}|, \tag{4.10}$$

$$|B(x,t,s)-B(x,t,\overline{s})|\leq L_D(x,t)|s-\overline{s}|, \tag{4.11}$$

for almost every $(x,t)\in Q_T$ and for every $\xi\in I\!R^N$. Then the problem (1.1) has a unique renormalized solution.

Proof. Let define a smooth approximation of T_n by $\widetilde{S}_n^\sigma$ such that for all $n>0$ and $\sigma>0$, we have $\widetilde{S}_n^\sigma(0)=0$ and

$$(\widetilde{S}_n^\sigma)'(r)=\begin{cases}0 & \text{for}\quad |r|\geq n+\sigma,\\ \frac{n+\sigma-|r|}{\sigma} & \text{for}\quad n\leq|r|\leq n+\sigma,\\ 1 & \text{for}\quad |r|\leq n.\end{cases} \tag{4.12}$$

For a fixed $n>0$, we have

$$\lim_{\sigma\to 0}(\widetilde{S}_n^\sigma)'(z)=\chi_{|z|\leq n}\quad\text{a.e. in}\quad Q_T. \tag{4.13}$$

and

$$\lim_{\sigma\to 0}\widetilde{S}_n^\sigma(z)=T_n(z)\quad\text{a.e. in}\quad Q_T. \tag{4.14}$$

Consider now two renormalized solutions u and v of (3.7)- (3.11) for the data f and u_0. Since $\widetilde{S}_n^\sigma \in W^{2,\infty}(I\!R)$ and $supp(\widetilde{S}_n^\sigma)' \subset [-n-\sigma, n+\sigma]$, then we take $S = \widetilde{S}_n^\sigma$ and we use $\frac{1}{k}T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(u))$ as a test function in the difference of equation (3.9) for u and v , we get

$$\frac{1}{k}\int_0^T\int_0^t < \frac{\partial\Big(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)\Big)}{\partial t}; T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)\Big) > dsdt$$
$$+ I_{1,n}^\sigma + I_{2,n}^\sigma + I_{3,n}^\sigma + I_{4,n}^\sigma = I_{5,n}^\sigma, \tag{4.15}$$

where

$$I_{1,n}^\sigma = \frac{1}{k}\int_0^T\int_0^t\int_\Omega [(\widetilde{S}_n^\sigma)'(u)A(u,\nabla u) - (\widetilde{S}_n^\sigma)'(v)A(v,\nabla v)]\nabla T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))dxdsdt,$$
$$I_{2,n}^\sigma = \frac{1}{k}\int_0^T\int_0^t\int_\Omega [(\widetilde{S}_n^\sigma)''(u)A(u,\nabla u)\nabla u - (\widetilde{S}_n^\sigma)''(v)A(v,\nabla v)\nabla v]T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))dxdsdt,$$
$$I_{3,n}^\sigma = \frac{1}{k}\int_0^T\int_0^t\int_\Omega [(\widetilde{S}_n^\sigma)'(u)B(u) - (\widetilde{S}_n^\sigma)'(v)B(v)]\nabla T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))dxdsdt,$$
$$I_{4,n}^\sigma = \frac{1}{k}\int_0^T\int_0^t\int_\Omega [(\widetilde{S}_n^\sigma)''(u)B(u)\nabla u - (\widetilde{S}_n^\sigma)''(v)B(v)\nabla v]T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))dxdsdt,$$
$$I_{5,n}^\sigma = \frac{1}{k}\int_0^T\int_0^t\int_\Omega f[(\widetilde{S}_n^\sigma)'(u) - (\widetilde{S}_n^\sigma)'(v)]T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))dxdsdt.$$

for any $k > 0$, $n > 0$, $\sigma > 0$.

Firstly we give this lemma.

Lemma 4.3.

$$\lim_{n\to+\infty}\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_0^T\int_0^t < \frac{\partial\Big(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)\Big)}{\partial t}; T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)) > dsdt = \int_{Q_T}|u-v|dxdt. \tag{4.16}$$

Proof.
Remark that $\widetilde{S}_n^\sigma(u)(t=0) = \widetilde{S}_n^\sigma(v)(t=0) = \widetilde{S}_n^\sigma(u_0)$ a.e. in Ω, then

$$\int_0^t < \frac{\partial\Big(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)\Big)}{\partial t}; T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)\Big) > dsdt = \int_\Omega \widetilde{T}_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))(t)dx,$$

and

$$\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_0^T\int_0^t < \frac{\partial\Big(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)\Big)}{\partial t}; T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v)\Big) > dsdt$$
$$= \lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T}\widetilde{T}_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))dxdt$$
$$= \int_{Q_T}|T_n(u) - T_n(v)|dxdt,$$

where $\widetilde{T}_k(r) = \displaystyle\int_0^r T_k(z)dz$.
We pass to the limit as $n \to +\infty$ in the last equality and we deduce (4.16). □

Secondly we will proof the limit of $I_{1,n}^\sigma, I_{2,n}^\sigma, I_{3,n}^\sigma, I_{4,n}^\sigma, I_{5,n}^\sigma$ respectively.
The limit of $I_{1,n}^\sigma$***:***
Let define

$$I_{1,n}^\sigma = \frac{1}{k}\int_0^T\int_0^t\int_\Omega Q_n^\sigma dxdsdt = \frac{1}{k}\int_{Q_T}(T-t)Q_n^\sigma dxdt$$

where $Q_n^\sigma = [(\widetilde{S}_n^\sigma)'(u)A(u,\nabla u) - (\widetilde{S}_n^\sigma)'(v)A(v,\nabla v)]\nabla T_k(\widetilde{S}_n^\sigma(u) - \widetilde{S}_n^\sigma(v))$.

Since $supp((\widetilde{S}_n^\sigma)') \subset [-n-\sigma, n+\sigma]$, we get

$$(\widetilde{S}_n^\sigma)'(u)A(u,\nabla u) = (\widetilde{S}_n^\sigma)'(u)A(T_{n+1}(u), \nabla T_{n+1}(u))$$

and

$$(\widetilde{S}_n^\sigma)'(v)A(v,\nabla v) = (\widetilde{S}_n^\sigma)'(u)A(T_{n+1}(v), \nabla T_{n+1}(v)).$$

Then by (4.12), (4.13) and (4.14), we have

$$\begin{cases} Q_n^\sigma \quad \text{converges to} \quad [\chi_{|u|\le n}A(u,\nabla u) - \chi_{|v|\le n}A(v,\nabla v)]\nabla T_k(T_n(u) - T_n(v)), \\[2mm] \qquad |Q_n^\sigma| \le \Big[|A(T_{n+1}(u), \nabla T_{n+1}(u))| + |A(T_{n+1}(v), \nabla T_{n+1}(v))|\Big] \\ \qquad\qquad \times(|\nabla T_{n+1}(u)| + |\nabla T_{n+1}(v)|)\chi_{|T_n(u)-T_n(v)|\le k} = R_n. \end{cases}$$

Since $R_n \in L^1(Q_T)$ we use the Lebesgue's Dominated convergence Theorem to have

$$\begin{aligned} \lim_{\sigma\to 0} I_{1,n}^\sigma &= \lim_{\sigma\to 0} \frac{1}{k}\int_{Q_T} (T-t)Q_n^\sigma dxdt \\ &= \frac{1}{k}\int_{Q_T} (T-t)[\chi_{|u|\le n}A(u,\nabla u) - \chi_{|v|\le n}A(v,\nabla v)]\nabla T_k(T_n(u) - T_n(v))dxdt \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} J_1 &= \frac{1}{k}\int_{\{|u-v|\le k, |u|\le n, |v|\le n\}} (T-t)\Big(A(u,\nabla u) - A(u,\nabla v)\Big)(\nabla u - \nabla v)dxdt, \\ J_2 &= \frac{1}{k}\int_{\{|u-v|\le k, |u|\le n, |v|\le n\}} (T-t)\Big(A(u,\nabla v) - A(v,\nabla v)\Big)(\nabla u - \nabla v)dxdt, \\ J_3 &= \frac{1}{k}\int_{\{|T_n(u)-T_n(v)|\le k, |u|> n, |v|\le n\}} (T-t)A(v,\nabla v)\nabla v dxdt, \\ J_4 &= \frac{1}{k}\int_{\{|T_n(u)-T_n(v)|\le k, |u|\le n, |v|> n\}} (T-t)A(u,\nabla u)\nabla u dxdt. \end{aligned}$$

Since $A(u,\nabla u)$ check the condition (3.3), one can have immediately

$$J_1 \ge 0. \tag{4.18}$$

On the other hand by (4.10) we have

$$\begin{aligned} |J_2| &\le \frac{T}{k}\int_{Q_T} \chi_{\{|u-v|\le k\}}|u-v|\Big(L_D(x,t) + \rho_D\overline{P}^{-1}P(|v|)\Big)(|\nabla u| + |\nabla v|)dxdt \\ &\le T\int_{\{|u-v|\le k\}} \Big(L_D(x,t) + \rho_D\overline{P}^{-1}P(|v|)\Big)(|\nabla u| + |\nabla v|)dxdt. \end{aligned}$$

Since $L_D(x,t) \in E_{\overline{M}}(Q_T)$, u and v in $W^{1,x}L_M(Q_T)$ and using (2.1), one can have

$$(L_D(x,t) + \rho_D\overline{P}^{-1}P(|v|))(|\nabla u| + |\nabla v|) \in L^1(Q_T)$$

and the Lebesgue Dominated Convergence Theorem allows us to conclude that for all $n \ge 1$

$$\limsup_{k\to 0} J_2 = 0. \tag{4.19}$$

In view of the definition of T_n, we have

$$J_3 = \frac{1}{k}\int_{\substack{\{n-k\le v\le n\}\\ \cup\{-n\le v\le -n+k\}}} (T-t)A(v,\nabla v)\nabla v dxdt,$$

and using (3.3) we deduce

$$\liminf_{n\to+\infty}\limsup_{k\to 0} J_3 \ge 0. \tag{4.20}$$

Similarly we have

$$J_4 = \frac{1}{k}\int_{\substack{\{n-k\le u\le n\}\\ \cup\{-n\le u\le -n+k\}}} (T-t)A(u,\nabla u)\nabla u dxdt,$$

and

$$\liminf_{n\to+\infty}\limsup_{k\to 0} J_4 \ge 0. \tag{4.21}$$

Now from (4.17)-(4.21) we obtain

$$\liminf_{n\to+\infty}\limsup_{k\to 0}\lim_{\sigma\to 0} I_{1,n}^{\sigma} \ge 0. \tag{4.22}$$

The limit of $I_{2,n}^{\sigma}$ ***and*** $I_{4,n}^{\sigma}$ ***:***
Now we claim that

$$|I_{2,n}^{\sigma}| + |I_{4,n}^{\sigma}| \le \frac{T}{\sigma}\Gamma(u,v,n,\sigma), \tag{4.23}$$

A simple derivation of the function $(\widetilde{S}_n^{\sigma})'$ one have for any $\sigma>0$ and $k>0$

$$\begin{aligned} |I_{2,n}^{\sigma}| &\le \frac{T}{\sigma}\int_{\substack{\{n-\sigma\le u\le n\}\\ \cup\{-n\le u\le -n+\sigma\}}} A(u,\nabla u))\nabla u dxdt \\ &+ \frac{T}{\sigma}\int_{\substack{\{n-\sigma\le v\le n\}\\ \cup\{-n\le v\le -n+\sigma\}}} A(v,\nabla v))\nabla v dxdt, \end{aligned} \tag{4.24}$$

Similarly we have

$$\begin{aligned} |I_{4,n}^{\sigma}| &\le \frac{T}{\sigma}\int_{\substack{\{n-\sigma\le u\le n\}\\ \cup\{-n\le u\le -n+\sigma\}}} B(u)\nabla u dxdt \\ &+ \frac{T}{\sigma}\int_{\substack{\{n-\sigma\le v\le n\}\\ \cup\{-n\le v\le -n+\sigma\}}} B(v)\nabla v dxdt. \end{aligned} \tag{4.25}$$

Combine (4.24) and (4.25) we deduce (4.23).
The limit of $I_{3,n}^{\sigma}$***:***
Let prove that

$$\limsup_{\sigma\to 0} |I_{3,n}^{\sigma}| \le \frac{T}{k}\Gamma(u,v,n,k) + \epsilon(k), \tag{4.26}$$

where $\epsilon(k)$ is a positive function such that $\lim_{k\to 0}\epsilon(k) = 0$.
For $n \ge 0$ we have

$$\begin{aligned} \limsup_{\sigma\to 0} |I_{3,n}^{\sigma}| &= |\frac{1}{k}\int_{Q_T} (T-t)(\chi_{\{|u|\le n\}}B(u) - \chi_{\{|v|\le n\}}B(v))\nabla T_k(T_n(u)-T_n(v))dxdt|. \\ &\le K_1 + K_2 + K_3, \end{aligned}$$

where

$$K_1 = \frac{T}{k}\int_{Q_T} \chi_{\{|u|\le n,|v|>n\}}|B(u)||\nabla T_k(T_n(u) - nsgn(v)|dxdt,$$

$$K_2 = \frac{T}{k}\int_{Q_T} \chi_{\{|u|>n,|v|\le n\}}|B(v)||\nabla T_k(T_n(v) - nsgn(u)|dxdt,$$

$$K_3 = \frac{T}{k}\int_{Q_T} \chi_{\{|u|\le n,|v|\le n\}}|B(u) - B(v)||\nabla T_k(T_n(u) - T_n(v))|dxdt.$$

We estimate K_1 and K_2 by (3.4) we have

$$K_1 \le \frac{T}{k}\int_{Q_T} \chi_{\{|u|\le n,|v|>n\}}\chi_{\{|u-nsgn(v)|\le k\}}|B(u)||\nabla u|dxdt$$

$$\le \frac{T}{k}\int_{\substack{\{n-k\le u\le n\}\\ \cup\{-n\le u\le -n+k\}}} |B(u)||\nabla u|dxdt, \tag{4.27}$$

and similarly

$$K_2 \le \frac{T}{k}\int_{\substack{\{n-k\le v\le n\}\\ \cup\{-n\le v\le -n+k\}}} |B(v)||\nabla v|dxdt. \tag{4.28}$$

On the other hand, by (4.11) one have since $L_D \in L_{\overline{M}}(Q_T)$,

$$\begin{aligned} K_3 &\le \frac{T}{k}\int_{\{|T_n(u)-T_n(v)|\le k\}\cap\{|u|\le n,|v|\le n\}} L_D(x,t)|u-v||\nabla T_k(T_n(u) - T_n(v))|dxdt \\ &= \frac{T}{k}\int_{\{|T_n(u)-T_n(v)|\le k\}\cap\{|u|\le n,|v|\le n\}} L_D(x,t)|T_n(u) - T_n(v)||\nabla T_k(T_n(u) - T_n(v))|dxdt \\ &\le T\int_{\{|T_n(u)-T_n(v)|\le k\}\cap\{|u|\le n,|v|\le n\}} L_D(x,t)(|\nabla T_n(u)| + |\nabla T_n(v)|)dxdt = \epsilon(k). \end{aligned}$$

Since L_D in $L_{\overline{M}}(Q_T)$ and due to (3.7), the function $L_D(x,t)(|\nabla T_n(u)| + |\nabla T_n(v)|) \in L^1(Q_T)$. Using the Lebesgue's Dominated Convergence Theorem we obtain $\lim_{k\to 0}\epsilon(k) = 0$ and

$$\lim_{k\to 0}|K_3| = 0. \tag{4.29}$$

Estimates (4.27)-(4.29) imply (4.26).

The limit of $I^{\sigma}_{5,n}$:

Using the Lebesgue's Theorem and (4.13) and (4.14), it is possible to have

$$\lim_{\sigma\to 0}|I^{\sigma}_{5,n}| \le \frac{T}{k}\int_{Q_T} |T_k(T_n(u) - T_n(v))| \times |f||\chi_{\{|u|\le n\}} - \chi_{\{|v|\le n\}}|dxdt.$$

Since $\lim_{k\to 0}\frac{T_k(z)}{k} = sign(z)$ in $I\!R$ and weakly-* in L^∞ then

$$\lim_{k\to 0}\lim_{n\to +\infty}\lim_{\sigma\to 0}|I^{\sigma}_{5,n}| \le \lim_{n\to +\infty}(\int_{\{|u|\ge n\}}|f|dxdt + \int_{\{|v|\ge n\}}|f|dxdt) = 0.$$

Then

$$\lim_{k\to 0}\liminf_{n\to +\infty}\lim_{\sigma\to 0} I^{\sigma}_{5,n} = 0. \tag{4.30}$$

Finally, let's go back to (4.15) and using Lemma (4.1), one have collected all the data to show that $u = v$ a.e. in Q_T. □

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