Three Methods to Solve Two Classes of Integral Equations of the Second Kind

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ABSTRACT: Three methods to solve two classes of integral equations of the second kind are introduced in this paper. Firstly, two Kantorovich methods are proposed and examined to numerically solving an integral equation appearing from mathematical modeling in biology. We use a sequence of orthogonal finite rank projections. The first method is based on general grid projections. The second one is established by using the shifted Legendre polynomials. We present a new convergence analysis results and we prove the associated theorems. Secondly, a new Nyström method is introduced for solving Fredholm integral equation of the second kind.

Key Words: Nyström method, Kantorovich method, Integral equations, Orthogonal finite rank projections.

Contents

1. Introduction

Over the last decades, lots of important problems in applied mathematics, science, physics, engineering , biology, electrodynamics, mechanics, economics and other different fields of computer, science and engineering are written and modeled in the form of integral equations. However, there are many obstacles to directly solve these equations . Thus, we should solve these equations by using numerical methods. Recently, several numerical results have been developed for solving integral equations. Among the most approximation schemes, Kantorovich method is the most efficient.

More recently, Mennouni established an improved convergence analysis via Kulkarni method (cf. [7]) to approximate the solution of integro-differential equation in $L^2([-1, 1], \mathbb{C})$ by using the Legendre polynomials. In [8], the author introduced an efficient Galerkin method for a class of Cauchy singular integral equations of the second kind with constant coefficients in $L^2([0, 1], \mathbb{C})$, the author used a sequence of orthogonal finite rank projections. The aim of [6] is to applied the Kulkarni method and a Galerkin method for solving second kind noncompact bounded operator equations. Moreover, the author used a sequence of orthogonal finite rank projections to approximate the solution of singular integral equations of the second kind with Cauchy kernel. The main idea of [10] is to propose a collocation method for solving singular integro-differential equations with logarithmic kernel using airfoil polynomials. The goal of [9] is to numerically solve the Cauchy integro-differential equations using the projection method based on the Legendre polynomials.

In this paper, we introduce three methods to solve two classes of integral equations of the second kind. The first main idea of this work is to extend and improve the results of previous works via two Kantorovich methods for solving an integral equation arising from a problem in mathematical biology. The second one is to develop new Nyström method to solve a Fredholm integral equation of the second kind. In the first Kantorovich method we use a general grid projections. In the second one we exploit the shifted Legendre polynomials in our approach. The convergence analysis and new results are presented in this work.

2. Two Kantorovich methods for solving an integral equation appearing from mathematical modeling in biology

2.1. Integral equation appearing from mathematical modeling in biology

Let us consider the following integral equation of the second kind

$$x(s) \int_0^1 k(s-t)dt = \int_0^1 x(\tau)k(\tau-s)d\tau + g(s), \quad 0 \le s \le 1, \tag{2.1}$$

where $k(\cdot, \cdot)$ is a Fredholm kernel, and g is a known function.

Equation (2.1) reads as

$$x(s) - \int_0^1 \frac{x(\tau)k(\tau-s)d\tau}{\int_0^1 k(s-t)dt} = f(s), \quad 0 \le s \le 1, \tag{2.2}$$

where

$$f(s) := \frac{g(s)}{\int_0^1 k(s-t)dt}.$$

Define the integral operator T:

$$Tx(s) := \int_0^1 \frac{x(\tau)k(\tau-s)d\tau}{\int_0^1 k(s-t)dt}, \quad 0 \le s \le 1.$$

Set $\mathcal{H} := L^2[0,1]$. Suppose that $k \in \mathrm{L}^1[0,1]$, $k > 0$ almost everywhere. We recall that for each $f \in \mathcal{H}$, T is compact from $\mathcal{H}$ into itself, (see [4]). Hence, the integral equation (2.1) has a unique solution $x \in \mathcal{H}$.

Let I denote the identity operator on $\mathcal{H}$. Eq. (2.2) can be rewritten in operator form as follows:

$$(I - T) x = f.$$

The purpose of this work is to approximate x through the solution x_n of the Kantorovich equation

$$(I - \pi_n T) x_n = f. \tag{2.3}$$

2.2. Projection approximations using general grids

Let $(s_{n,j})_{j=0}^n$ be a grid on $[0,1]$ such that

$$0 < s_{n,0} < s_{n,1} < \ldots < s_{n,n} < 1.$$

Set

$$h_{n,i} := s_{n,i} - s_{n,i-1}, \quad i \in [\![1,n]\!], \quad h_n := (h_{n,1}, h_{n,2}, \ldots, h_{n,n}).$$

Let us consider $(\pi_n)_{n \ge 1}$, a sequence of bounded projections each one of finite rank, such that

$$\pi_n x := \sum_{j=1}^n \langle x, e_{n,j} \rangle e_{n,j},$$

where

$$e_{n,j} := \frac{\phi_{n,j}}{\sqrt{h_{n,j}}}, \quad \phi_{n,j}(s) := \begin{cases} 1 \text{ for } s \in]s_{n,j-1}, s_{n,j}[, \\ 0 \text{ otherwise.} \end{cases}$$

Let

$$J_n := \{s_{n,j}, \quad j \in [\![0, n]\!]\}.$$

Define the modulus of continuity of the function $\psi \in \mathcal{H}$ relative to h_n as follows:

$$\omega_2(\psi, J_n) := \sup_{0 \le \delta \le h_n} \left(\int_0^1 |\psi(\tau + \delta) - \psi(\tau)|^2 \, d\tau \right)^{\frac{1}{2}}.$$

All functions are extended by 0 outside $[0, 1]$. We recall that

$$\omega_2(\psi, J_n) \to 0 \text{ as } n \to \infty \text{ for all } \psi \in \mathcal{H},$$

and that, for all $\psi \in \mathcal{H}$ (cf. [2]),

$$\|(I - \pi_n)\psi\| \le \omega_2(\psi, J_n). \tag{2.4}$$

2.3. First Kantorovich method via general grids

We have

$$\pi_n T x := \sum_{j=1}^n \langle Tx, e_{n,j} \rangle e_{n,j}.$$

Applying T to both sides of equation (2.3), and performing the inner product with $e_{n,i}$ to both sides of this equation, we get

$$\langle Tx_n, e_{n,i} \rangle - \sum_{j=1}^n \langle Tx_n, e_{n,j} \rangle \langle Te_{n,j}, e_{n,i} \rangle = \langle Tf, e_{n,i} \rangle,$$

or, equivalently,

$$(I_n - A_n)X_n = b_n, \tag{2.5}$$

where

$$X_n(j) := \langle Tx_n, e_{n,j} \rangle,$$

and

$$A_n(i, j) := \langle Te_{n,j}, e_{n,i} \rangle,$$

$$b_n(i) := \langle Tf, e_{n,i} \rangle.$$

Hence

$$A_n(i, j) := \frac{1}{\sqrt{h_{n,j} h_{n,i}}} \int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_j} \frac{k(\tau - s)}{\int_0^1 k(s-t)dt} d\tau ds,$$

$$b_n(i) := \frac{1}{\sqrt{h_{n,i}}} \int_{s_{i-1}}^{s_i} \int_0^1 \frac{f(\tau)k(\tau - s)}{\int_0^1 k(s-t)dt} d\tau ds.$$

2.4. Convergence analysis of the first Kantorovich method

For all $x \in \mathcal{H}$,

$$\lim_{n \to \infty} \|\pi_n T x - T x\| = 0,$$

and since T is compact,

$$\lim_{n \to \infty} \|(\pi_n T - T) T\| = 0, \quad \lim_{n \to \infty} \|(\pi_n T - T) \pi_n T\| = 0.$$

Theorem 2.1. *There exists a positive constant M, such that*

$$\|x_n - x\| \leq M \left[\omega_2(x, J_n) + \omega_2(f, J_n)\right].$$

Proof. In fact

$$\pi_n x = \pi_n T x + \pi_n f.$$

Since

$$\begin{aligned}
x - \pi_n x &= x - x_n + x_n - \pi_n x \\
&= x - x_n + (\pi_n T x_n + f) - (\pi_n T x + \pi_n f) \\
&= x - x_n + \pi_n T (x_n - x) + (I - \pi_n) f \\
&= (I - \pi_n T)(x - x_n) + (I - \pi_n) f.
\end{aligned}$$

Hence

$$x - x_n = (I - \pi_n T)^{-1} \left[(I - \pi_n) x - (I - \pi_n) f\right],$$

and since T is compact, the

$$M := \sup_{n \geq N} \left\|(I - \pi_n T)^{-1}\right\|,$$

is finite. Using (2.4), we get the desired result. □

2.5. Second Kantorovich method via shifted Legendre polynomials

The aim of this section is to use Kantorovich method for solving (2.1) via shifted Legendre polynomials. For this purpose, let $(L_n)_{n \geq 0}$ denote the sequence of Legendre polynomials. The Shifted Legendre Polynomial $\widetilde{L}_n(s)$ is defined as

$$\widetilde{L}_n(s) := L_n(2s - 1).$$

Let us consider

$$\widetilde{e}_{n,j} := \sqrt{2j + 1}\, \widetilde{L}_j,$$

the corresponding normalized sequence. Let $(\pi_n)_{n \geq 0}$ be the sequence of bounded finite rank orthogonal projections defined by

$$\pi_n x := \sum_{j=0}^{n-1} \langle x, \widetilde{e}_{n,j} \rangle \widetilde{e}_{n,j}.$$

Hence, for $y \in \mathcal{H}$,

$$\lim_{n \to \infty} \|\pi_n y - y\| = 0.$$

We recall that (cf. [3]) there exists $C > 0$ such that, for all $y \in H^r([0,1], \mathbb{C})$,

$$\|(I - \pi_n)y\| \leq C n^{-r} \|y\|_r. \tag{2.6}$$

It follows from (2.3) that

$$(I_n - \widetilde{A}_n)\widetilde{X}_n = \widetilde{b}_n,$$

where

$$\widetilde{A}_n(i,j) \quad := \quad \sqrt{2j+1}\sqrt{2i+1}\int_0^1\int_0^1 \frac{\widetilde{L}_j(\tau)k(\tau-s)d\tau}{\int_0^1 k(s-t)dt}\widetilde{L}_i(s)dtds,$$

$$\widetilde{b}_n(i) \quad := \quad \sqrt{2i+1}\int_0^1\int_0^1 \frac{f(\tau)k(\tau-s)d\tau}{\int_0^1 k(s-t)dt}\widetilde{L}_i(s)dtds.$$

Once the above system is solved, x_n is recovered as

$$x_n = \sum_{j=0}^{n-1}\widetilde{X}_n(j)\sqrt{2j+1}\widetilde{L}_j + f.$$

2.6. Convergence analysis of second Kantorovich method

Theorem 2.2. *Assume that $f \in H^r([0,1],\mathbb{C})$ for some $r > 0$. Then, there exists $\alpha > 0$ such that*

$$\frac{\|x_n - x\|}{\|x\|_r} \le \alpha\|T\|n^{-r}.$$

Proof. We have

$$\begin{aligned} x_n - x &= \left[(I - \pi_n T)^{-1}f - (I - T)^{-1}f\right] \\ &= (I - \pi_n T)^{-1}\left[(\pi_n - I)Tx\right], \end{aligned}$$

and hence

$$\|x_n - x\| \le MC_0 n^{-r}\|Tx\|_r, \quad \text{for some positive constant} \quad C_0,$$

so that

$$\frac{\|x_n - x\|}{\|x\|_r} \le \alpha\|T\|n^{-r}, \quad \alpha := MC_0.$$

$\square$

Theorem 2.3. *Assume that $f \in H^r([0,1],\mathbb{C})$ for some $r > 0$. Then, there exists $\beta > 0$ such that*

$$\|x_n - x\| \le \beta n^{-r}\left[\|x\|_r + \|f\|_r\right]$$

Proof. Recall that

$$\pi_n x = \pi_n Tx + \pi_n f.$$

As above and as in [1], we have

$$x - \pi_n x \quad = \quad (I - \pi_n T)(x_n - x) + (I - \pi_n)f.$$

Hence

$$x_n - x = (I - \pi_n T)^{-1}\left[(I - \pi_n)x + (\pi_n - I)f\right],$$

using (2.6), we get

$$\|x_n - x\| \le Mn^{-r}\left[C_1\|x\|_r + C_2\|f\|_r\right], \quad \text{for some positive constants} \quad C_1, C_2,$$

as we wanted to prove.

$\square$

3. A new Nyström method for solving Fredholm integral equation

3.1. Nyström method

Let us consider the following Fredholm integral equation of of the second kind

$$u(s) - \int_{-1}^{1} k(s,t)u(t)dt = f(s), \quad s \in I := [-1,1], \tag{3.1}$$

We recall the quadrature formula introduced in [11] as follows

$$\int_{-1}^{1} h(s)ds \approx \overline{\omega}_c^{(+)}h(\tau_c) + \sum_{\nu=1}^{n} \overline{\omega}_\nu h(\tau_\nu) + \overline{\omega}_c^{(-)}h(-\tau_c),$$

where

$$\tau_\nu = \cos\theta_\nu, \qquad \theta_\nu = \frac{\nu}{n+1}\pi, \qquad \nu = 1,2,\ldots,n$$

$$\pm\tau_c = \pm\cos\theta_c, \qquad \theta_c = \frac{\pi}{2(n+1)}.$$

In this section, we introduce a new Nyström method for solving the integral equation (3.1)

$$\breve{u}(s) - \left[\overline{\omega}_c^{(+)}k(s,\tau_c)\breve{u}(\tau_c) + \sum_{\nu=1}^{n}\overline{\omega}_\nu k(s,\tau_\nu)\breve{u}(\tau_\nu) + \overline{\omega}_c^{(-)}k(s,-\tau_c)\breve{u}(-\tau_c)\right] = f(s). \tag{3.2}$$

Collocating (3.2) at τ_i pour $i = 1,\ldots,n$, we get the following linear system

$$\breve{u}(\tau_i) - \left[\overline{\omega}_c^{(+)}k(\tau_i,\tau_c)\breve{u}(\tau_c) + \sum_{\nu=1}^{n}\overline{\omega}_\nu k(\tau_i,\tau_\nu)\breve{u}(\tau_\nu) + \overline{\omega}_c^{(-)}k(\tau_i,-\tau_c)\breve{u}(-\tau_c)\right] = f(\tau_i),$$

that is

$$\breve{u}(\tau_i) - \left[\sum_{\nu=0}^{n+1}\overline{\omega}_\nu k(\tau_i,\tau_\nu)\breve{u}(\tau_\nu)\right] = f(\tau_i), \tag{3.3}$$

with

$$\overline{\omega}_0 = \overline{\omega}_c^{(+)}, \qquad \tau_0 = \tau_c,$$

and

$$\overline{\omega}_{n+1} = \overline{\omega}_c^{(-)}, \qquad \tau_{n+1} = -\tau_c,$$

which is a system of $n+2$ linear equations with the unknown

$$\breve{u}_n := [\breve{u}(\tau_0), \breve{u}(\tau_1), \ldots, \breve{u}(\tau_{n+1})],$$

$$\breve{f}_n := [f(\tau_0), f(\tau_1), \ldots, f(\tau_{n+1})],$$

and

$$\breve{A}_n := \breve{A}_n(i,\nu) = \overline{\omega}_\nu k(\tau_i,\tau_\nu).$$

Hence the following linear system

$$(I - \breve{A}_n)\breve{u}_n = \breve{f}_n.$$

3.2. Explicit linear system

The corresponding explicit formulae for the weights of formula is presented in [11] as follows

$$\omega_\nu = \frac{2}{n+1}\left\{1 - 2\sum_{k=1}^{[(n-1)/2)]} \frac{\cos 2k\theta_\nu}{4k^2 - 1} - \frac{\cos 2[(n+1)/2]\theta_\nu}{2[(n+1)/2] - 1}\right\},$$

hence

$$\omega_\nu = \frac{2}{n+1}\left\{1 - 2\sum_{k=1}^{[(n+1)/2)]} \frac{\cos 2k\theta_\nu}{4k^2 - 1} - \frac{\cos 2[(n+1)/2]\theta_\nu}{2[(n+1)/2] + 1}\right\}, \qquad \nu = 1, 2, \ldots, n,$$

where

$$\omega_\nu = \frac{4\sin\theta_\nu}{n+1} \sum_{k=1}^{[(n+1)/2)]} \frac{\sin(2k-1)\theta_\nu}{2k-1}, \qquad \nu = 1, 2, \cdots, n.$$

Moreover, the author of [11] proved that the weights $\overline{\omega}_\nu$, $\overline{\omega}_c^{(+)}$ and $\overline{\omega}_c^{(-)}$ are given by

$$\overline{\omega}_\nu = \omega_\nu + \frac{2\sin^2\theta_\nu \cos 2[(n+1)/2]\theta_\nu}{(2[n/2]+1)(2[(n+1)/2]+1)\sin(\theta_\nu + \theta_c)\sin(\theta_\nu - \theta_c)},$$

with $\nu = 1, 2, \ldots, n$,

$$\overline{\omega}_c^+ = \overline{\omega}_c^- = \begin{cases} \dfrac{\sin\theta_c}{n+1}, & \text{if } n \text{ is even,} \\ \dfrac{(n+1)\tan\theta_c}{n(n+2)}, & \text{if } n \text{ is odd.} \end{cases}$$

Equation (3.3) reads as

$$\breve{u}(\tau_i) - \left[\sum_{\nu=1}^n \left[\omega_\nu + \frac{2\sin^2\theta_\nu\cos 2[(n+1)/2]\theta_\nu}{(2[n/2]+1)(2[(n+1)/2]+1)\sin(\theta_\nu + \theta_c)\sin(\theta_\nu - \theta_c)}\right] k(\tau_i, \tau_\nu)\breve{u}(\tau_\nu)\right]$$
$$- \frac{\sin\theta_c}{n+1}\left[k(\tau_i, \tau_c)\breve{u}(\tau_c) + k(\tau_i, -\tau_c)\breve{u}(-\tau_c)\right]$$
$$= f(\tau_i) \quad \text{for} \quad n \quad \text{even,}$$

and

$$\breve{u}(\tau_i) - \left[\sum_{\nu=1}^n \left[\omega_\nu + \frac{2\sin^2\theta_\nu\cos 2[(n+1)/2]\theta_\nu}{(2[n/2]+1)(2[(n+1)/2]+1)\sin(\theta_\nu + \theta_c)\sin(\theta_\nu - \theta_c)}\right] k(\tau_i, \tau_\nu)\breve{u}(\tau_\nu)\right]$$
$$- \frac{(n+1)\tan\theta_c}{n(n+2)}\left[k(\tau_i, \tau_c)\breve{u}(\tau_c) + k(\tau_i, -\tau_c)\breve{u}(-\tau_c)\right]$$
$$= f(\tau_i) \quad \text{for} \quad n \quad \text{odd.}$$

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