Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.45291

On Semi-Symmetric and Locally Symmetric Submanifolds of Conformal Kenmotsu Manifolds

Roghayeh Abdi

ABSTRACT: In this paper, we consider semi symmetric submanifolds of conformal Kenmotsu manifolds and then using obtained results for this type of submanifolds, we characterize locally symmetric submanifolds of conformal Kenmotsu manifolds.

Key Words: Kenmotsu manifold, Conformal Kenmotsu manifold.

Contents

1. Introduction

In [8], K. Kenmotsu defined and studied a new class of almost contact manifolds called Kenmotsu manifolds.
Let (M, J, g) be an almost Hermitian manifold of dimension $2n$, where J denotes the almost complex structure and g the Hermitian metric. Then (M, J, g) is called a locally conformal Kaehler manifold if for each point p of M, there exists an open neighborhood U of p and a positive function f_U on U so that the local metric $g_U = \exp(-f)g_{|U}$ is Kaehlerian. If $U = M$, then the manifold (M, J, g) is said to be a globally conformal Kaehler manifold. The 1-form $\omega = df$ is called the Lee form and its metrically equivalent vector field $\omega^\sharp = grad\ f$, where $\sharp$ means the rising of the indices with respect to g, namely $g(X, \omega^\sharp) = \omega(X)$ for all X tangent to M, is called Lee vector field [7].
We have introduced conformal Kenmotsu manifolds by using an idea of globally conformal Kaehler manifolds. Also, we have given an example of a conformal Kenmotsu manifold that is not Kenmotsu. Hence the category of conformal Kenmotsu manifolds and Kenmotsu manifolds is not the same (see [1]-[5]).
In [9], Kobayashi has proved: let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ such that the structural vector field $\xi\mid_M$ is tangent to M, then

$$\nabla_X\xi = X - \eta(X)\xi, \qquad h(X, \xi) = 0, \qquad R(X, Y)\xi = \eta(X)Y - \eta(Y)X$$

for all vector fields X and Y tangent to M, where ∇, h and R are the Riemannian connection, the second fundamental form and the curvature tensor of M, respectively.
In this paper, as a generalization of these results, we state Lemmas 3.1, 3.2, 3.3 and 3.4 for a submanifold of a conformal Kenmotsu manifold.
A Riemannian manifold (M, g) is called a locally symmetric if its Riemannian curvature tensor R saties-fies $\nabla R = 0$ where ∇ denotes its Riemannian connection. This notion of locally symmetric manifold has been weakened by many authors in several ways for example the notion of semi symmetric manifolds.

2010 *Mathematics Subject Classification:* 53C25, 53C40.
Submitted November 09, 2018. Published October 29, 2019

In this paper, we present the following problem:
Can we characterize locally symmetric submanifolds in conformal Kenmotsu manifolds such that the structural vector field ξ is tangent to the submanifold and the Lee vector field $\omega^\sharp$ is either tangent or normal to the submanifold?
Before considering the answer of the above question, an example for the existence of submanifolds in conformal Kenmotsu manifolds tangent to ξ and either tangent or normal to $\omega^\sharp$ is constructed (see Section 3).
Corresponding to the above problem, first we consider semi symmetric submanifolds in conformal Kenmotsu manifolds and then using obtained results for this type of submanifolds, we give the following theorems for locally symmetric submanifolds in conformal Kenmotsu manifolds.

- Let $\acute{M}^m$ be a locally symmetric submanifold of a conformal Kenmotsu manifold M normal to $\omega^\sharp$. Then $\acute{M}$ is locally isometric to the hyperbolic space $\mathbb{H}^m(-\exp(f))$.
- There is not any locally symmetric submanifold $\acute{M}$ of a conformal Kenmotsu manifold M tangent to $\omega^\sharp$.

The present paper is organized as follows. In Section 2, we recall some definitions and notions about conformal Kenmotsu manifolds. Section 3 gives some preliminary lemmas on submanifolds of a conformal Kenmotsu manifold. Also, we present an example for the existence of submanifolds in conformal Kenmotsu manifolds tangent to ξ and either tangent or normal to $\omega^\sharp$. In sections 4 and 5, we consider semi-symmetric and locally symmetric submanifolds of a conformal Kenmotsu manifold tangent (normal) to the Lee vector field.

2. Conformal Kenmotsu Manifolds

A $(2n+1)$-dimensional differentiable manifold M is an almost contact metric manifold, if it admits an almost contact metric structure (φ, ξ, η, g) consisting of a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g and satisfying following conditions:

$$\varphi^2 = -Id + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\varphi\xi = 0, \qquad \eta o \varphi = 0, \qquad \eta(X) = g(X, \xi)$$

for all vector fields X, Y on the module of the vector fields $\chi(M)$ [6].
An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a Kenmotsu manifold and an α-Kenmotsu manifold if the following relations

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X \tag{2.1}$$

and

$$(\nabla_X \varphi)Y = \alpha\{-g(X, \varphi Y)\xi - \eta(Y)\varphi X\} \tag{2.2}$$

hold on M, respectively, where ∇ denotes the Riemannian connection of g and α is a constant function on M. From (2.1) for a Kenmotsu manifold, we have

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.3}$$

For a Kenmotsu manifold, we also have

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{2.4}$$

for all vector fields X, Y tangent to M, where R is the curvature tensor of M (see [8]).
A $(2n+1)$-dimensional smooth manifold M with almost contact metric structure (φ, η, ξ, g) is called a conformal Kenmotsu manifold if there exists a positive smooth function $f : M \to \mathbb{R}$ such that

$$\tilde{g} = \exp(f)g, \qquad \tilde{\xi} = (\exp(-f))^{\frac{1}{2}}\xi, \qquad \tilde{\eta} = (\exp(f))^{\frac{1}{2}}\eta, \qquad \tilde{\varphi} = \varphi$$

is a Kenmotsu structure on M (see [1]-[5]).
Let M be a conformal Kenmotsu manifold, with $\tilde{\nabla}$ and ∇ denote the Riemannian connections of M with respect to the metrics $\tilde{g}$ and g, respectively. Using the Koszul formula, one can simply obtain the following relation between $\tilde{\nabla}$ and ∇:

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp\} \tag{2.5}$$

for all vector fields X, Y on M, where $\omega(X) = g(grad\ f, X) = X(f)$. Note that the vector field $\omega^\sharp = grad\ f$ is called the Lee vector field of the conformal Kenmotsu manifold M. Then from $\eta(X) = g(X,\xi)$, we have the equality $\eta(\omega^\sharp) = \omega(\xi)$. Although $\omega(\omega^\sharp) = \|\omega^\sharp\|^2$, it is not necessarily $\|\omega^\sharp\|^2 = 1$, that is, $\omega^\sharp$ is not necessarily a unit vector field.
Assuming that $\tilde{R}$ and R are the curvature tensors of $(M, \varphi, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ and $(M, \varphi, \eta, \xi, g)$, respectively. We have the following relation between $\tilde{R}$ and R:

$$\begin{aligned}
\exp(-f)\tilde{g}(\tilde{R}(X,Y)Z,W) =& g(R(X,Y)Z,W) \\
&+ \frac{1}{2}\{B(X,Z)g(Y,W) - B(Y,Z)g(X,W) \\
&+ B(Y,W)g(X,Z) - B(X,W)g(Y,Z)\} \\
&+ \frac{1}{4}\|\omega^\sharp\|^2\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}
\end{aligned} \tag{2.6}$$

for all vector fields X, Y, Z, W on M, where B satisfies

$$B := \nabla\omega - \frac{1}{2}\omega \otimes \omega. \tag{2.7}$$

Obviously, B is a symmetric tensor field of type (0,2). On the other hand, from equations (2.1), (2.3) and (2.5), we get

$$\begin{aligned}
(\nabla_X\varphi)Y &= (\exp(f))^{\frac{1}{2}}\{-g(X,\varphi Y)\xi - \eta(Y)\varphi X\} \\
&- \frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X + g(X,Y)\varphi\omega^\sharp - g(X,\varphi Y)\omega^\sharp\},
\end{aligned} \tag{2.8}$$

$$\nabla_X\xi = (\exp(f))^{\frac{1}{2}}\{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\} \tag{2.9}$$

for all vector fields X, Y on M.
Note that, if the function of conformal change f be constant on the conformal Kenmotsu manifold M, i.e. $\omega^\sharp = 0$, then M is an α-Kenmotsu manifold in view of (2.2) and (2.8). In this paper, we suppose that the conformal Kenmotsu manifold M is non-α-Kenmotsu, that is, f is non-constant, so $\omega^\sharp$ is a non-zero vector field on M. Also, in the definition of the conformal Kenmotsu manifold M, we have assumed that f is non-zero, hence M is non-Kenmotsu by (2.8).

3. Submanifolds of Conformal Kenmotsu Manifolds

Let $(\acute{M}, \acute{g})$ be an m-dimensional submanifold of a $(2n+1)$-dimensional conformal Kenmotsu manifold (M, g). The Gauss and Weingarten formulas are given as

$$\nabla_X Y = \acute{\nabla}_X Y + h(X,Y), \qquad \nabla_X N = -A_N X + \nabla^\perp_X N$$

for all vector fields X, Y tangent to $\acute{M}$ and each vector field N normal to $\acute{M}$, where $\acute{\nabla}$ is the Riemannian connection of $\acute{M}$ determined by the induced metric $\acute{g}$ and $\nabla^\perp$ is the normal connection of $T^\perp\acute{M}$. It is known that $g(h(X,Y),N) = \acute{g}(A_N X, Y)$, where A_N is the shape operator of $\acute{M}$ with respect to unit normal vector field N.

The Gauss equation is given as

$$\begin{aligned}\exp(-f)\tilde{g}(\tilde{R}(X,Y)Z,W) =& \acute{g}(\acute{R}(X,Y)Z,W)\\ &- \acute{g}(h(X,W),h(Y,Z)) + \acute{g}(h(Y,W),h(X,Z))\\ &+ \frac{1}{2}\{B(X,Z)\acute{g}(Y,W) - B(Y,Z)\acute{g}(X,W)\\ &+ B(Y,W)\acute{g}(X,Z) - B(X,W)\acute{g}(Y,Z)\}\\ &+ \frac{1}{4}\|\omega^\sharp\|^2\{\acute{g}(X,Z)\acute{g}(Y,W) - \acute{g}(Y,Z)\acute{g}(X,W)\} \qquad (3.1)\end{aligned}$$

for all X, Y, Z, W tangent to $\acute{M}$, where $\acute{R}$ is the curvature tensor of $\acute{M}$.
In this paper, we assume that $\xi\mid_{\acute{M}}$ is tangent to $\acute{M}$.

3.1. Example

In this subsection, we construct an example of a five-dimensional conformal Kenmotsu manifold which is not Kenmotsu. Also, we present two submanifolds M_1 and M_2 in M such that the structural vector field ξ is tangent to both M_1 and M_2 and the Lee vector field $\omega^\sharp$ is tangent to M_1 and normal to M_2. We consider the five-dimensional manifold

$$M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5 \mid x_1 > 0, z \neq 0\},$$

where (x_1, x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$\begin{aligned}&e_1 = \exp(-z)\frac{\partial}{\partial x_1}, \qquad e_2 = \exp(-z)\frac{\partial}{\partial x_2}, \qquad e_3 = \exp(-z)\frac{\partial}{\partial y_1},\\ &e_4 = \exp(-z)\frac{\partial}{\partial y_2}, \qquad e_5 = (\exp(x_1))^{\frac{1}{2}}\frac{\partial}{\partial z},\end{aligned}$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = \exp(-x_1), \qquad g(e_5, e_5) = 1$$

and the remaining $g(e_i, e_j) = 0$, $i, j : 1, \cdots, 5$. Let η be the 1-form defined by $\eta(X) = g(X, e_5)$ for each vector field X on M. Thus, we have

$$\eta(e_1) = 0, \quad \eta(e_2) = 0, \quad \eta(e_3) = 0, \quad \eta(e_4) = 0, \quad \eta(e_5) = 1.$$

We define the $(1, 1)$-tensor field φ as

$$\varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2, \quad \varphi e_5 = 0.$$

Then using the linearity of φ and g, we have

$$\varphi^2 X = -X + \eta(X)e_5, \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y on M. Thus, for $e_5 = \xi$, (φ, ξ, η, g) defines an almost contact metric structure on M. Moreover, by the definition of bracket on manifolds we get

$$\begin{aligned}&[e_1, e_5] = (\exp(x_1))^{\frac{1}{2}}e_1 + \frac{1}{2}\exp(-z)e_5, \qquad [e_2, e_5] = (\exp(x_1))^{\frac{1}{2}}e_2,\\ &[e_3, e_5] = (\exp(x_1))^{\frac{1}{2}}e_3, \qquad [e_4, e_5] = (\exp(x_1))^{\frac{1}{2}}e_4\end{aligned}$$

and the remaining $[e_i, e_j] = 0$, $i, j : 1, \cdots, 5$. The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)\\ &- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),\end{aligned}$$

which is known as Koszul formula. By using this formula, we obtain

$$\begin{aligned}
&\nabla_{e_1}e_1=-\frac{1}{2}\exp(-z)e_1-(\exp(-x_1))^{\frac{1}{2}}e_5, \qquad \nabla_{e_1}e_2=-\frac{1}{2}\exp(-z)e_2,\\
&\nabla_{e_1}e_3=-\frac{1}{2}\exp(-z)e_3, \qquad \nabla_{e_1}e_4=-\frac{1}{2}\exp(-z)e_4,\\
&\nabla_{e_1}e_5=(\exp(x_1))^{\frac{1}{2}}e_1, \qquad \nabla_{e_2}e_1=-\frac{1}{2}\exp(-z)e_2,\\
&\nabla_{e_2}e_2=\frac{1}{2}\exp(-z)e_1-(\exp(-x_1))^{\frac{1}{2}}e_5, \qquad \nabla_{e_2}e_5=(\exp(x_1))^{\frac{1}{2}}e_2,\\
&\nabla_{e_3}e_1=-\frac{1}{2}\exp(-z)e_3, \qquad \nabla_{e_3}e_3=\frac{1}{2}\exp(-z)e_1-(\exp(-x_1))^{\frac{1}{2}}e_5,\\
&\nabla_{e_3}e_5=(\exp(x_1))^{\frac{1}{2}}e_3, \qquad \nabla_{e_4}e_1=-\frac{1}{2}\exp(-z)e_4,\\
&\nabla_{e_4}e_4=\frac{1}{2}\exp(-z)e_1-(\exp(-x_1))^{\frac{1}{2}}e_5, \qquad \nabla_{e_5}e_1=-\frac{1}{2}\exp(-z)e_5,\\
&\nabla_{e_5}e_5=\frac{1}{2}\exp(x_1-z)e_1, \qquad \nabla_{e_4}e_5=(\exp(x_1))^{\frac{1}{2}}e_4
\end{aligned}$$

and the remaining $\nabla_{e_i}e_j=0$, $i,j:1,\cdots,5$. By the following conformal change

$$\tilde{g}=\exp(x_1)g, \qquad \tilde{\xi}=(\exp(-x_1))^{\frac{1}{2}}\xi, \qquad \tilde{\eta}=(\exp(x_1))^{\frac{1}{2}}\eta, \qquad \tilde{\varphi}=\varphi,$$

$(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is a Kenmotsu manifold that we verify it as follows. The above conformal change can be written as

$$\begin{aligned}
&\tilde{\xi}=\frac{\partial}{\partial z}, \qquad \tilde{g}(e_1,e_1)=\tilde{g}(e_2,e_2)=\tilde{g}(e_3,e_3)=\tilde{g}(e_4,e_4)=\tilde{g}(\tilde{\xi},\tilde{\xi})=1,\\
&\tilde{\eta}(e_1)=\tilde{\eta}(e_2)=\tilde{\eta}(e_3)=\tilde{\eta}(e_4)=0, \qquad \tilde{\eta}(\tilde{\xi})=1,\\
&\tilde{\varphi}e_1=e_3, \quad \tilde{\varphi}e_2=e_4, \quad \tilde{\varphi}e_3=-e_1, \quad \tilde{\varphi}e_4=-e_2, \quad \tilde{\varphi}\tilde{\xi}=0.
\end{aligned}$$

Also, we have

$$[e_1,\tilde{\xi}]=e_1, \qquad [e_2,\tilde{\xi}]=e_2, \qquad [e_3,\tilde{\xi}]=e_3, \qquad [e_4,\tilde{\xi}]=e_4$$

and the remaining $[e_i,e_j]=0$, $i,j:1,\cdots,4$. Suppose that $\tilde{\nabla}$ is the Riemannian connection of the metric $\tilde{g}$. Using the Koszul formula, we get

$$\begin{aligned}
&\tilde{\nabla}_{e_1}e_1=-\tilde{\xi}, \qquad \tilde{\nabla}_{e_1}\tilde{\xi}=e_1, \qquad \tilde{\nabla}_{e_2}e_2=-\tilde{\xi},\\
&\tilde{\nabla}_{e_2}\tilde{\xi}=e_2, \qquad \tilde{\nabla}_{e_3}e_3=-\tilde{\xi}, \qquad \tilde{\nabla}_{e_3}\tilde{\xi}=e_3,\\
&\tilde{\nabla}_{e_4}e_4=-\tilde{\xi}, \qquad \tilde{\nabla}_{e_4}\tilde{\xi}=e_4, \qquad \tilde{\nabla}_{\tilde{\xi}}\tilde{\xi}=0
\end{aligned}$$

and the remaining $\tilde{\nabla}_{e_i}e_j=0$, $i,j:1,\cdots,4$. It can be easily considered that (2.1) holds on $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$. Thus, (M,φ,ξ,η,g) is a conformal Kenmotsu manifold but is not Kenmotsu, Since we have

$$(\nabla_X\varphi)Y\neq -g(X,\varphi Y)\xi-\eta(Y)\varphi X$$

for some vector fields X,Y on M (for instance, $(\nabla_{e_4}\varphi)e_2\neq -g(e_4,\varphi e_2)\xi-\eta(e_2)\varphi e_4$).
Suppose $M_1=\{(x_1,y_1,y_2,z)\in\mathbb{R}^4 \mid (x_1,y_1,y_2,z)\neq 0\}$ is a four-dimensional submanifold of M with the isometric immersion defined by

$$\begin{aligned}
&\iota_1: M_1\to M\\
&\iota(x_1,y_1,y_2,z)=(x_1,0,y_1,y_2,z),
\end{aligned}$$

where (x_1,y_1,y_2,z) are the standard coordinates in $\mathbb{R}^4$. We choose the vector fields

$$\begin{aligned}
&e_1=\exp(-z)\frac{\partial}{\partial x_1}, \qquad e_3=\exp(-z)\frac{\partial}{\partial y_1},\\
&e_4=\exp(-z)\frac{\partial}{\partial y_2}, \qquad e_5=(\exp(x_1))^{\frac{1}{2}}\frac{\partial}{\partial z},
\end{aligned}$$

which are linearly independent at each point of M_1. Then, e_1, e_3, e_4 and e_5 form a basis for the tangent space of M_1 and e_2 spans the normal space of M_1 in M. Let g_1 be the induced metric on M_1. Thus, we have

$$g_1(e_1, e_1) = g_1(e_3, e_3) = g_1(e_4, e_4) = \exp(-x_1), \qquad g_1(e_5, e_5) = 1.$$

Using $\omega(Y) = Y(x_1)$, for each vector field Y on M, it can be easily calculated that

$$\omega(e_1) = e_1(x_1) = \exp(-z),\ \ \omega(e_2) = 0,\ \ \omega(e_3) = 0,\ \ \omega(e_4) = 0,\ \ \omega(e_5) = 0$$

then $\omega^\sharp = \exp(x_1 - z)e_1$. We see that M_1 is a submanifold of the conformal Kenmotsu manifold M such that $\omega^\sharp \mid_{M_1}$ and $\xi \mid_{M_1}$ are tangent to M_1.
Now, let $M_2 = \{(x_2, y_1, y_2, z) \in \mathbb{R}^4 \mid (x_2, y_1, y_2, z) \neq 0\}$ be a four-dimensional submanifold of M with the isometric immersion defined by

$$\begin{aligned} &\iota_2 : (M_2, g_2) \to (M, g) \\ &\iota_2(x_2, y_1, y_2, z) = (2, x_2, y_1, y_2, z), \end{aligned}$$

where (x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^4$. We choose the vector fields

$$\begin{aligned} e_2 &= \exp(-z)\frac{\partial}{\partial x_2}, \qquad & e_3 &= \exp(-z)\frac{\partial}{\partial y_1}, \\ e_4 &= \exp(-z)\frac{\partial}{\partial y_2}, \qquad & e_5 &= \exp(1)\frac{\partial}{\partial z}, \end{aligned}$$

which are linearly independent at each point of M_2. Then, e_2, e_3, e_4 and e_5 form a basis for the tangent space of M_2 and e_1 spans the normal space of M_2 in M. Suppose g_2 is the induced metric on M_2. Then, we have

$$g_2(e_2, e_2) = g_2(e_3, e_3) = g_2(e_4, e_4) = \exp(-2), \qquad g_2(e_5, e_5) = 1.$$

Thus, M_2 is a submanifold of the conformal Kenmotsu manifold M such that $\xi \mid_{M_2}$ and $\omega^\sharp \mid_{M_2} = \exp(2 - z)e_1$ are tangent and normal to M_2, respectively, in view of the values $\omega(e_i)$ for all $i : 1, \cdots, 5$.
Now, we give some preliminary lemmas on the submanifold $\acute{M}$ of the conformal Kenmotsu manifold M tangent to ξ and either tangent or normal to $\omega^\sharp$.

Lemma 3.1. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp \mid_{\acute{M}}$ is normal to $\acute{M}$. Then*

$$B(X, Y) = -\omega(h(X, Y)), \tag{3.2}$$

$$h(X, \xi) = \frac{1}{2}\eta(X)\omega^\sharp, \tag{3.3}$$

$$\acute{\nabla}_X \xi = (\exp(f))^{\frac{1}{2}}\{X - \eta(X)\xi\} \tag{3.4}$$

for all vector fields X, Y tangent to $\acute{M}$.

Proof. From (2.7) we have

$$B(X, Y) = (\nabla_X \omega)Y - \frac{1}{2}\omega(X)\omega(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y) - \frac{1}{2}\omega(X)\omega(Y)$$

for all X, Y tangent to $\acute{M}$. Since $\omega^\sharp \mid_{\acute{M}}$ is normal to $\acute{M}$, the above equation can be written as

$$B(X, Y) = -\omega(\nabla_X Y)$$

for all X, Y on $\acute{M}$. Then by using the Gauss formula, we obtain (3.2).
Taking $Y = \xi$ in the Gauss formula and using (2.9), we have

$$\acute{\nabla}_X \xi + h(X, \xi) = \nabla_X \xi = (\exp(f))^{\frac{1}{2}}\{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\}$$

for each X tangent to $\acute{M}$. Since $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$, comparing the tangential part and the normal part in the above equation, we obtain (3.3) and (3.4). $\square$

Lemma 3.2. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is tangent to $\acute{M}$. Then*

$$B(X,Y) = \acute{g}(\acute{\nabla}_X\omega^\sharp, Y) - \frac{1}{2}\omega(X)\omega(Y), \tag{3.5}$$

$$h(X,\xi) = 0, \tag{3.6}$$

$$\acute{\nabla}_X\xi = (\exp(f))^{\frac{1}{2}}\{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\} \tag{3.7}$$

for all vector fields X, Y tangent to $\acute{M}$.

Proof. Similarly to the Lemma 3.1, equations (3.5), (3.6) and (3.7) are immediate results of (2.7), (2.9) and the Gauss formula. $\square$

Lemma 3.3. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$. Then*

$$\acute{R}(X,Y)\xi = \exp(f)\{\eta(X)Y - \eta(Y)X\}, \tag{3.8}$$

$$\acute{R}(X,\xi)Y = \exp(f)\{\acute{g}(X,Y)\xi - \eta(Y)X\} \tag{3.9}$$

for all vector fields X, Y tangent to $\acute{M}$.

Proof. Equation (3.8) follows from (2.4), (3.1), (3.2) and (3.3). Using the relation (3.8) and the symmetric property of $\acute{R}$, we get (3.9). $\square$

Lemma 3.4. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is tangent to $\acute{M}$. Then*

$$\begin{aligned}\acute{R}(X,Y)\xi =&(\exp(f) - \frac{1}{4}\parallel\omega^\sharp\parallel^2)\{\eta(X)Y - \eta(Y)X\}\\ &-\frac{1}{2}\{\eta(\acute{\nabla}_X\omega^\sharp)Y - \frac{1}{2}\omega(X)\omega(\xi)Y - \eta(\acute{\nabla}_Y\omega^\sharp)X + \frac{1}{2}\omega(Y)\omega(\xi)X\\ &+\eta(X)\acute{\nabla}_Y\omega^\sharp - \frac{1}{2}\eta(X)\omega(Y)\omega^\sharp - \eta(Y)\acute{\nabla}_X\omega^\sharp + \frac{1}{2}\eta(Y)\omega(X)\omega^\sharp\}\end{aligned} \tag{3.10}$$

for all vector fields X, Y tangent to $\acute{M}$.

Proof. Equations (2.4), (3.1), (3.5) and (3.6) yield (3.10). $\square$

Corollary 3.5. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then*

$$\begin{aligned}\acute{R}(X,Y)\xi =&(\exp(f) - \frac{1}{4}\parallel\omega^\sharp\parallel^2)\{\eta(X)Y - \eta(Y)X\}\\ &+\frac{1}{4}\{\omega(X)\omega(\xi)Y - \omega(Y)\omega(\xi)X + \eta(X)\omega(Y)\omega^\sharp - \eta(Y)\omega(X)\omega^\sharp\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\acute{R}(X,\xi)Y =&(\exp(f) - \frac{1}{4}\parallel\omega^\sharp\parallel^2)\{\acute{g}(X,Y)\xi - \eta(Y)X\}\\ &+\frac{1}{4}\{\omega(\xi)\acute{g}(X,Y)\omega^\sharp - \omega(\xi)\omega(Y)X + \omega(X)\omega(Y)\xi - \omega(X)\eta(Y)\omega^\sharp\}\end{aligned} \tag{3.12}$$

for all vector fields X, Y tangent to $\acute{M}$.

Proof. Equation (3.11) is an immediate resulte of (3.10). We obtain (3.12) by (3.11) and the symmetric property of $\acute{R}$. □

Lemma 3.6. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp |_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then*

$$\omega(\xi) \neq 0, \tag{3.13}$$
$$(\exp(f))^{\frac{1}{2}} - \frac{1}{2}\omega(\xi) \neq 0, \tag{3.14}$$
$$\alpha exp(f) + \beta(\|\omega^\sharp\|^2 - \omega(\xi)^2) \neq 0. \tag{3.15}$$

In (3.15), α and β are some non-zero constants.

Proof. The proof of the relations (3.13), (3.14) and (3.15) are given by contradiction.
Suppose $\omega(\xi) = 0$. Taking the covariant differentation of $\omega(\xi) = 0$ with respect to ξ and using $\acute{\nabla}\omega^\sharp = 0$, we obtain

$$\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = 0.$$

Using (3.7) in the above equation, we get

$$\|\omega^\sharp\|^2 = \omega(\xi)^2.$$

Since we have assumed that $\omega(\xi) = 0$, from the above equation it follows that $\|\omega^\sharp\|^2 = 0$ which contradicts the hypothesis $\omega^\sharp \neq 0$. Hence (3.13) holds on $\acute{M}$.
Now, we assume

$$(exp(f))^{\frac{1}{2}} - \frac{1}{2}\omega(\xi) = 0. \tag{3.16}$$

Taking the covariant differentation of (3.16) along vector field ξ, we have

$$\frac{1}{2}\omega(\xi)(exp(f))^{\frac{1}{2}} - \frac{1}{2}\xi(\omega(\xi)) = 0. \tag{3.17}$$

Using (3.16) and $\acute{\nabla}\omega^\sharp = 0$ in (3.17), we get $2\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = \omega(\xi)^2$. Then by making use of (3.7) in $2\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = \omega(\xi)^2$, we find $2\omega(\xi)^2 = \|\omega^\sharp\|^2$. As $\omega^\sharp$ is parallel on $\acute{M}$, it follows that $\|\omega^\sharp\|^2$ is constant on $\acute{M}$. Thus, $\omega(\xi)^2$ is constant on $\acute{M}$. Then from (3.17) we have $\omega(\xi) = 0$ which is a contradiction in view of (3.13).
Finally, we suppose

$$\alpha exp(f) + \beta(\|\omega^\sharp\|^2 - \omega(\xi)^2) = 0. \tag{3.18}$$

Taking the covariant differentation of (3.18) along vector field ξ and using $\acute{\nabla}\omega^\sharp = 0$, we find

$$\alpha\omega(\xi)exp(f) - 2\beta\omega(\xi)\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = 0.$$

Making use of (3.7) and (3.13) in the above equation, we get

$$\alpha exp(f) + \beta(\omega(\xi)^2 - \|\omega^\sharp\|^2) = 0. \tag{3.19}$$

Summing (3.18) to (3.19), we have $\alpha exp(f) = 0$ which is a contradiction.
Hence (3.13), (3.14) and (3.15) hold on $\acute{M}$. □

4. Semi-Symmetric Submanifolds

A submanifold $\acute{M}$ of a conformal Kenmotsu manifold M is said to be semi-symmetric if $\acute{R}(X,Y).\acute{R}=0$, for all vector fields X,Y tangent to $\acute{M}$ where $\acute{R}$ denotes the curvature tensor of $\acute{M}$.

Theorem 4.1. *Let $\acute{M}$ be a semi-symmetric submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$. Then $K(X,Y)=-\exp(f)$ for all vector fields X,Y orthogonal to ξ.*

Proof. Since $\acute{M}$ is semi-symmetric, we have

$$\begin{aligned} 0 &= \acute{R}(X,\xi)\acute{R}(X,Y)Y-\acute{R}(\acute{R}(X,\xi)X,Y)Y \\ &- \acute{R}(X,\acute{R}(X,\xi)Y)Y-\acute{R}(X,Y)\acute{R}(X,\xi)Y \end{aligned} \tag{4.1}$$

for all vector fields X,Y on $\acute{M}$. From (3.9), we obtain

$$\begin{aligned} 0 &= g(\acute{R}(X,Y)Y,X)\xi-\eta(\acute{R}(X,Y)Y)\xi-g(X,X)\acute{R}(\xi,Y)Y \\ &+ \eta(X)\acute{R}(X,Y)Y+\eta(Y)\acute{R}(X,Y)X \end{aligned}$$

for each X orthogonal to Y. From inner product the above equation with vector field ξ and by using (3.9), we get

$$g(\acute{R}(X,Y)Y,X)=-\exp(f)g(X,X)g(Y,Y)$$

for each plane $\{X,Y\}$ orthogonal to ξ. The above equation compelets the proof of the theorem. □

Theorem 4.2. *Let $\acute{M}$ be a semi-symmetric submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then*

$$K(X,Y)=-\exp(f)+\frac{1}{4}\parallel\omega^\sharp\parallel^2-\frac{\frac{1}{4}\exp(f)\parallel\omega^\sharp\parallel^2\,cos^2\theta}{\exp(f)-\frac{1}{4}\parallel\omega^\sharp\parallel^2\,sin^2\theta} \tag{4.2}$$

for all X,Y orthogonal to both ξ and $\omega^\sharp\mid_{\acute{M}}$, where θ denotes angle between ξ and $\omega^\sharp\mid_{\acute{M}}$.

Proof. Since $\acute{M}$ is semi-symmetric, we have

$$\begin{aligned} 0 &= \acute{R}(X,\xi)\acute{R}(X,Y)Y-\acute{R}(\acute{R}(X,\xi)X,Y)Y \\ &- \acute{R}(X,\acute{R}(X,\xi)Y)Y-\acute{R}(X,Y)\acute{R}(X,\xi)Y \end{aligned} \tag{4.3}$$

for all vector fields X,Y on $\acute{M}$. Since $\omega^\sharp\mid_{\acute{M}}$ is parallel on $\acute{M}$, so by using (3.12), we get

$$\begin{aligned} 0 &= (\exp(f)-\frac{1}{4}\parallel\omega^\sharp\parallel^2)(g(\acute{R}(X,Y)Y,X)\xi-\eta(\acute{R}(X,Y)Y)X) \\ &- \frac{1}{4}\{\omega(\acute{R}(X,Y)Y)\omega(\xi)X-\omega(\xi)g(\acute{R}(X,Y)Y,X)\omega^\sharp \\ &- \omega(\acute{R}(X,Y)Y)\omega(X)\xi+\eta(\acute{R}(X,Y)Y)\omega(X)\omega^\sharp\} \\ &- (\exp(f)-\frac{1}{4}\parallel\omega^\sharp\parallel^2)g(X,X)\acute{R}(\xi,Y)Y \\ &+ \frac{1}{4}\{\omega(X)\omega(\xi)\acute{R}(X,Y)Y-g(X,X)\omega(\xi)\acute{R}(\omega^\sharp,Y)Y \\ &- (\omega(X))^2\acute{R}(\xi,Y)Y-\omega(X)\omega(Y)\acute{R}(X,\xi)Y \\ &+ \omega(Y)\omega(\xi)\acute{R}(X,Y)X-\omega(X)\omega(Y)\acute{R}(X,Y)\xi\} \end{aligned}$$

for each plane $\{X, Y\}$ orthogonal to ξ. Taking X and Y orthogonal to $\omega^\sharp\mid_{\acute{M}}$ in the above equation, we obtain

$$\begin{aligned} 0 &= (\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2)(g(\acute{R}(X,Y)Y,X)\xi - \eta(\acute{R}(X,Y)Y)X) \\ &- \frac{1}{4}\{\omega(\acute{R}(X,Y)Y)\omega(\xi)X - \omega(\xi)g(\acute{R}(X,Y)Y,X)\omega^\sharp \\ &+ g(X,X)\omega(\xi)\acute{R}(\omega^\sharp,Y)Y\} - (\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2)g(X,X)\acute{R}(\xi,Y)Y. \end{aligned}$$

From inner product, the above equation with vector field ξ, we conclude

$$\begin{aligned} &(\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2 + \frac{1}{4}(\omega(\xi))^2)g(R(X,Y)Y,X) \\ = \ &(\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2)g(X,X)g(R(\xi,Y)Y,\xi) + \frac{1}{4}g(X,X)\omega(\xi)g(R(\omega^\sharp,Y)Y,\xi). \end{aligned}$$

Putting (3.12) in the above equation, we get

$$\begin{aligned} &(\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2 + \frac{1}{4}(\omega(\xi))^2)g(R(X,Y)Y,X) \\ = \ &-(\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2 + \frac{1}{4}(\omega(\xi))^2)(\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2)g(X,X)g(Y,Y) \\ - \ &\frac{1}{4}g(X,X)g(Y,Y)(\omega(\xi))^2. \end{aligned}$$

Then making use of (3.15), we find

$$\begin{aligned} g(R(X,Y)Y,X) &= -(\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2)g(X,X)g(Y,Y) \\ &- \frac{\frac{1}{4}g(X,X)g(Y,Y)(\omega(\xi))^2}{\exp(f) - \frac{1}{4}\parallel \omega^\sharp \parallel^2 + \frac{1}{4}(\omega(\xi))^2} \end{aligned}$$

for each plane $\{X, Y\}$ orthogonal to both ξ and $\omega^\sharp\mid_{\acute{M}}$. Using $\omega(\xi) = \parallel \omega^\sharp \parallel cos\theta$ (where θ denotes angle between ξ and $\omega^\sharp\mid_{\acute{M}}$) in the above equation, we get (4.2). Thus theorem is proved. □

5. Locally Symmetric Submanifolds

A submanifold $\acute{M}$ of a conformal Kenmotsu manifold M is said to be locally symmetric such that $\acute{\nabla}\acute{R} = 0$ where $\acute{\nabla}$ and $\acute{R}$ denote the Riemannian connention and curvature tensor of $\acute{M}$, respectively.

Theorem 5.1. *Let $\acute{M}^m$ be a locally symmetric submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$. Then $\acute{M}$ is locally isometric to the hyperbolic space $\mathbb{H}^m(-\exp(f))$.*

Proof. Since $\acute{M}$ is locally symmetric, we can write

$$0 = \acute{\nabla}_Z(\acute{R}(X,Y)\xi) - \acute{R}(\acute{\nabla}_Z X,Y)\xi - \acute{R}(X,\acute{\nabla}_Z Y)\xi - \acute{R}(X,Y)\acute{\nabla}_Z\xi.$$

From (3.8) and (3.4), we get

$$\begin{aligned} \acute{R}(X,Y)Z &= (\exp(f))^{\frac{1}{2}}\omega(X)\{\eta(Y)X - \eta(X)Y\} \\ &- \exp(f)\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned}$$

Since $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$, we obtain

$$\acute{R}(X,Y)Z = -\exp(f)\{g(Y,Z)X - g(X,Z)Y\} \tag{5.1}$$

for all vector fields X, Y, Z on $\acute{M}$. Note that f is constant on $\acute{M}$ because $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$. Thus the proof of the theorem is completed. □

Lemma 5.2. *Let $\acute{M}$ be a locally symmetric submanifold of a conformal Kenmotsu manifold M such that $\omega^{\sharp}\mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then*

$$\begin{aligned} & \acute{R}(X,Y)Z = (\exp(f) - \frac{1}{4} \parallel \omega^{\sharp} \parallel^2)\{g(X,Z)Y - g(Y,Z)X\} \\ + & \frac{2\exp(f)}{2(\exp(f))^{\frac{1}{2}} - \omega(\xi)}(\omega(Z) - \eta(Z)\omega(\xi))\{\eta(X)Y - \eta(Y)X\} \\ - & \frac{1}{4}\{\omega(Y)\omega(Z)X - \omega(X)\omega(Z)Y - g(X,Z)\omega(Y)\omega^{\sharp} + g(Y,Z)\omega(X)\omega^{\sharp}\} \end{aligned} \tag{5.2}$$

for all vector fields X, Y, Z on $\acute{M}$.

Proof. Since $\omega^{\sharp}\mid_{\acute{M}}$ is parallel on $\acute{M}$, taking the covariant differentiation of (3.11) along vector field Z tangent to $\acute{M}$, we have

$$\begin{aligned} & \acute{R}(X,Y)\acute{\nabla}_Z\xi = \exp(f)\omega(Z)\{\eta(X)Y - \eta(Y)X\} \\ + & (\exp(f) - \frac{1}{4} \parallel \omega^{\sharp} \parallel^2)\{g(X,\acute{\nabla}_Z\xi)Y - g(Y,\acute{\nabla}_Z\xi)X\} - \frac{1}{4}\{\omega(Y)\omega(\acute{\nabla}_Z\xi)X \\ - & \omega(X)\omega(\acute{\nabla}_Z\xi)Y - g(X,\acute{\nabla}_Z\xi)\omega(Y)\omega^{\sharp} + g(Y,\acute{\nabla}_Z\xi)\omega(X)\omega^{\sharp}\}, \end{aligned}$$

then making use of (3.7) in the above equation, it follows that

$$\begin{aligned} & ((\exp(f))^{\frac{1}{2}} - \frac{1}{2}\omega(\xi))\acute{R}(X,Y)Z \\ = & \exp(f)\omega(Z)\{\eta(X)Y - \eta(Y)X\} \\ + & ((\exp(f))^{\frac{1}{2}} - \frac{1}{2}\omega(\xi))\{(\exp(f) - \frac{1}{4} \parallel \omega^{\sharp} \parallel^2)\{g(X,Z)Y - g(Y,Z)X\} \\ - & \frac{1}{4}\{\omega(Y)\omega(Z)X - \omega(X)\omega(Z)Y + \omega(X)g(Y,Z)\omega^{\sharp} - \omega(Y)g(X,Z)\omega^{\sharp}\}\} \end{aligned}$$

for each vector field Z orthogonal to ξ. Then by using (3.14) in the above equation, we obtain

$$\begin{aligned} & \acute{R}(X,Y)Z = (\exp(f) - \frac{1}{4} \parallel \omega^{\sharp} \parallel^2)\{g(X,Z)Y - g(Y,Z)X\} \\ + & \frac{2\exp(f)}{2(\exp(f))^{\frac{1}{2}} - \omega(\xi)}\omega(Z)\{\eta(X)Y - \eta(Y)X\} - \frac{1}{4}\{\omega(Y)\omega(Z)X - \omega(X)\omega(Z)Y \\ + & \omega(X)g(Y,Z)\omega^{\sharp} - \omega(Y)g(X,Z)\omega^{\sharp}\} \end{aligned}$$

for each vector field Z orthogonal to ξ. For all vector fields X, Y, Z, using (3.11), we get (5.2). □

Corollary 5.3. *Let $\acute{M}$ be a locally symmetric submanifold of a conformal Kenmotsu manifold M such that $\omega^{\sharp}\mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then*

$$K(X,Y) = -\exp(f) + \frac{1}{4} \parallel \omega^{\sharp} \parallel^2 \tag{5.3}$$

for all vector fields X, Y orthogonal to both ξ and $\omega^{\sharp}\mid_{\acute{M}}$.

Proof. Equation (5.3) yields (5.2). □

Theorem 5.4. *There is no locally symmetric submanifold $\acute{M}$ of a conformal Kenmotsu manifold M such that $\omega^{\sharp}\mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$.*

Proof. Let $\acute{M}$ be locally symmetric. Since any Locally symmetric manifold is semi-symmetric, from Theorem 4.2, we have

$$K(X,Y) = -\exp(f) + \frac{1}{4} \parallel \omega^\sharp \parallel^2 - \frac{\frac{1}{4}\exp(f) \parallel \omega^\sharp \parallel^2 cos^2\theta}{\exp(f) - \frac{1}{4} \parallel \omega^\sharp \parallel^2 sin^2\theta} \tag{5.4}$$

for all vector fields X, Y orthogonal to both ξ and $\omega^\sharp \mid_{\acute{M}}$. Comparing (5.3) and (5.4), we get $cos^2\theta = 0$. Hence, $\omega(\xi) = 0$, that is a contradiction in view of (3.13). □

References

1. Abdi, R. and Abedi, E., Invariant and anti-invariant submanifolds of a conformal Kenmotsu manifold. Azerbaijan J. Math. 5, 54-63, (2015).

2. Abdi, R. and Abedi, E., *CR-hypersurfaces of a conformal Kenmotsu manifold satisfying certain shape operator conditions*, Periodica Mathematica Hungarica 73, 83-92, (2016).

3. Abdi, R. and Abedi, E., *On the Ricci tensor of submanifolds in conformal Kenmotsu manifolds*, Kyushu Journal of Math. 71, 257-269, (2017).

4. Abdi, R. and Abedi, E., *On normal Jacobi operator of CR-hypersurfaces in conformal Kenmotsu space forms*, Siberian Math. Journal 58, 923-931, (2017).

5. Abdi, R., External geometry of submanifolds in conformal Kenmotsu manifolds, Bulletin of the Iranian Mathematical Society, DOI: 10.1007/s41980-018-0082-1.

6. Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Birkhauser, Boston-Basel-Berlin, (2002).

7. Gray A. and Hervella, L. M., *Sixteen classes of almost Hermitian manifold and their linear invariants*, Ann. Math. Pure Appl. 123, 35-58, (1980).

8. Kenmotsu, K., *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. 24, 93-103, (1972).

9. Kobayashi, M., *Semi-invariant submanifolds of a certain class of almost contact manifolds*, Tensor (N.S.) 43, 28-36, (1986).

Roghayeh Abdi,
Department of Mathematics,
Azarbaijan Shahid Madani University,
Iran.
E-mail address: `rabdi@azaruniv.ac.ir`