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Three Nontrivial Solutions of Boundary Value Problems for Semilinear $\Delta_\gamma-$Laplace Equation *

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ABSTRACT: In this paper, we study the multiplicity of weak solutions to the boundary value problem

$$\Delta_\gamma u + f(x,u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$ and Δ_γ is the subelliptic operator of the type

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \ \partial_{x_j} := \frac{\partial}{\partial x_j}, \gamma = (\gamma_1, \gamma_2, ..., \gamma_N),$$

the nonlinearity $f(x,\xi)$ is subcritical growth and may be not satisfy the Ambrosetti-Rabinowitz (AR) condition. We establish the existence of three nontrivial solutions by using Morse theory.

Key Words: Semilinear degenerate elliptic equations, Morse theory, Three solutions, Multiple solutions.

Contents

1. Introduction

In the last decades, the boundary value problem for semilinear elliptic equations

$$-\Delta u = f(x,u), \quad x \in \Omega, \qquad u \in H_0^1(\Omega),$$

has been studied by many authors, see, for example [1,20] and the references therein. The following (AR) condition introduced in [1]

(AR) For some $\theta > 2$ and $R > 0$, we have

$$\theta F(x,\xi) \leq f(x,\xi)\xi, \quad \forall\, |\xi| \geq R, \quad \forall\, x \in \Omega,$$

where $F(x,\xi) = \int_0^\xi f(x,\tau)\,d\tau$, plays an important role in their studies. Boundary value problems for nonlinear degenerate elliptic differential equations were treated in [10] and then subsequently in [8,5]. In [25,26], the critical exponent phenomenon was observed for a model of the Grushin type operators. The results were then generalized in [23] to a large class of semilinear degenerate elliptic differential equations. Recently, in [23,24] the second author of this paper and his colaborator have extended the research to a more complicated class of nonlinear degenerate elliptic differential operators. Very recently, the authors of [11] investigated the $\Delta_\gamma-$Laplace operator under the additional assumption that the operator is homogeneous of degree two with respect to a semigroup of dilations in $\mathbb{R}^N$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [27,28] (see also some recent results in [2,3,11,12,13,14,15,16,17,18,19,22,24,26]).

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In this paper, we study multiplicity of weak solutions to the following problem

$$\Delta_\gamma u + f(x,u) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$
$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, Δ_γ (see the definition of this function space below) and $f(x,\xi) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x,0) = 0$.

Let $F(x,\xi) = \int_0^\xi f(x,\tau)d\tau$ and suppose that the non-linearity f satisfies the following conditions:

(A1) $f \in C(\bar{\Omega}\times\mathbb{R},\mathbb{R})$ with $f(x,0) = 0$ and satisfies the improved subcritical polynomial growth condition, i.e.

$$\lim_{\xi\to\infty} \frac{f(x,\xi)}{|\xi|^{2^*_\gamma-1}} = 0 \quad \text{uniformly for } x \in \bar{\Omega},$$

where $2^*_\gamma := 2\widetilde{N}/(\widetilde{N}-2)$;

(A2) $\lim\limits_{|\xi|\to 0} \frac{f(x,\xi)}{\xi} = p(x)$, uniformly for $x \in \Omega$, where $p \in L^\infty(\Omega)$ satisfies $p(x) \le \lambda_1$ for all $x \in \Omega$ and $p(x) < \lambda_1$ on some $\Omega_0 \subset \Omega_1$ with $|\Omega_0| > 0$, where $\Omega_1 := \{x \in \Omega : \phi_1(x) \neq 0\}$ and $\lambda_1 > 0$ that has an associated eigenfunction ϕ_1 is the first eigenvalue of $-\Delta_\gamma$ with homogeneous Dirichlet boundary data;

(A3) $f(x,\xi)$ is superlinear at infinity, i.e. $\lim\limits_{|\xi|\to+\infty} f(x,\xi)/\xi = +\infty$ uniformly for all $x \in \Omega$;

(A4) There exist $\theta \ge 1$ and $C(x) \in L^1_+(\Omega)$ such that $\theta\mathcal{F}(x,\xi) \ge \mathcal{F}(x,s\xi) - C(x)$ for $(x,\xi) \in \Omega\times\mathbb{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,\xi) = f(x,\xi) - 2F(x,\xi)$.

The condition (A4) was first introduced by L. Jeanjean [7], there are many functions which satisfy (A4), but do not satisfy the (AR) condition. An example of such function is

$$f(x,\xi) = \xi\ln(1+|\xi|).$$

Our main result is given by the following theorem.

Theorem 1.1. *Assume conditions* (A1)-(A4) *hold. Then the problem* (1.1)–(1.2) *has at least three nontrivial solutions.*

The structure of our note is as follows: In Section 2, we give some preliminary results. In Section 3, we proved Theorem 1.1.

2. Preliminary results

First of all, let us collect some concepts and results of Morse theory that will be used below. For the details, we refer to [4]. Let $\mathbb{X}$ be a real Banach space and $\Phi \in C^1(\mathbb{X},\mathbb{R})$. $K = \{u \in \mathbb{X} | \Phi'(u) = 0\}$ is the critical set of Φ. Let $u \in K$ be an isolated critical point of Φ with $\Phi(u) = c \in \mathbb{R}$, and U be an isolated neighborhood of u, i.e. $K \cap U = \{u\}$. The group

$$C_m(\Phi,u) = H_m(\Phi^c \cap U, \Phi^c \cap U\backslash\{u\}), \quad m = 0,1,2,\ldots,$$

is called the m-th critical group of Φ at u, where $\Phi^c = \{u \in \mathbb{X} | \Phi(u) \le c\}$.

$H_m(\cdot,\cdot)$ is the singular relative homology group of Φ at infinity is defined by

$$C_m(\Phi,\infty) = H_m(\mathbb{X},\Phi^a), \quad m = 0,1,2,\ldots.$$

We denote

$$P(u,t) = \sum_i \operatorname{rank} C_i(\Phi,u)t^i, \quad P(\infty,t) = \sum_i \operatorname{rank} C_i(\Phi,\infty)t^i.$$

Let $\alpha < \beta$ be the regular values of Φ and set

$$P(\alpha, \beta, t) = \sum_i \operatorname{rank} C_i(\Phi, \infty) t^i.$$

If $K = \{u_1, u_2, \ldots, u_k\}$, then there is a polynomial $Q(t)$ with nonnegative integer as its coefficients such that

$$\sum_j P(u_j, t) = P(\infty, t) + (1+t)Q(t), \tag{2.1}$$

$$\sum_{\alpha<\Phi(u_j)<\beta} P(u_j, t) = P(\alpha, \beta, t) + (1+t)Q(t). \tag{2.2}$$

Throughout the paper Ω denotes a bounded domain with smooth boundary in $\mathbb{R}^N, N \geq 2$. As in [11], we consider the operators of the form

$$\Delta_\gamma := \sum_{j=1}^N \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right),\ \partial_{x_j} := \frac{\partial}{\partial x_j}, j = 1, 2, \ldots, N.$$

Here, the functions $\gamma_j : \mathbb{R}^N \longrightarrow \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \backslash \Pi$, where

$$\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^N x_j = 0 \right\}.$$

Moreover, we assume the following properties:
i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that

$$\delta_t : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \delta_t(x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N), 1 = \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N,$$

such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.,

$$\gamma_j(\delta_t(x)) = t^{\varepsilon_j - 1}\gamma_j(x), \forall x \in \mathbb{R}^N, \forall t > 0,\ j = 1, \ldots, N.$$

The number

$$\widetilde{N} := \sum_{j=1}^N \varepsilon_j$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.
ii)

$$\gamma_1 = 1, \gamma_j(x) = \gamma_j(x_1, x_2, \ldots, x_{j-1}),\ j = 2, \ldots, N.$$

iii) There exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k}\gamma_j(x) \leq \rho\gamma_j(x), \forall k \in \{1, 2, \ldots, j-1\}, \forall j = 2, \ldots, N,$$

and for every $x \in \overline{\mathbb{R}}_+^N := \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \geq 0, \forall j = 1, 2, \ldots, N\right\}$.
iv) Equalities $\gamma_j(x) = \gamma_j(x^*)\ (j = 1, 2, \ldots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = (|x_1|, \ldots, |x_N|) \text{ if } x = (x_1, x_2, \ldots, x_N).$$

Definition 2.1. *By $S_\gamma^p(\Omega)$ $(1 \leq p < +\infty)$ we will denote the set of all functions $u \in L^p(\Omega)$ such that $\gamma_j \partial_{x_j} u \in L^p(\Omega)$ for all $j = 1, \ldots, N$. We define the norm in this space as follows*

$$\|u\|_{S_\gamma^p(\Omega)} = \left\{ \int_\Omega \left(|u|^p + \sum_{j=1}^N \left|\gamma_j \partial_{x_j} u\right|^p \right) dx \right\}^{\frac{1}{p}}.$$

If $p = 2$ we can also define the scalar product in $S^2_\gamma(\Omega)$ as follows

$$(u, v)_{S^2_\gamma(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{j=1}^{N} (\gamma_j \partial_{x_j} u, \gamma_j \partial_{x_j} v)_{L^2(\Omega)}.$$

The space $S^p_{\gamma,0}(\Omega)$ is defined as the closure of $C^1_0(\Omega)$ in the space $S^p_\gamma(\Omega)$.

Set

$$\nabla_\gamma u := (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u), |\nabla_\gamma u| := \Big(\sum_{j=1}^{N} \left|\gamma_j \partial_{x_j} u\right|^2 \Big)^{\frac{1}{2}}.$$

From Proposition 3.2 and Theorem 3.3 in [11], we have the following embedding result.

Proposition 2.1. *Assume that $\widetilde{N} > 2$. Then $S^2_{\gamma,0}(\Omega) \hookrightarrow L^p(\Omega)$, where $1 \le p \le \dfrac{2\widetilde{N}}{\widetilde{N} - 2}$. Moreover, the number $2^*_\gamma = \dfrac{2\widetilde{N}}{\widetilde{N} - 2}$ is the critical Sobolev exponent of the embedding $S^2_{\gamma,0}(\Omega) \hookrightarrow L^p(\Omega)$ and when $1 \le p < 2^*_\gamma$, the embedding is compact.*

We now give some examples of the $\Delta_\gamma-$Laplace operator. We use the following notations: we split $\mathbb{R}^N$ into

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

and write

$$x = \left(x^{(1)}, x^{(2)}, x^{(3)}\right),\ x^{(i)} = \left(x^{(i)}_1, x^{(i)}_2, \ldots, x^{(i)}_{N_i}\right) \in \mathbb{R}^{N_i},$$

$$|x^{(i)}|^2 = \sum_{j=1}^{N_i} |x^{(i)}_j|^2,\ i = 1, 2, 3.$$

We denote the classical Laplace operator in $\mathbb{R}^{N_i}$ by

$$\Delta_{x^{(i)}} := \sum_{j=1}^{N_i} \partial^2_{x^{(i)}_j}.$$

Example 2.2. *Let α be a real positive number. The operator*

$$\Delta_\gamma := \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} (\Delta_{x^{(2)}} + \Delta_{x^{(3)}}),$$

where

$$\gamma = (\underbrace{1, 1, \ldots, 1}_{N_1-times}, \underbrace{|x^{(1)}|^\alpha, \ldots, |x^{(1)}|^\alpha}_{(N_2+N_3)-times}),$$

is called the Grushin operator (see [6]).

Example 2.3. *Let α, β be nonnegative real numbers. The operator*

$$\Delta_\gamma := \Delta_{x^{(1)}} + \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha} |x^{(2)}|^{2\beta} \Delta_{x^{(3)}},$$

where

$$\gamma = (\underbrace{1, 1, \ldots, 1}_{(N_1+N_2)-times}, \underbrace{|x^{(1)}|^\alpha |x^{(2)}|^\beta, \ldots, |x^{(1)}|^\alpha |x^{(2)}|^\beta}_{N_3-times}),$$

is called the strongly degenerate elliptic operators (see [24,28]).

Definition 2.4. *A function $u \in S^2_{\gamma,0}(\Omega)$ is called a weak solution of the problem (1.1)–(1.2) if the identity*

$$\int\limits_{\Omega} \nabla_\gamma u \cdot \nabla_\gamma \varphi \, dx - \int\limits_{\Omega} f(x,u)\, \varphi \, dx = 0,$$

holds for every $\varphi \in C_0^\infty(\Omega)$.

Definition 2.5. *Let $\mathbb{X}$ be a real Banach space with its dual space $\mathbb{X}^*$ and $\Phi \in C^1(\mathbb{X},\mathbb{R})$. The functional Φ is said to satisfy Cerami condition at level $c \in \mathbb{R}$ ($(C)_c$ condition for short) if for any sequence $\{x_m\}_{m=1}^\infty \subset \mathbb{X}$ with*

$$\Phi(x_m) \to c \text{ and } (1 + \|x_m\|_{\mathbb{X}}) \|\Phi'(x_m)\|_{\mathbb{X}^*} \to 0,$$

then there exists a subsequence $\{x_{m_k}\}_{k=1}^\infty$ that converges strongly in $\mathbb{X}$. Φ satisfies the (C) condition if Φ satisfies $(C)_c$ condition at every $c \in \mathbb{R}$.

3. Proof of the main result

First, we observe that the problem (1.1)–(1.2) has a variational structure. Indeed it is the Euler-Lagrange equation of the functional $\Phi : S^2_{\gamma,0}(\Omega) \to \mathbb{R}$ defined as follows:

$$\Phi(u) = \frac{1}{2} \int\limits_{\Omega} |\nabla_\gamma u|^2 dx - \int\limits_{\Omega} F(x,u)\, dx,$$

By the hypotheses on f , we can see that the functional Φ is Frechét differentiable in $S^2_{\gamma,0}(\Omega)$ and for any $\varphi \in S^2_{\gamma,0}(\Omega)$,

$$\langle \Phi'(u), \varphi \rangle = \int\limits_{\Omega} \nabla_\gamma u \cdot \nabla_\gamma \varphi dx - \int\limits_{\Omega} f(x,u)\varphi dx.$$

Thus, critical points of Φ are solutions of problem (1.1)–(1.2).

Let

$$f_+(x,\xi) = \begin{cases} f(x,\xi), & \xi > 0, \\ 0, & \xi \le 0; \end{cases}$$

$$\Phi_\pm(u) = \frac{1}{2} \int\limits_{\Omega} |\nabla_\gamma u|^2 dx - \int\limits_{\Omega} F_\pm(x,u)\, dx,$$

where $F_\pm(x,\xi) = \int_0^\xi f_\pm(x,\tau) d\tau$. Now, we prove the following compactness condition for Φ and $\Phi_\pm$.

Lemma 3.1. *Let* (A1)-(A4) *be satisfied. Then the functionals Φ and $\Phi_\pm$ satisfies the (C) condition on $S^2_{\gamma,0}(\Omega)$.*

Proof. We only give the proof for Φ_+, the cases of Φ and Φ_- are similar. Let $\{u_n\}_{n=1}^\infty \subset S^2_{\gamma,0}(\Omega)$ be a sequence such that

$$\Phi_+(u_n) \to c, \ \left(1 + \|u_n\|_{S^2_{\gamma,0}(\Omega)}\right) \|\Phi'_+(u_n)\|_{(S^2_{\gamma,0}(\Omega))^*} \to 0, \quad \text{as } n \to \infty. \tag{3.1}$$

The proof of this lemma, we divide two steps:

Step 1. We first prove that $\{u_n\}_{n=1}^\infty$ is bounded in $S^2_{\gamma,0}(\Omega)$. Let $u_n^+ = \max\{u_n, 0\}$, $u_n^- = \min\{u_n, 0\}$. From (3.1), we obtain

$$|\langle \Phi'_+(u_n), \varphi \rangle| \le \epsilon_n \|\varphi\|_{S^2_{\gamma,0}(\Omega)} \quad \text{for any } \varphi \in S^2_{\gamma,0}(\Omega), \tag{3.2}$$

where $\epsilon_n \to 0$ as $n \to \infty$, then the boundedness of u_n^- can be directly obtained. For the case of u_n^+, by contradiction, we assume that $\|u_n^+\|_{S^2_{\gamma,0}(\Omega)} \to \infty$ as $n \to \infty$. Let $v_n = \|u_n^+\|^{-1}_{S^2_{\gamma,0}(\Omega)} u_n^+$, then

$\|v_n\|_{S^2_{\gamma,0}(\Omega)} = 1$. By Proposition 2.1, up to a subsequence, we have

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{weakly in } S^2_{\gamma,0}(\Omega) \text{ as } n\to\infty,\\ v_n &\to v \quad \text{strongly in } L^q(\Omega) \text{ as } n\to\infty,\\ v_n &\to v \quad \text{a.e. in } \Omega \text{ as } n\to\infty. \end{aligned}$$

Case 1. If $v \neq 0$ then the Lebesgue measure of $\Omega_0 = \{x\in\Omega : v(x)\neq 0\}$ is positive. Using (3.1), we obtain

$$\langle \Phi'_+(u_n), u_n^+\rangle = o(1),$$

which implies that

$$\int_\Omega \frac{f_+(x,u_n^+)u_n^+}{\|u_n^+\|^2_{S^2_{\gamma,0}(\Omega)}}dx = \int_\Omega \frac{f_+(x,u_n^+)u_n^+}{|u_n^+|^2}|v_n|^2dx = 1+o(1). \tag{3.3}$$

By (A3), there is a constant $M>0$ such that

$$f_+(x,u_n^+)u_n^+ > 0, \quad \text{as } |u_n| > M,$$

then we have

$$\int_{\Omega\setminus\Omega_0} \frac{f_+(x,u_n^+)u_n^+}{(u_n^+)^2}|v_n|^2dx \geq -C. \tag{3.4}$$

On the other hand, for $x\in\Omega_0$, $u_n^+\to\infty$ as $n\to\infty$. Then by the Fatou's lemma and (A3) we have

$$\int_{\Omega_0} \frac{f_+(x,u_n^+)u_n^+}{(u_n^+)^2}|v_n|^2dx \to \infty, \quad \text{as } n\to\infty.$$

Combining this with (3.4) gives

$$\int_{\Omega} \frac{f_+(x,u_n^+)u_n^+}{(u_n^+)^2}|v_n|^2dx \to \infty, \quad \text{as } n\to\infty. \tag{3.5}$$

This contradicts (3.3). Then this case is impossible.

Case 2. If $v\equiv 0$ then for any $n\in\mathbb{N}$ there exists $t_n\in[0,1]$ such that

$$\Phi_+(t_nu_n^+) = \max_{t\in[0,1]} \Phi_+(tu_n^+).$$

For any $R>0$, we assume that $w_n = 2\sqrt{R}v_n$. Then $w_n\to 0$ in $L^q(\mathbb{R}^N)$. So from conditions (A1) and (A2), for every $\epsilon>0$, we can find a constant $C(\epsilon)>0$ such that

$$F(x,w_n) \leq C(\epsilon)(w_n)^2 + \epsilon(w_n)^{2^*_\gamma}, \tag{3.6}$$

which implies

$$\lim_{n\to\infty}\int_\Omega F_+(x,w_n)dx = 0. \tag{3.7}$$

Since $2\sqrt{R}\|u_n^+\|^{-1}_{S^2_{\gamma,0}(\Omega)} \in (0,1)$ for n large enough, by (3.7) we obtain

$$\Phi_+(t_nu_n^+) \geq \Phi_+(w_n) = 2R - \int_\Omega F_+(x,w_n)dx \geq R,$$

which implies

$$\Phi_+(t_nu_n^+)\to\infty, \quad \text{as } n\to\infty. \tag{3.8}$$

From $\Phi_+(0) = 0$ and $\Phi_+(u_n^+) \to c$ we have $t_n \in (0,1)$, then

$$\langle \Phi'_+(t_n u_n^+), t_n u_n^+ \rangle = t_n \frac{d}{dt}\Big|_{t=t_n} \Phi_+(t u_n) = 0.$$

Then, from (A4) it follows that

$$\begin{aligned}\frac{1}{\theta}\Phi_+(t_n u_n^+) &= \frac{1}{\theta}\Big(\Phi_+(t_n u_n^+) - \frac{1}{2}\langle \Phi'_+(t_n u_n^+), t_n u_n^+\rangle\Big)\\ &= \frac{1}{2\theta}\int\limits_\Omega \mathcal{F}(x, t_n u_n^+)dx\\ &\le \frac{1}{2}\int\limits_\Omega \mathcal{F}(x, u_n^+)dx + \frac{1}{2\theta}\int\limits_\Omega C(x)dx\\ &= \Phi_+(u_n^+) - \frac{1}{2}\langle \Phi'_+(u_n^+), u_n^+\rangle + c \to C.\end{aligned}$$

This contradicts that $\Phi_+(t_n u_n^+) \to \infty$. Hence $\{u_n\}_{n=1}^\infty$ is bounded; that is, there exists a positive constant M such that

$$\|u_n\|_{S^2_{\gamma,0}(\Omega)} \le M, \quad \text{for all } n \in \mathbb{N}.$$

Step 2. We prove $\{u_n\}_{n=1}^\infty$ has a convergent subsequence. In fact, we can suppose that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } S^2_{\gamma,0}(\Omega) \text{ as } n\to\infty,\\ u_n &\to u \quad \text{strongly in } L^q(\Omega) \text{ as } n\to\infty,\\ u_n &\to u \quad \text{a.e. in } \Omega \text{ as } n \to \infty.\end{aligned}$$

Now, since Ω is a bounded set, for every $\epsilon > 0$, we can find a constant $C(\epsilon) > 0$ such that

$$f_+(x,s) \le C(\epsilon) + \epsilon |s|^{2^*_\gamma - 1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

then

$$\begin{aligned}&\Big|\int\limits_\Omega f_+(x,u_n)(u_n - u)dx\Big|\\ &\le C(\epsilon)\int\limits_\Omega |u_n - u|dx + \epsilon\int\limits_\Omega |u_n - u||u_n|^{2^*_\gamma - 1}dx\\ &\le C(\epsilon)\int\limits_\Omega |u_n - u|dx + \epsilon\Big(\int\limits_\Omega \big(|u_n|^{2^*_\gamma-1}\big)^{\frac{2^*_\gamma}{2^*_\gamma-1}}dx\Big)^{\frac{2^*_\gamma-1}{2^*_\gamma}}\Big(\int\limits_\Omega |u_n - u|^{2^*_\gamma}dx\Big)^{1/2^*_\gamma}\\ &\le C(\epsilon)\int\limits_\Omega |u_n - u|dx + \epsilon C(\Omega).\end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in $S^2_{\gamma,0}(\Omega)$, it follows that $\int\limits_\Omega |u_n - u|dx \to 0$. Since $\epsilon > 0$ is arbitrary, we can conclude that

$$\int\limits_\Omega (f_+(x,u_n) - f_+(x,u))(u_n - u)dx \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

By (3.9), we have

$$\langle \Phi'_+(u_n) - \Phi'_+(u), (u_n - u)\rangle \to 0 \quad \text{as } n\to\infty. \tag{3.10}$$

From (3.9) and (3.10), we obtain $\|u_n\|_{S^2_{\gamma,0}(\Omega)} \to \|u\|_{S^2_{\gamma,0}(\Omega)}$, as $n \to \infty$. Thus we have

$$\|u_n - u\|_{S^2_{\gamma,0}(\Omega)} \to 0, \text{ as } n \to \infty,$$

which means that Φ_+ satisfies condition (C). □

Lemma 3.2. *Assume that conditions* (A1), (A3), (A4) *hold. Then we have*

$$C_m(\Phi,\infty) = C_m(\Phi_\pm,\infty) = \{0\}, \quad m = 0,1,2,\dots.$$

Proof. We only give the proof of Φ_+; the others are similar. Let $S = \{u \in S^2_{\gamma,0}(\Omega) : \|u\|_{S^2_{\gamma,0}(\Omega)} = 1,\ u^+ \neq 0\}$ and $B^\infty = \{u \in S^2_{\gamma,0}(\Omega) : \|u\|_{S^2_{\gamma,0}(\Omega)} \leq 1\}$. By (A3), for any $M > 0$ there exists $c > 0$, such that $F(x,t) \geq Mt^2 - c$, for $(x,t) \in \Omega \times \mathbb{R}$, which implies $\Phi_+(tu) \to -\infty$, as $t \to +\infty$, for any $u \in S$. Using (A4), we have

$$f_+(x,t)t - 2F_+(x,t) \geq -\frac{C(x)}{\theta}, \quad \text{for } (x,t) \in \Omega \times \mathbb{R}. \tag{3.11}$$

Choose

$$a < \min\big\{\inf_{u \in B^\infty} \Phi_+(u),\ -\frac{C_*}{2\theta}\big\},$$

where $C_* = \int\limits_\Omega C(x)dx$. Then for any $u \in S$, there exists $t > 1$ such that $\Phi_+(tu) \leq a$, that is

$$\Phi_+(tu) = \frac{t^2}{2} - \int_\Omega F_+(x,tu)dx \leq a,$$

which (3.11) implies

$$\frac{d}{dt}\Phi_+(tu) = t - \int_\Omega f_+(x,tu)u \leq \frac{1}{t}\big(2a + \frac{C_*}{\theta}\big) < 0\,.$$

Therefore, by the implicit function theorem, there exists a unique $T \in C(S,\mathbb{R})$ such that

$$\Phi_+(T(u)u) = a, \quad \text{for } u \in S.$$

Let $S_1 = \{u \in S^2_{\gamma,0}(\Omega) : \|u\|_{S^2_{\gamma,0}(\Omega)} \geq 1,\ u^+ \neq 0\}$. We construct a strong deformation retract $\tau : [0,1] \times S_1 \to S_1$ which satisfies $\tau(s,u) = (1-s)u + sT\big(\frac{u}{\|u\|_{S^2_{\gamma,0}(\Omega)}}\big)\frac{u}{\|u\|_{S^2_{\gamma,0}(\Omega)}}$ if $\Phi_+(u) \geq a$ and $\tau(s,u) = u$ if $\Phi_+(u) < a$. Hence, It follows from the construction of τ that Φ_+^a is a strong deformation retract of S_1, which is homotopy equivalent to the set S. By the homotopy invariance of homology group, we have

$$\begin{aligned} C_m(\Phi_+,\infty) =& H_m(S^2_{\gamma,0}(\Omega), \Phi_+^a) \\ \cong& H_m(S^2_{\gamma,0}(\Omega), S) \\ \cong& H_m(S^2_{\gamma,0}(\Omega), S^2_{\gamma,0}(\Omega) \setminus \{0\}) \\ =& 0. \end{aligned}$$

□

Proof of Theorem 1.1. By Lemma 3.1, we know that Φ and $\Phi_\pm$ satisfy the (C) condition. By conditions (A1) and (A2), we can easily prove that 0 is a local minimum of Φ and $\Phi_\pm$. So, we have

$$C_m(\Phi,0) = C_m(\Phi_\pm,0) = \delta_{m,0}G. \tag{3.12}$$

Using the mountain pass theorem in [21], we obtain Φ_+ (Φ_-) has a critical point $u_+ > 0$ ($u_- < 0$), and $u_\pm$ are also the nontrivial critical points of the functional Φ. Without loss of generality, we assume that $u_\pm$ are isolated and the only nontrivial critical points of the functional Φ. Now we claim that

$$C_m(\Phi_\pm, u_\pm) = \delta_{m,1}G. \tag{3.13}$$

Indeed, using the methods of [9], we let $\Phi_+(u_+) = c > 0$. It follows from the homology exact sequence of the triple $\Phi_+^A \subset \Phi_+^{\frac{c}{2}} \subset S^2_{\gamma,0}(\Omega)$, we have

$$\begin{aligned} \cdots \to H_m(S^2_{\gamma,0}(\Omega), \Phi_+^A) \to H_m(S^2_{\gamma,0}(\Omega), \Phi_+^{\frac{c}{2}}) \to H_{m-1}(\Phi_+^{\frac{c}{2}}, \Phi_+^A) \to \\ \to H_{m-1}(S^2_{\gamma,0}(\Omega), \Phi_+^A) \to \dots, \end{aligned} \tag{3.14}$$

where $A < 0$ is a constant. Since 0 is the only critical point of Φ_+ in the set $\Phi_+^{\frac{c}{2}}$, by (3.12), we obtain

$$H_m(\Phi_+^{\frac{c}{2}}, \Phi_+^A) = C_m(\Phi_+, 0) = \delta_{m,0}G. \tag{3.15}$$

Similarly, since u_+ is the only critical point of Φ_+ in the set $\{u \in S^2_{\gamma,0}(\Omega) | \Phi_+(u) \geq \frac{c}{2}\}$, we have

$$H_m(S^2_{\gamma,0}(\Omega), \Phi_+^{\frac{c}{2}}) = C_m(\Phi_+, u_1), \quad m = 0, 1, 2, \ldots. \tag{3.16}$$

From Lemma 3.2, we have

$$H_m(S^2_{\gamma,0}(\Omega), \Phi_+^A) = C_m(\Phi_+, \infty) = 0, \quad m = 0, 1, 2, \ldots. \tag{3.17}$$

From (3.14) to (3.17), we deduce that

$$C_m(\Phi_+, u_+) = C_{m-1}(\Phi_+, 0) = \delta_{m,1}G.$$

The case for u_- is similar, that is

$$C_m(\Phi_-, u_-) = C_{m-1}(\Phi_-, 0) = \delta_{m,1}G.$$

Hence

$$C_m(\Phi, u_\pm) = \delta_{m,1}G.$$

The Morse equality (2.1) with $t = -1$ implies that

$$(-1)^0 + (-1)^1 + (-1)^1 = 0,$$

which is a contradiction. Then the problem (1.1)–(1.2) has at least three nontrivial solutions.

References

1. A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Funct. Anal. **14** (1973), 349–381.
2. C. T. Anh and B. K. My, *Existence of solutions to* Δ_λ*−Laplace equations without the Ambrosetti – Rabinowitz condition*, Complex Var. Elliptic Equ. **61**(2016), no. 1, 137–150.
3. C. T. Anh and B. K. My, *Liouville – type theorems for elliptic inequalities involving the* Δ_λ*−Laplace operator,* Complex Var. Elliptic Equ. **61**(2016), no. 7, 1002–1013.
4. K. C. Chang; *Infinite-dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications, 6. Birkhäuser Boston, Inc., Boston, MA, 1993. x+312 pp.
5. N. Garofalo and E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group,* Indiana Univ. Math. J. **41**(1992), no. 1, 71–98.
6. V. V. Grushin, *A certain class of hypoelliptic operators*, Mat. Sb. (N.S.) **83** (1970), no. 125, 456–473 [in Russian].
7. L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on* $\mathbb{R}^n$, Proc. Roy. Soc. Edinburgh Sect. A **129**(1999), no. 4, 787–809.
8. D. S. Jerison and J. M. Lee, *The Yamabe problem on CR manifolds*, J. Differential Geom. **25**(1987),no. 2, 167–197.
9. M. Y. Jiang, *Critical groups and multiple solutions of the* p*−Laplacian equations*, Nonlinear Anal., 59 (2004), 1221–1241.
10. D. Jerison, *The Dirichlet problem for the Kohn Laplacian on the Heisenberg group*, II, J. Funct. Anal. **43**(1981), no. 2, 224–257.
11. A. E. Kogoj and E. Lanconelli, *On semilinear* Δ_λ*−Laplace equation,* Nonlinear Analysis. **75** (2012), no. 12, 4637–4649.
12. A. E. Kogoj and E. Lanconelli, *Linear and semilinear problems involving* Δ_λ*−Laplacians,* Electron. J. Differential Equations 2018, no. 25, 12 pp.
13. D. T. Luyen, D. T. Huong and L. T. H. Hanh, *Existence of infinitely many solutions for* Δ_γ*−Laplace problems* Math. Notes **103** (2018), no. 5, 724–736.
14. D. T. Luyen, *Two nontrivial solutions of boundary value problems for semilinear* Δ_γ *differential equations,* Math. Notes **101** (2017), no. 5, 815–823.
15. D. T. Luyen and N. M. Tri, *Existence of solutions to boundary value problems for semilinear* Δ_γ *differential equations*, Math. Notes **97** (2015), no. 1, 73–84.

16. D. T. Luyen and N. M. Tri, *Large–time behavior of solutions to damped hyperbolic equation involving strongly degenerate elliptic differential operators*, Siberian Math. J. **57** (2016), no. 4, 632–649.

17. D. T. Luyen and N. M. Tri, *Global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator*, Ann. Pol. Math. **117** (2016), no. 2, 141–162.

18. D. T. Luyen and N. M. Tri, *Existence of infinitely many solutions for semilinear degenerate Schrödinger equations,* J. Math. Anal. Appl. **461** (2018), no. 2, 1271–1286.

19. D. T. Luyen and N. M. Tri, *On the existence of multiple solutions to boundary value problems for semilinear elliptic degenerate operators.* Complex Var. Elliptic Equ. **64** (2019), no. 6, 1050–1066.

20. S. I. Pohožaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$. (Russian) Dokl. Akad. Nauk SSSR, **165**, (1965), 36–39.

21. P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. viii+100 pp.

22. B. Rahal and M. K. Hamdani, *Infinitely many solutions for* $\Delta_\alpha-$*Laplace equations with sign-changing potential,* J. Fixed Point Theory Appl. **20** (2018), no. 4, 20:137.

23. N. T. C. Thuy and N. M. Tri, *Some existence and nonexistence results for boundary value problems for semilinear elliptic degenerate operators*, Russ. J. Math. Phys. **9**(2002), no. 3, 365–370.

24. P. T. Thuy and N. M. Tri, *Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations*, NoDEA Nonlinear Differential Equations Appl. **19** (2012), no. 3, 279–298.

25. N. M. Tri, *On the Grushin equation*, Mat. Zametki **63**(1998), no. 1 , 95–105.

26. N. M. Tri, *Critical Sobolev exponent for hypoelliptic operators,* Acta Math. Vietnam. **23** (1998), no. 1, 83–94.

27. N. M. Tri, *Semilinear Degenerate Elliptic Differential Equations, Local and global theories* (Lambert Academic Publishing, 2010).

28. N. M. Tri, *Recent Progress in the Theory of Semilinear Equations Involving Degenerate Elliptic Differential Operators* (Publishing House for Science and Technology of the Vietnam Academy of Science and Technology, 2014).

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