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Full Discretization to an Hyperbolic Equation with Nonlocal Coefficient

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ABSTRACT: We present full discretization of the telegraph equation with nonlocal coefficient using Rothe-finite element method. For solving the equation numerically we use the Newton Raphson method, but the nonlocal term causes difficulties because the Jacobien matrix is full. To remedy these difficulties we apply the technique used by Sudhakar [4]. The optimal a priori error estimates for both semi discrete and fully discrete schemes are derived in V, introduced in (1.4), and $H^1(\Omega)$ and a numerical experiment is described to support our theoretical result.

Key Words: Roth's method, Finite element method, Telegraph equation, Nonlocal term and a priori estimate.

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1. Introduction and Preliminaries

Let Ω is a simply connected bounded domain of $\mathbb{R}^d, d \geq 2$ with Lipschitz continuous boundary $\partial\Omega$. Consider the following nonlocal hyperbolic problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + a(l(u))(Au) = f(x,t,u) \text{ in } Q = \Omega \times [0,T], \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \times [0,T]. \end{cases} \tag{1.1}$$

Where $T < \infty, a$ is a function depends of $l(u)$ with

$$l(u) = \int_\Omega u(x,t)dx. \tag{1.2}$$

We introduce the elliptic differential operator A defined by

$$Au := -div(A(x)\nabla u) + b(x)u, \tag{1.3}$$

where $A(x)$ is a symmetric matrix with entries that are uniformly bounded and measurable, $b(x)$ is a bounded positive function and we assume that f, u_0, u_1 and $A(x)$ are smooth enough functions.
The acoustic telegraph equation (1.1) with nonlocal term and constant coefficients is used to model the effects of diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation (see [1]). The function a in equation (1.1) is the diffusion depends an a nonlocal quantity $\int_\Omega u(x,t)dx$ and assumed to depend on the entire population in the domain Ω. Recent years have seen an increasing interest in studying nonlocal problems, of this type of problems [[4], [5], [8]].
One of the more popular methods for solving partial differential equation is the Roth method (or the

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method of lines), this method is used in the time discretization of evolution equations where the derivatives with respect to one variable are replaced by the corresponding difference quotients that finally leads to systems of differential equations for functions of the remaining variables. Roth's method was introduced by E. Roth in his the pioneer work1930, it has been adopted and developed by many authors for example O.A. Ladyzenskaja [9], [10] and K. Rektorys [[13], [14]] for solving second order linear and quasilinear parabolic problems. Recently Roth's method has been studied linear and quasilinear hyperbolic equations [[6], [15], [3]].
The purpose of this work is to combine Rothe's method with finite element. The fully discrete scheme for problem (1.1) gives a system of nonlinear equations, we use Newton Raphson method to solve this system. It is known that the Newton Raphson iteration is the most popular for solving nonlinear algebraic equations because it is fast convergent in a small number of iteration. One of the main difficulties of using Newton's is the fully Jacobien matrix, this difficulty can be addressed by reformulate the system as [4].
The paper is organized as follows : In section 1, we present some basic notations needed material. Section 2 contains the weak formulation, the discretization scheme based on Rothe's method and a priori estimates. In section 3 we give the fully discrete scheme and a priori error estimates. Finally, a numerical example is presented in section 4.
Let $(.,.)$ denote the inner product in $L^2(\Omega)$, and let $(.,.)_A$ be the inner product of

$$V = \{u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega)\}, \tag{1.4}$$

its norm is defined by

$$(u,v)_A = (A(x)\nabla u, \nabla v) + (b(x)u, v) \quad \forall u, v \in V, \tag{1.5}$$

and the norms on $L^2(\Omega)$, V are denoted $\|.\|$, $\|.\|_A$ respectively. We take $C_\epsilon = C(\epsilon^{-1})$ with ϵ is small. For $m \geq 0$, we use $H^m(\Omega)$ to denote the Sobolev space on Ω of order m with the norm

$$\|w\|_m = \Big(\sum_{0\leq\alpha\leq m} \|\frac{\partial^\alpha w}{\partial x^\alpha}\|^2\Big)^{\frac{1}{2}}.$$

Along this work we shall always assume the following assumptions:

1. $(H1)$ $u^0 \in V$, $u^1 \in L^2(\Omega)$
2. $(H2)$ $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in the sense
$$\|f(x,t,s) - f(x,t',s')\| \leq C\{|t-t'|(|s| + |s'| + |s-s'|)\}, \tag{1.6}$$
and satisfies the condition of growth
$$\|f(x,t,\xi)\| \leq C(1 + |\xi|), \quad \forall (x,t,\xi) \in \Omega \times [0,T] \times \mathbb{R}. \tag{1.7}$$
3. $(H3)$ $a : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with the Lipschitz constant L_M, this means
$$|a\big(l(u)\big) - a\big(l(v)\big)| \leq L_M\|u - v\|, \quad \forall u, v \in V. \tag{1.8}$$
and satisfies
$$0 < m \leq a(s) \leq M < \infty, \quad \forall s \in \mathbb{R}. \tag{1.9}$$
4. $(H4)$ $A(x)$ is symmetric matrix satisfies:
$$(A\xi, \xi) \geq C\|\xi\|^2. \tag{1.10}$$
and let $(.,.)_A$ be a bounded, coercive and symmetric bilinear form according to choose the coefficients $A(x)$, i.e.,
$$|(u,v)_A| \leq C\|u\|_A\|V\|_A, \quad (u,u)_A \geq C\|u\|_A^2, \quad \forall u, v \in V. \tag{1.11}$$

Definition 1.1. *A function u is said a weak solution of (1.1) if*

$$\begin{cases} 1)u : Q = \Omega \times [0,T] \to \mathbb{R} \text{ and } u \in H^1\big([0,T], L^2(\Omega)\big) \cap L^2([0,T], V) \text{such that,} \\ \forall v \in H^1\big([0,T]; L^2(\Omega)\big) \cap L^2([0,T], V) \text{ with } v(x,T) = 0, \\ 2) - \int_{[0,T]} \big(\partial_t u, \partial_t v\big) - \big(u_1, v(.,0)\big) + \int_{[0,T]} \big(\partial_t u, v\big) + \int_{[0,T]} a(l(u))\big(u, v\big)_A = \int_{[0,T]} \big(f, v\big), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \end{cases}$$

2. Time Discretization

We divide the interval $[0,T]$ into n subintervals of length $\tau = \frac{T}{n}$ and denote $u^i = u(t_i, x)$, $t_i = i\tau, i = 0, 1, ..., n$. Let u^{-1} be defined as

$$u^{-1}(x) = u^0(x) - \tau u^1(x),$$

the recurrent approximation scheme for $i = 1, ..., n$ becomes

$$\begin{cases} \text{Find } u^i \cong u(., t_i) \in V, i = 1, 2, ..., n, \text{such that,} \\ \big(\delta^2 u^i, v\big) + \big(\delta u^i, v\big) + a(l(u^i))\big(u^i, v\big)_A = \big(f^i, v\big) \end{cases} \tag{2.1}$$

We define the Roth's functions by a piecewise linear interpolation with respect to the time t,

$$u^n = u^{i-1} + (t - t_{i-1})\delta u^i, \qquad \forall t \in [t_{i-1}, t_i],\ \ 1 \le i \le n) \tag{2.2}$$

$$\delta u^n = \delta u^{i-1} + (t - t_{i-1})\delta^2 u^i, \qquad \forall t \in [t_{i-1}, t_i],\ \ 1 \le i \le n, \tag{2.3}$$

together with the step function

$$\bar{u}^n = \begin{cases} u^i & t \in [t_{i-1}, t_i], i = 1, ..., n, \\ u^0 & t \in [-\tau, 0]. \end{cases} \tag{2.4}$$

We denote by $\bar{f}^n$ the function

$$\bar{f}^n = \begin{cases} f^i & t \in [t_{i-1}, t_i], i = 1, ..., n, \\ 0 & t = 0. \end{cases} \tag{2.5}$$

Then, the problem (2.1) can be takes the form:

$$\begin{cases} \forall v \in H^1\big([0,T]; L^2(\Omega)\big) \cap L^2([0,T], v) \text{ with } v(x,T) = 0. \\ \big(\partial_t \delta u^n, v\big) + \big(\partial_t u^n, v\big) + a(l(\bar{u}^n))\big(\bar{u}^n, v\big)_A = \big(\bar{f}^n, v\big). \end{cases} \tag{2.6}$$

By integrating the above equation over $[0,T]$, we get

$$\begin{cases} \forall v \in H^1\big([0,T]; L^2(\Omega)\big) \cap L^2([0,T], v) \text{ with } v(x,T) = 0. \\ - \int_{[0,T]} \big(\delta u^n, \partial_t v\big) - \big(\delta u^n(0), v(.,0)\big) + \int_{[0,T]} \big(\partial_t u^n, v\big) + \int_{[0,T]} a(l(\bar{u}^n))\big(\bar{u}^n, v\big)_A = \int_{[0,T]} \big(\bar{f}^n, v\big) \end{cases} \tag{2.7}$$

Lemma 2.1. *For $1 \le i \le s \le n$, the estimates*

$$\|\delta u^s\|^2 + \sum_{i=1}^{s} \|\delta u^i - \delta u^{i-1}\|^2 + \sum_{i=1}^{s} \tau \|\delta u^i\|^2 + m\|u^s\|_A^2$$
$$+ m \sum_{i=1}^{s} \|u^i - u^{i-1}\|_A^2 \le C\tau. \tag{2.8}$$

Proof. Choose $v = \delta u^i$ in the equation (2.1), we get

$$(\delta u^i - \delta u^{i-1}, \delta u^i) + \tau\|\delta u^i\|^2 + m(u^i, u^i - u^{i-1})_A \leq \tau\|f^i\|\|\delta u^i\|.$$

Using Young, we obtain

$$\|\delta u^i\|^2 - \|\delta u^{i-1}\|^2 + \|\delta u^i - \delta u^{i-1}\|^2 + \tau\|\delta u^i\|^2 + m\big(\|u^i\|_A^2 - \|u^{i-1}\|_A^2 + \|u^i - u^{i-1}\|_A^2\big) \leq \tau\|f^i\|\|\delta u^i\|.$$

Taking summation from $i = 1$ to s, we get

$$\|\delta u^s\|^2 - \|\delta u^0\|^2 + \sum_{i=1}^{s}\|\delta u^i - \delta u^{i-1}\|^2 + \sum_{i=1}^{s}\tau\|\delta u^i\|^2 + m\|u^s\|_A^2 - m\|u^0\|_A^2$$
$$+m\sum_{i=1}^{s}\|u^i - u^{i-1}\|_A^2 \leq \sum_{i=1}^{s}\tau\|f^i\|\|\delta u^i\|.$$

Applying the Abel's summing formula, we obtain

$$\begin{aligned}
&\|\delta u^s\|^2 + \sum_{i=1}^{s}\|\delta u^i - \delta u^{i-1}\|^2 + \sum_{i=1}^{s}\tau\|\delta u^i\|^2 + m\|u^s\|_A^2 + m\sum_{i=1}^{s}\|u^i - u^{i-1}\|_A^2 \\
&\leq C + \epsilon\sum_{i=1}^{s}\tau\|f^i\|^2 + C_\epsilon\sum_{i=1}^{s}\tau\|\delta u^i\|^2, \\
&\leq \epsilon\Big(1 + \sum_{i=1}^{s}\sum_{r=1}^{i-1}\tau^2\|\delta u^r\|^2\Big) + C_\epsilon\sum_{i=1}^{s}\tau\|\delta u^i\|^2. \qquad (2.9)
\end{aligned}$$

Using the Gronwall's Lemma (see, e.g. [11]) inequality and choosing $\epsilon = \tau$ to get

$$\|\delta u^s\|^2 + \sum_{i=1}^{s}\|\delta u^i - \delta u^{i-1}\|^2 + \sum_{i=1}^{s}\tau\|\delta u^i\|^2 + m\|u^s\|_A^2 + m\sum_{i=1}^{s}\|u^i - u^{i-1}\|_A^2 \leq C\tau$$

□

Corollary 2.2. *There exists a positive constant C such that*

$$\|\partial_t u^n\|^2_{L^2([0,T];L^2(\Omega))} \leq C, \|u^n\|^2_{L^2([0,T];V)} \leq C, \qquad (2.10)$$

$$\|u^n - \bar{u}^n\|^2_{L^2([0,T];V)} \leq \frac{C}{n}, \qquad (2.11)$$

$$\|u^n - \bar{u}^n\|^2_{L^2([0,T];L^2(\Omega))} \leq \frac{C}{n^2}, \|u^n - \bar{u}^n_\tau\|^2_{L^2([0,T];L^2(\Omega))} \leq \frac{C}{n^2}, \qquad (2.12)$$

$$\|\delta u^n - \partial_t u^n\|^2_{L^2([0,T];L^2(\Omega))} \leq \frac{C}{n}. \qquad (2.13)$$

We denote by $e_u = u - u^n$ and $e_f = f - \bar{f}^n$.

Theorem 2.3. *[1] Under the assumptions (H1)-(H4), we have*

$$\|e_u\|^2_{C([0,T];L^2(\Omega))} + m\|e_u\|^2_{L^2([0,T];V)} \leq C(\tau^2 + \tau). \qquad (2.14)$$

3. Full Discretization

At each time t_i, $0 \leq i \leq n$, we consider a triangulation Υ_h^i made of triangles T^i such that no nodes of every triangle lies in the interior of a side of another triangle. Let V_h^i be the discrete space of V^i defined by

$$V_h^i = \Big\{ \Phi_h \in C^0(\bar{\Omega}) \text{ tel que } \Phi_h|_{T^i} \text{ is polynomial of degree one } \forall T^i \in \Upsilon_h^i \Big\}.$$

Let $\{p_j\}_{j=1}^N$ be interior nodes of Υ_h^i et $\{\Phi_j(x)\}_{j=1}^N$ be the basic functions for the space V_h^i such that any function will be the pyramid form in V_h^i and which takes the value 1 at $\{p_j\}_{j=1}^N$ and vanishes at exterior nodes. We can write the solution u_h^i as

$$u_h^i(t) = \sum_{j=1}^N \alpha_j^i \Phi_j(x) \in V_h^i.$$

Let X be a Banach space, we use the following norm in discrete version.

$$\|u\|_{L^\infty(0,T,\tau;X)} = \max_{1\leq m\leq J} \big\|u^m\big\|_X, \tag{3.1}$$

$$\|u\|^2_{L^2(0,T,\tau;X)} = \tau \sum_{m=1}^J \big\|u^m\big\|_X^2.$$

Then, the fully discrete scheme for problem (1.1) reads as

$$\begin{cases} \text{Find } u_h^i \in V_h^i(\Omega) \text{ such that :} \\ u_h(0) = u_h^0,\ u_{ht}(0) = u_h^1 \text{ and } u_h^{-1} = u_h^0 - \tau u_h^1, \\ \text{and, } \forall v \in V_h^i, \\ \big(\delta^2 u_h^i, v_h\big) + \big(\delta u_h^i, v_h\big) + a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_A = \big(f^i, v_h\big). \end{cases} \tag{3.2}$$

We introduce the orthogonal projection operator $\Pi_h^i : H_0^1(\Omega) \longrightarrow V_h^i(\Omega)$ such that :

$$(\nabla w, \nabla v) = (\nabla \Pi_h^i w, \nabla v) \ \ \forall w \in H_0^1(\Omega), v \in V_h^i(\Omega). \tag{3.3}$$

From fully discrete weak formulation of (3.2), we have

$$\begin{cases} \text{Find } u_h^i \in V_h^i(\Omega) \text{ such that :} \\ u_h(0) = u_h^0,\ u_{ht}(0) = u_h^1 \text{ and } u_h^{-1} = u_h^0 - \tau u_h^1, \\ \text{and, } \ \forall v \in V_h^i, \\ \Big(\dfrac{u_h^i - \Pi_h^i u_h^{i-1}}{\tau} - \dfrac{u_h^{i-1} - \Pi_h^{i-1} u_h^{i-2}}{\tau}, v_h\Big) + \tau\Big(\dfrac{u_h^i - \Pi_h^i u_h^{i-1}}{\tau}, v_h\Big) + \tau a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_A = \tau\big(f^i, v_h\big). \end{cases} \tag{3.4}$$

This implies,

$$\begin{cases} \text{Find } u_h^i \in V_h^i(\Omega) \text{ such that :} \\ u_h(0) = u_h^0,\ u_{ht}(0) = u_h^1 \text{ and } u_h^{-1} = u_h^0 - \tau u_h^1, \\ \text{and, } \ \forall v \in V_h^i, \\ (1+\tau)\big(u_h^i, v_h\big) + \tau^2 a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_A f = \tau^2\big(f^i, v_h\big) + \big((1+\tau)\Pi_h^i u_h^{i-1} + \big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), v_h\big). \end{cases} \tag{3.5}$$

The problem (3.5) give as a system of nonlinear algebraic equations by using finite element, then can be given this system as follows :

$$F_j(\bar{\alpha}^i) = F_j(u_h^i) = 0 \quad 1 \leq j \leq N, \tag{3.6}$$

where $\bar{\alpha}^i = [\alpha_1^i, \alpha_2^i, ..., \alpha_N^i]$, and

$$\begin{aligned} F_j(u_h^i) &= (1+\tau)\big(u_h^i, v_h\big) + \tau^2 a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_A - \tau^2\big(f^i, v_h\big) \\ &-\big((1+\tau)\Pi_h^i u_h^{i-1} + \big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), v_h\big). \end{aligned} \tag{3.7}$$

We use Newton-Raphson method to solve (3.5), but the presence of nonlocal term in the equation destroys the sparsity of Newton-Raphson method.
We compute the Jacobian matrix J To get the value of α_j^i by Newton's method, every element of the Jacobian matrix takes the form

$$\begin{aligned} \frac{\partial F_j(u_h^i)}{\partial \alpha_j^i} &= (1+\tau)\big(\phi_j, \phi_l\big) + \tau^2\Big(\int_\Omega \phi_j\Big) a'\big(l(u_h^i)\big)\big(u_h^i, \phi_l\big)_A \\ &+\tau^2 a\big(l(u_h^i)\big)\big(\phi_j, \phi_l\big)_A - \tau^2\big(f'(u_h^i)\phi_j, \phi_l\big). \end{aligned} \tag{3.8}$$

In order to ensure the sparsity of the Jacobian matrix we modify the scheme (3.5) according to the technic used by Chaudhary in [4]. Then the problem (3.5) can be rewritten as follows :
Find $d \in \mathbb{R}$, and $u_h^i \in V_h^i$ such that

$$l(u_h^i) - d = 0. \tag{3.9}$$

$$\begin{aligned} &(1+\tau)\big(u_h^i, v_h\big) + \tau^2 a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_A - \tau^2\big(f^i, v_h\big) - \big((1+\tau)\Pi_h^i u_h^{i-1} \\ &+\big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), v_h\big) = 0 \quad \forall v_h \in V_h^i. \end{aligned} \tag{3.10}$$

Take $v_h = \phi_j$, and reformulate the equations (3.9)-(3.10) as follows:

$$F_j(u_h^i, d) = (1+\tau)\big(u_h^i, \phi_l\big) + \tau^2 a\big(d\big)\big(u_h^i, \phi_l\big)_A - \tau^2\big(f^i, \phi_l\big) - \big((1+\tau)\Pi_h^i u_h^{i-1} + \big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), \phi_l\big).$$

$$F_{N+1}^i = l(u_h^i) - d. \tag{3.11}$$

This implies

$$J \begin{bmatrix} \bar{\alpha}^i \\ \beta \end{bmatrix} = \begin{bmatrix} A & b \\ c & \delta_{11} \end{bmatrix} \begin{bmatrix} \bar{\alpha}^i \\ \beta \end{bmatrix} = \begin{bmatrix} \bar{F}^i \\ F_{N+1}^i \end{bmatrix}, \tag{3.12}$$

where $A = A_{N\times N}$, $b = b_{N\times 1}$ and $c = c_{1\times N}$ take the form

$$\begin{aligned} A_{jl} &= (1+\tau)\big(\phi_j, \phi_l\big) + \tau^2 a\big(d\big)\big(\phi_j, \phi_l\big)_A - \tau^2\big(f'(u_h^i)\phi_j, \phi_l\big), \\ b_{j1} &= \tau^2 a'\big(d\big)\big(u_h^i, \phi_l\big)_A, \\ c_{1l} &= \Big(\int_\Omega \phi_j\Big), \\ \delta_{11} &= -1, \end{aligned}$$

and $\bar{\alpha}^i = [\alpha_1^i, \alpha_2^i, ..., \alpha_N^i]^T$, $\bar{F}^i = [F_1^i, F_2^i, ..., F_N^i]^T$.
The matrix system (3.12) can be solved by using the Sherman-Morrison Woodbury formula and block elimination with one-refinement algorithm in [8], [7].
We introduce the orthogonal projection to get an optimal convergence between u^i, u_h^i. Therefor, we can take the error as follows.

$$\begin{aligned} e^i = u^i - u_h^i &= u^i - \Pi_h^i u^i + \Pi_h^i u^i - u^i \\ &= \rho_h^i + \theta_h^i. \end{aligned} \tag{3.13}$$

Theorem 3.1. *[12] : There exists a positive constant C, independent of h such that*

$$\|v - \Pi_h^i v\|_j \leq C h_j^i \|v\|_i \quad \forall v \in H^i \cap H_0^1 \quad j = 0, 1; i = 1, 2 \tag{3.14}$$

$$\|v_t - \Pi_h^i v_t\|_j \leq C h_j^i \|v_t\|_i \quad \forall v \in H^i \cap H_0^1 \quad j = 0, 1; i = 1, 2 \tag{3.15}$$

$$\|v_{tt} - \Pi_h^i v_{tt}\|_j \leq C h_j^i \|v_{tt}\|_i \quad \forall v \in H^i \cap H_0^1 \quad j = 0, 1; i = 1, 2 \tag{3.16}$$

Lemma 3.2. *The estimates*

$$\|\nabla \Pi_h^i u^i\| \leq C \tag{3.17}$$

$$\|\Pi_h^i u^i\|_A \leq c \tag{3.18}$$

Proof:
For $w = u^i$ in (3.2), we have

$$(\nabla u^i, \nabla v_h) = (\nabla \Pi_h^i u^i, \nabla v_h).$$

Choosing $v_h = \Pi_h^i u^i$, to get

$$\begin{aligned} \|\nabla \Pi_h^i u^i\|^2 &= (\nabla u^i, \nabla \Pi_h^i u^i) \\ &\leq \|\nabla u^i\| \|\nabla \Pi_h^i u^i\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|\nabla \Pi_h^i u^i\| &= \|\nabla u^i\| \\ &\leq C. \end{aligned}$$

Further

$$\begin{aligned} \|\Pi_h^i u^i\|_A^2 &= (\Pi_h^i u^i, \Pi_h^i u^i)_A \\ &= (A \nabla \Pi_h^i u^i, \nabla \Pi_h^i u^i) + (a(x) \Pi_h^i u^i, \Pi_h^i u^i) \\ &\leq C(\|\nabla \Pi_h^i u^i\|^2 + \|\Pi_h^i u^i\|^2). \end{aligned}$$

Using Poincaré inequality, we obtain

$$\begin{aligned} \|\Pi_h^i u^i\|_A &\leq C(\|\nabla \Pi_h^i u^i\|) \\ &\leq c. \end{aligned}$$

where c and C are some positive constants.

Lemma 3.3. *Let $u_h^0 \in V_h^0$ and $u_h^1 \in V_h^0$ and for $1 \leq i \leq s \leq n$, then the solution $u_h^i \in V_h^i$ of the problem (3.2) satisfied*

$$\|\delta u_h^s\|_{L^2(0,T;L^2(\Omega))}^2 + m \|u_h^s\|_{L^2(0,T;V)}^2 \leq C. \tag{3.19}$$

We use the same proof in Lemma 2.1 to obtain the existence of u_h^i and a priori estimates.

$$\|\delta u_h^s\|^2 + \sum_{i=1}^{s} \|\delta u_h^i - \delta u_h^{i-1}\|^2 + \sum_{i=1}^{s} \tau \|\delta u_h^i\|^2 + m \|u_h^s\|_A^2 + m \sum_{i=1}^{s} \|u_h^i - u_h^{i-1}\|_A^2 \leq C\tau.$$

This means

$$\|\delta u_h^s\|^2 + m \|u_h^s\|_A^2 \leq C.$$

We integrate from 0 to T, to obtain

$$\|\delta u_h^s\|_{L^2(0,T;L^2(\Omega))}^2 + m \|u_h^s\|_{L^2(0,T;V)}^2 \leq C.$$

Theorem 3.4. *We assume that* $\frac{\mathbf{m}\min(b(x))}{2} \geq \frac{16e^3c^2L_M}{m}$ *where c is given in Eq.(3.18). Then, there exists a positive constant C such that*

$$\left\|u^i - u_h^i\right\|_{L^2(0,T,\tau,H^1(\Omega))} \leq C(h+h^2). \tag{3.20}$$

Proof:
From equations (2.1), (3.1), we have

$$\begin{aligned}
&\Big(\partial_t\delta\theta_h^i, v_h\Big) + \Big(\partial_t\theta_h^i, v_h\Big) + a_h^i\Big(\theta_h^i, v_h\Big)_A \\
&= \Big(\partial_t\delta\Pi_h^i u^i, v_h\Big) + \Big(\partial_t\Pi_h^i u^i, v_h\Big) + a_h^i\Big(\Pi_h^i u^i, v_h\Big)_A \\
&-\Big(\partial_t\delta u_h^i, v_h\Big) - \Big(\partial_t u_h^i, v_h\Big) - a_h^i\Big(u_h^i, v_h\Big)_A \\
&= -\Big(f^i, v_h\Big) + \Big(\partial_t\delta\Pi_h^i u^i, v_h\Big) + \Big(\partial_t\Pi_h^i u^i, v_h\Big) \\
&+a_h^i\Big(\Pi_h^i u^i, v_h\Big)_A \\
&= -\Big(\partial_t\delta u^i, v_h\Big) - \Big(\partial_t u^i, v_h\Big) - a^i\Big(u^i, v_h\Big)_A \\
&+\Big(\partial_t\delta\Pi_h^i u^i, v_h\Big) + \Big(\partial_t\Pi_h^i u^i, v_h\Big) + a_h^i\Big(\Pi_h^i u^i, v_h\Big)_A \\
&+a^i\Big(\Pi_h^i u^i, v_h\Big) - a^i\Big(\Pi_h^i u^i, v_h\Big) \\
&= -\Big(\partial_t\delta(u^i - \Pi_h^i u^i), v_h\Big) - \Big(\partial_t(u^i - \Pi_h^i u^i), v_h\Big) \\
&-a^i\Big((u^i - \Pi_h^i u^i), v_h\Big)_A + (a_h^i - a^i)\Big(\Pi_h^i u_h^i, v_h\Big)_A.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big(\partial_t\delta\theta_h^i, v_h\Big) + \Big(\partial_t\theta_h^i, v_h\Big) + a_h^i\Big(\theta_h^i, v_h\Big)_A = -\Big(\partial_t\delta\rho_h^i, v_h\Big) - \Big(\partial_t\rho_h^i, v_h\Big) \\
&-a^i\Big(\rho_h^i, v_h\Big)_A + (a_h^i - a^i)\Big(\Pi_h^i u^i, v_h\Big)_A.
\end{aligned} \tag{3.21}$$

Choosing $v_h = \tau^2\delta\theta_h^i$ in (3.21), we obtain

$$\begin{aligned}
&\tau^2\Big(\partial_t\delta\theta_h^i, \delta\theta_h^i\Big) + \tau^2\Big(\partial_t\theta_h^i, \delta\theta_h^i\Big) + \tau^2 a_h^i\Big(\theta_h^i, \delta\theta_h^i\Big)_A \\
&= -\tau^2\Big(\partial_t\delta\rho_h^i, \delta\theta_h^i\Big) - \tau^2\Big(\partial_t\rho_h^i, \delta\theta_h^i\Big) \\
&-\tau^2 a^i\Big(\rho_h^i, \delta\theta_h^i\Big)_A + \tau^2(a_h^i - a^i)\Big(\Pi_h^i u^i, \delta\theta_h^i\Big)_A.
\end{aligned} \tag{3.22}$$

New left-hand side of (3.22) can be estimated as follows.

$$\begin{aligned}
&\tau^2\Big(\partial_t\delta\theta_h^i, \delta\theta_h^i\Big) + \tau^2\Big(\partial_t\theta_h^i, \delta\theta_h^i\Big) + \tau^2 a_h^i\Big(\theta_h^i, \delta\theta_h^i\Big)_A \\
&\geq \tau^2\frac{\partial_t}{2}\left\|\delta\theta_h^i\right\|^2 + \tau^2\left\|\delta\theta_h^i\right\|^2 + \tau^2 m\frac{\delta}{2}\left\|\theta_h^i\right\|_A, \\
&\geq \frac{\tau^2}{2}\partial_t\left\|\delta\theta_h^i\right\|^2 + \tau^2\left\|\delta\theta_h^i\right\|^2 + \frac{\tau}{2}m(\left\|\theta_h^i\right\|_A^2 - \left\|\theta_h^{i-1}\right\|_A^2), \\
&\geq \frac{\tau^2}{2}\partial_t\left\|\delta\theta_h^i\right\|^2 + \tau^2\left\|\delta\theta_h^i\right\|^2 + \frac{\tau}{2}m\left\|\theta_h^i\right\|_A^2 - \frac{\tau}{2}m\left\|\theta_h^{i-1}\right\|_A^2.
\end{aligned} \tag{3.23}$$

To estimate the right-hand side of (3.22), we need the following steps.
Step1. We estimate $\Big| -\tau^2\Big(\partial_t\delta\rho_h^i, \delta\theta_h^i\Big) - \tau^2\Big(\partial_t\rho_h^i, \delta\theta_h^i\Big)\Big|$.
Using Cauchy-schwarz, we get

$$\Big| -\tau^2\Big(\partial_t\delta\rho_h^i, \delta\theta_h^i\Big) - \tau^2\Big(\partial_t\rho_h^i, \delta\theta_h^i\Big)\Big| \leq \tau\left\|\partial_t\delta\rho_h^i\right\|\tau\left\|\delta\theta_h^i\right\| + \tau\left\|\partial_t\rho_h^i\right\|\tau\left\|\delta\theta_h^i\right\|.$$

Thus,

$$\Big| -\tau^2\Big(\partial_t\delta\rho_h^i,\delta\theta_h^i\Big)-\tau^2\Big(\partial_t\rho_h^i,\delta\theta_h^i\Big)\Big| \le \frac{\tau^2}{2}\|\partial_t\delta\rho_h^i\|^2+\tau^2\|\delta\theta_h^i\|^2+\frac{\tau^2}{2}\|\partial_t\rho_h^i\|^2 \tag{3.24}$$

Step2. We estimate

$$\Big| -\tau^2 a^i\Big(\rho_h^i,\delta\theta_h^i\Big)_A+\tau^2\big(a_h^i-a^i\big)\Big(\Pi_h^i u^i,\delta\theta_h^i\Big)_A\Big|.$$

Applying Cauchy-schwarz inequality and Using the inequality

$$ab\le \frac{\omega}{2}a^2+\frac{1}{2\omega}b^2$$

with $\omega=\frac{m}{8}$, we obtain

$$\begin{aligned}
&\Big| -\tau^2 a^i\Big(\rho_h^i,\delta\theta_h^i\Big)_A+\tau^2\big(a_h^i-a^i\big)\Big(\Pi_h^i u^i,\delta\theta_h^i\Big)_A\Big| \le M\tau\|\rho_h^i\|_A\|\theta_h^i-\theta_h^{i-1}\|_A\\
&+c\tau|a_h^i-a^i|\|\theta_h^i-\theta_h^{i-1}\|_A\\
&\le \frac{m}{16}\tau\|\theta_h^i-\theta_h^{i-1}\|_A^2+\frac{4M^2}{m}\tau\|\rho_h^i\|_A^2\\
&+\frac{4c^2}{m}\tau|a_h^i-a^i|^2+\frac{m}{16}\tau\|\theta_h^i-\theta_h^{i-1}\|_A^2,\\
&\le \frac{4M^2}{m}\tau\|\rho_h^i\|_A^2+\frac{4c^2}{m}\tau|a_h^i-a^i|^2\\
&+\frac{m}{8}\tau\big(\|\theta_h^i\|_A+\|\theta_h^{i-1}\|_A\big)^2.
\end{aligned}$$

Using Lipschitz continuity of a, we have

$$\begin{aligned}
|a_h^i-a^i| &\le L_M\|u_h^i-u^i\|\\
&\le L_M\|u_h^i-\Pi_h^i u^i+\Pi_h^i u^i-u^i\|\\
&\le L_M\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big| -\tau^2 a^i\Big(\rho_h^i,\delta\theta_h^i\Big)_A+\tau^2\big(a_h^i-a^i\big)\Big(\Pi_h^i u^i,\delta\theta_h^i\Big)_A\Big|\\
&\le \frac{4M^2}{m}\tau\|\rho_h^i\|_A^2+\frac{4c^2}{m}\tau L_M^2\Big(\|\theta_h^i\|\\
&+\|\rho_h^i\|\Big)^2+\frac{m}{8}\tau\big(\|\theta_h^i\|_A+\|\theta_h^{i-1}\|_A\big)^2.
\end{aligned} \tag{3.25}$$

From (3.23), (3.24) and (3.25), we get

$$\begin{aligned}
&\frac{\tau^2}{2}\partial_t\|\delta\theta_h^i\|^2+\tau^2\|\delta\theta_h^i\|^2+\frac{\tau}{2}m\|\theta_h^i\|_A^2-\frac{\tau}{2}m\|\theta_h^{i-1}\|_A^2\\
&\le \frac{\tau^2}{2}\|\partial_t\delta\rho_h^i\|^2+\tau^2\|\delta\theta_h^i\|^2+\frac{\tau^2}{2}\|\partial_t\rho_h^i\|^2\\
&+\frac{4M^2}{m}\tau\|\rho_h^i\|_A^2+\frac{4c^2}{m}\tau L_M^2\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big)^2\\
&+\frac{m}{4}\tau\|\theta_h^i\|_A^2+\frac{m}{4}\tau\|\theta_h^{i-1}\|_A^2.
\end{aligned}$$

This implies,

$$\begin{aligned}
&\tau^2\partial_t\|\delta\theta_h^i\|^2+\tau\frac{m}{2}\|\theta_h^i\|_A^2\\
&\le \tau^2\|\partial_t\delta\rho_h^i\|^2+\tau^2\|\partial_t\rho_h^i\|^2+\frac{8M^2}{m}\tau\|\rho_h^i\|_A^2\\
&+\frac{8c^2}{m}\tau L_M^2\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big)^2+\frac{3m}{2}\tau\|\theta_h^{i-1}\|_A^2.
\end{aligned}$$

Taking sum from $i=1$ to n to get

$$
\begin{aligned}
&\tau^2\partial_t\|\delta\theta_h^n\|^2+\tau\frac{m}{2}\sum_{i=1}^{n}\|\theta_h^i\|_A^2\\
&\le \tau^2\sum_{i=1}^{n}\|\partial_t\delta\rho_h^i\|^2+\tau^2\sum_{i=1}^{n}\|\partial_t\rho_h^i\|^2+\frac{8M^2}{m}\tau\sum_{i=1}^{n}\|\rho_h^i\|_A^2\\
&+\frac{8c^2}{m}\tau L_M^2\sum_{i=1}^{n}\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big)^2+\frac{3m}{2}\tau\sum_{i=1}^{n-1}\|\theta_h^i\|_A^2.
\end{aligned}
$$

Now applying Gronwall's inequality, we get

$$
\begin{aligned}
&\tau^2\partial_t\|\delta\theta_h^n\|^2+\frac{m}{2}\tau\sum_{i=1}^{n-1}\|\theta_h^i\|_A^2+\frac{m}{2}\tau\|\theta_h^n\|_A^2\\
&\le e^3\Big(\tau^2\sum_{i=1}^{n}\|\partial_t\delta\rho_h^i\|^2+\tau^2\sum_{i=1}^{n}\|\partial_t\rho_h^i\|^2\\
&+\frac{8M^2}{m}\tau\sum_{i=1}^{n}\|\rho_h^i\|_A^2+\frac{8c^2}{m}\tau L_M^2\sum_{i=1}^{n}\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big)^2\Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\tau^2\partial_t\|\delta\theta_h^n\|^2+\frac{m}{2}\tau\sum_{i=1}^{n}\|\theta_h^i\|_A^2\\
&\le e^3\Big(\tau^2\sum_{i=1}^{n}\|\partial_t\delta\rho_h^i\|^2+\tau^2\sum_{i=1}^{n}\|\partial_t\rho_h^i\|^2+\frac{8M^2}{m}\tau\sum_{i=1}^{n}\|\rho_h^i\|_A^2\\
&+\frac{16c^2}{m}L_M^2\tau\sum_{i=1}^{n}\Big(\|\theta_h^i\|^2+\|\rho_h^i\|^2\Big)\Big).
\end{aligned}
$$

Again,

$$
\begin{aligned}
&\tau^2\partial_t\|\delta\theta_h^n\|^2+\frac{m}{2}\tau\sum_{i=1}^{n}\xi\|\nabla\theta_h^i\|^2+\frac{m}{2}\min\big(b(x)\big)\tau\sum_{i=1}^{n}\|\theta_h^i\|^2\\
&\le e^3\Big(\tau^2\sum_{i=1}^{n}\|\partial_t\delta\rho_h^i\|^2\\
&+\tau^2\sum_{i=1}^{n}\|\partial_t\rho_h^i\|^2+\frac{8M^2}{m}\tau\sum_{i=1}^{n}\|\rho_h^i\|_A^2\\
&+\frac{16c^2}{m}L_M^2\tau\sum_{i=1}^{n}\|\theta_h^i\|^2+\frac{16c^2}{m}L_M^2\tau\sum_{i=1}^{n}\|\rho_h^i\|^2\Big).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\tau^2\partial_t\|\delta\theta_h^n\|^2+\tau\sum_{i=1}^{n}\|\nabla\theta_h^i\|^2+\tau\sum_{i=1}^{n}\|\theta_h^i\|^2\\
&\le C\Big(\tau^2\sum_{i=1}^{n}\|\partial_t\delta\rho_h^i\|^2+\tau^2\sum_{i=1}^{n}\|\partial_t\rho_h^i\|^2\\
&+\tau\sum_{i=1}^{n}\|\rho_h^i\|_A^2+\tau\sum_{i=1}^{n}\|\rho_h^i\|^2\Big).
\end{aligned}
$$

Integrating inequality from 0 to T, we have

$$\begin{aligned}
&\tau^2\|\delta\theta_h^n\|^2+\tau\sum_{i=1}^n\|\nabla\theta_h^i\|^2+\tau\sum_{i=1}^n\|\theta_h^i\|^2\\
&\le\tau^2\|\delta\theta_h^0\|^2+C\Big(\tau^2\sum_{i=1}^n\|\partial_t\delta\rho_h^i\|^2\\
&+\tau^2\sum_{i=1}^n\|\partial_t\rho_h^i\|^2+\tau\sum_{i=1}^n\|\rho_h^i\|_A^2+\tau\sum_{i=1}^n\|\rho_h^i\|^2\Big).
\end{aligned}$$

This implies

$$\begin{aligned}
\|\theta_h\|_{L^2(0,T,\tau,H^1(\Omega))}&\le\tau^2\|\delta\theta_h^0\|^2+C\Big(\tau^2\sum_{i=1}^n\|\partial_t\delta\rho_h^i\|^2+\tau^2\sum_{i=1}^n\|\partial_t\rho_h^i\|^2\\
&+\tau\sum_{i=1}^n\|\rho_h^i\|_A^2+\tau\sum_{i=1}^n\|\rho_h^i\|^2\Big),
\end{aligned}\tag{3.26}$$

We have

$$\begin{aligned}
\delta\theta_h^i&=\frac{1}{\tau}\int_{t_{i-1}}^{t_i}\theta_{hs}(s)ds\\
\|\delta\theta_h^i\|^2&\le\frac{1}{\tau}\int_{t_{i-1}}^{t_i}\|\theta_{hs}(s)\|^2ds\\
\|\delta\theta_h^i\|^2&\le\frac{1}{\tau}\int_0^T\|\theta_{hs}(s)\|^2ds.
\end{aligned}$$

Thus,

$$\|\delta\theta_h^0\|_{L^2(\Omega)}^2\le\frac{1}{\tau}\int_0^T\|\theta_{hs}(0)\|_{L^2(\Omega)}^2ds.$$

If we take $u_h^0=\Pi_h^0u^0$, then

$$\begin{aligned}
\|\theta_{hs}(0)\|&=\|\partial_s(\Pi_h^0u^0-u_h^0)\|\\
&\le\|\partial_s(\Pi_h^0u^0-u^0)\|+\|\partial_s(u^0-u_h^0)\|\\
&\le Ch^2\|u_s^0\|_{H^2(\Omega)}^2.
\end{aligned}\tag{3.27}$$

So,

$$\tau^2\|\delta\theta_h^0\|^2\le Ch^4\|u_t^0\|_{H^2(\Omega)}^2.$$

Again, we note that

$$\delta\partial_t\rho_h^i=\frac{1}{\tau}\int_{t_{i-1}}^{t_i}\partial_s\rho_h(s)ds.$$

This shows

$$\|\delta\partial_t\rho_h^i\|^2\le\frac{1}{\tau}\|\rho_{htt}\|_{L^2(t_{i-1},t_i;L^2(\Omega))}^2,$$

and

$$\begin{aligned}
\tau^2\sum_{i=1}^n\|\delta\partial_t\rho_h^i\|^2&\le\tau\sum_{i=1}^n\|\rho_{htt}\|_{L^2(t_{i-1},t_i;L^2(\Omega))}^2,\\
&\le\tau\|\rho_{htt}\|_{L^2(0,T;L^2(\Omega))}^2,\\
&\le Ch^4\tau\|u_{htt}\|_{L^2(0,T;H^2(\Omega))}^2.
\end{aligned}$$

Thus,

$$\tau^2 \sum_{i=1}^{n} \left\|\delta\partial_t \rho_h^i\right\|^2 \leq Ch^4 \left\|u_{htt}\right\|^2_{L^2(0,T;H^2(\Omega))}. \tag{3.28}$$

Further

$$\begin{aligned}
\left\|\partial_t \rho_h^i\right\|^2 &\leq Ch^4 \left\|u_{ht}^i\right\|^2_{H^2(\Omega)}, \\
\tau^2 \sum_{i=1}^{n} \left\|\partial_t \rho_h^i\right\|^2 &\leq \tau^2 \sum_{i=1}^{n} Ch^4 \left\|u_{ht}^i\right\|^2_{H^2(\Omega)}, \\
\tau^2 \sum_{i=1}^{n} \left\|\partial_t \rho_h^i\right\|^2 &\leq \tau Ch^4 \left\|u_{ht}\right\|^2_{L^2(0,T,\tau;H^2(\Omega))}.
\end{aligned}$$

So,

$$\tau^2 \sum_{i=1}^{n} \left\|\partial_t \rho_h^i\right\|^2 \leq Ch^4 \left\|u_{ht}\right\|^2_{L^2(0,T,\tau;H^2(\Omega))}. \tag{3.29}$$

Also

$$\begin{aligned}
\left\|\rho_h^i\right\|^2 &\leq Ch^2 \left\|u_h^i\right\|^2_{H^2(\Omega)}, \\
\tau \sum_{i=1}^{n} \left\|\rho_h^i\right\|^2 &\leq Ch^4 \tau \sum_{i=1}^{n} \left\|u_h^i\right\|^2_{H^2(\Omega)}, \\
\tau \sum_{i=1}^{n} \left\|\rho_h^i\right\|^2 &\leq Ch^4 \left\|u_h\right\|^2_{L^2(0,T,\tau;H^2(\Omega))}. \qquad (3.30)
\end{aligned}$$

Finally

$$\begin{aligned}
\left\|\rho_h^i\right\|_A^2 &= \left(A\nabla\rho_h^i, \nabla\rho_h^i\right) + \left(b(x)\rho_h^i, \rho_h^i\right), \\
&\leq C\left(\left\|\nabla\rho_h^i\right\|^2 + \left\|\rho_h^i\right\|^2\right) \\
\tau \sum_{i=1}^{n} \left\|\rho_h^i\right\|_A &\leq C\tau \sum_{i=1}^{n} \left\|\rho_h^i\right\|^2_{H^1(\Omega)} \\
&\leq C\left\|\rho_h\right\|^2_{L^2(0,T,\tau;H^1(\Omega))}, \\
&\leq Ch^2 \left\|u_h\right\|^2_{L^2(0,T,\tau;H^2(\Omega))}. \qquad (3.31)
\end{aligned}$$

New using the estimates (3.27)-(3.31) in (3.26), we get

$$\tau \sum_{i=1}^{n} \left\|\theta_h^i\right\|^2_{H^1(\Omega)} \leq C(h^2 + h^4).$$

So,

$$\left\|\theta_h\right\|^2_{L^2(0,T,\tau,H^1(\Omega))} \leq C(h^2 + h^4).$$

Where c is is constant depending on $\left\|u_h\right\|^2_{L^2(0,T,\tau;H^2(\Omega))}$, $\left\|u_{htt}\right\|^2_{L^2(0,T;H^2(\Omega))}$, $\left\|u_{ht}\right\|^2_{L^2(0,T,\tau;H^2(\Omega))}$, $\left\|u_{htt}\right\|^2_{L^2(0,T;H^2(\Omega))}$ and $\left\|u_t^0\right\|^2_{H^2(\Omega)}$. We conclude

$$\left\|u^i - u_h^i\right\|_{L^2(0,T,\tau;H^1(\Omega))} \leq c(h + h^2).$$

4. Numerical experiment

In this section, we set up a numerical experiment to find an approximate solution of problem (1.1), if we use Roth's approximation in time discretization and finite element scheme in the spatial discretization in which we prescribe the computational domain $\Omega = (0, 1)$, the time interval $(0, 1)$ i.e. $T = 1$ and we take $A(x) = b(x) = 1$.

In order using Newton's we take initial guess u^0 and u^1 as follows

$$u^0 = 0,$$

and

$$u^1 = \begin{cases} 1, & \text{at interior node} \\ 0, & \text{at boundary node} \end{cases}$$

The tolerance for stopping iteration is defined to be 10^{-15}, we have considered the step length $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}$ and $\tau = 0.001$. We plot the error in log log-plot.
We choose $f(x, t, u)$ according to test solution $u(x, t) = x(1-x)te^{-t^2}$ and $a(l(u)) = 1 + \cos(l(u))$. The table below gives the numerical errors.

h	$\|u^i - u_h^i\|_{H^1(\Omega)}$
$\frac{1}{10}$	$9.8689e-003$
$\frac{1}{20}$	$5.6454e-003$
$\frac{1}{30}$	$3.9796e-003$
$\frac{1}{40}$	$3.0748e-003$

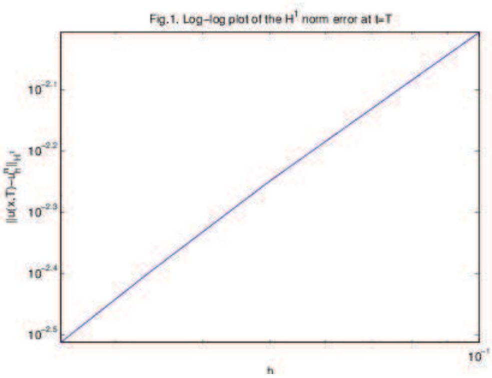

Figure 1: The results of error in log log-plot.

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