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Existence of Solutions of a Quasilinear Problem With Neumann Boundary Conditions

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ABSTRACT: This paper is devoted to study the existence of weak solutions of a quasilinear system of partial differential equations which are a combination of the Perona-Malik equation and the heat equation. The proof of the main results are based on the compactness method and the motonocity arguments.

Key Words: Topological degree, Quasilinear problem, Homotopy.

Contents

1. Introduction

In this article, we study the existence of the solutions for the following problem

$$\begin{cases} -\mathrm{div}\Big(g_1(|\nabla v|)\nabla u\Big) - \dfrac{1}{\lambda_1^2}\Delta u = f_1(x) - uh_1(x) & \text{in } \Omega, \\ -\mathrm{div}\Big(g_2(|\nabla u|)\nabla v\Big) - \dfrac{1}{\lambda_2^2}\Delta v = f_2(x) - vh_2(x) & \text{in } \Omega, \\ \Big(g_1(|\nabla v|) + \dfrac{1}{\lambda_1^2}\Big)\nabla u \cdot \vec{\eta} = \Big(g_2(|\nabla u|) + \dfrac{1}{\lambda_2^2}\Big)\nabla v \cdot \vec{\eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $f = (f_1, f_2)$ is function in $(L^2(\Omega))^2$ and $0 < \lambda \leq 1$ suth that $\lambda = (\lambda_1, \lambda_2)$, $h = (h_1, h_2)$ is function in $(L^\infty(\Omega))^2$ satisfy $h_i > 0$, $i = 1, 2$.
The function $g = (g_1, g_2)$ is defined by one of the following expressions:

$$g(s) = \frac{1}{1 + (\frac{s}{\lambda})^2} \quad \text{or} \quad g(s) = \exp\big(-\frac{s^2}{2\lambda^2}\big).$$

It is clear that the function $g(s)$ is a decreasing non-negative function satisfying the following conditions

$$\begin{cases} \lim\limits_{s\to 0} g(s) = 1, \\ \lim\limits_{s\to +\infty} g(s) = 0. \end{cases} \tag{1.2}$$

We remark that, if $g_i = 1$ for $i = 1, 2$ we recover the linear diffusion.

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In 2014, A. Atlas et al [1] proved the existence and the uniqueness of solutions of the problem

$$\begin{cases} -\mathrm{div}(g(|\nabla u|)\nabla u) - \dfrac{1}{\lambda^p}\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f - u \ \text{ in } \Omega, \\ \Big(g(|\nabla u|) + \dfrac{1}{\lambda^2}\Big)\nabla u \cdot \vec{\eta} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

they also studied the asymptotic behavior of the solution as $p \to \infty$. The solvability of the problem (1.3) in this setting was proved by S. Lecheheb et al [7] in the case where $p = 2$ and the right hand side is $f - k(x)u$, and they also solved this problem when the right hand side is $f(u)$, $p = 2$ see [8].

In this work, we extend the results obtained in [7] to the system (1.1). This type of systems has been extensively studied by several authors. In 2009, A. Moussaoui and B. Khodja [12] studied the existence of nontrivial solutions of semilinear elliptic systems. In 2013, H. Lakehal et al [5] proved the existence of solution for a nonlinear elliptic system through the Schauder's fixed point theorem and an appropriate choice of homotopy. Far from being complete, we refer readers to [3,6,9,11].

The aim of this work is to investigate the existence of solutions to the quasilinear system (1.1) with zero Neumann boundary conditions. This existence is obtained by using the compactness method and the monotonicity arguments. The corresponding method has been first introduced by Vishik and called the compacteness method by J.L. Lions [10]. Our problem is a combination of the Perona-Malik equation [1,4,13,14] and the heat equation [2].

The paper is organized as follows. In the next section we present the main result. In the section 3, we prove the existence of the solution of the problem (1.1) under the condition 1.2, using monotonicity arguments.

2. Main result

In this section, we discuss the notions of weak solutions and the main result. First, let

$$\mathrm{U} = H^1(\Omega) \times H^1(\Omega),$$

which is a Banach space endowed with the norm

$$\|(u,v)\|_{\mathrm{U}}^2 = \|u\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2,$$

and let $\widetilde{\mathrm{V}} = L^2(\Omega) \times L^2(\Omega)$, and $\widetilde{\mathrm{U}} = L^\infty(\Omega) \times L^\infty(\Omega)$. In the sequel, $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$ will denote the usual norms of $L^2(\Omega)$, $H^1(\Omega)$ and $L^\infty(\Omega)$, respectively.
We give now the:

Definition 2.1. *We say that $(u, v) \in U$ is a weak solution for the system (1.1) if for any $(\varphi, \psi) \in \mathrm{U}$ we have*

$$\begin{aligned} &\int_\Omega (g_1(|\nabla v|) + \frac{1}{\lambda_1^2})\nabla u \nabla\varphi \,\mathrm{d}x + \int_\Omega (g_2(|\nabla u|) + \frac{1}{\lambda_2^2})\nabla v \nabla\psi \,\mathrm{d}x \\ = &\int_\Omega f_1\varphi \,\mathrm{d}x + \int_\Omega f_2\psi \,\mathrm{d}x - \int_\Omega uh_1(x)\varphi \,\mathrm{d}x - \int_\Omega vh_2(x)\psi \,\mathrm{d}x. \end{aligned} \tag{2.1}$$

Our main result is the:

Theorem 2.2. *Under condition (1.2), the problem (1.1) has at least one solution.*

3. Proof of Theorem 2.2

Let V be a finite-dimensional subspace of U endowed with the U-norm, and V* its dual. Define the mappings $H : V \times [0, 1] \longrightarrow V^*$ by

$$\begin{aligned} \langle H(u,v,t),(\varphi,\psi)\rangle_U &= \int_\Omega \Big(g_1(t|\nabla v|) + \frac{1}{\lambda_1^2}\Big)\nabla u \nabla\varphi \,\mathrm{d}x + \int_\Omega \Big(g_2(t|\nabla u|) + \frac{1}{\lambda_2^2}\Big)\nabla v \nabla\psi \,\mathrm{d}x \\ &- \int_\Omega f_1(x)\varphi \,\mathrm{d}x - \int_\Omega f_2(x)\psi \,\mathrm{d}x + \int_\Omega uh_1(x)\varphi \,\mathrm{d}x + \int_\Omega vh_2(x)\psi \,\mathrm{d}x, \end{aligned}$$

for all $(\varphi,\psi)\in \mathrm{V}$, H is well defined.

3.1. A priori bounds.

Let us show now that

$$\Big\{(u,v)\in V : H(u,v,t)=0, \quad \text{for some } t\in[0,1]\Big\}\subset \bar{B}(0,\widetilde{\rho}) \quad \text{where}$$

$$\widetilde{\rho}=\frac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_{\widetilde{V}}.$$

Indeed, if $H(u,v,t)=0$ for same $(u,v,t)\in \mathrm{V}\times[0,1]$, then

$$0=\langle H(u,v,t),(u,v)\rangle_U \geq \min(c_1,c_2)\|(u,v)\|_U^2-2\|(f_1,f_2)\|_{\widetilde{V}}\|(u,v)\|_U,$$

which implies that

$$\|(u,v)\|_U\leq \frac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_{\widetilde{V}}.$$

Consequently, for any $R>\dfrac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_{\widetilde{V}}$, we have

$$H(u,v,t)\neq 0 \quad \text{ if } (u,v,t)\in \partial B^V(R)\times[0,1], \tag{3.1}$$

where $\partial B^V(R)$ is the boundary of the open ball of center 0 and radius R in the space V see [9].

3.2. H is bounded.

Now, if $(u,v,t)\in \bar{B}^V(R)\times[0,1]$, we have

$$\begin{aligned}
|\langle H(u,v,t),(\varphi,\psi)\rangle| &\leq \left(\max\Big(1+\frac{1}{\lambda_1^2},1+\frac{1}{\lambda_2^2},2\|(h_1,h_2)\|_{\widetilde{U}}\Big)\|(u,v)\|_U+2\|(f_1,f_2)\|_{\widetilde{V}}\right)\|(\varphi,\psi)\|_U\\
&\leq \Bigg(\underbrace{\max\Big(1+\frac{1}{\lambda_1^2},1+\frac{1}{\lambda_2^2},2\|(h_1,h_2)\|_{\widetilde{U}}\Big)R+2\|(f_1,f_2)\|_{\widetilde{V}}}_{\widetilde{R}}\Bigg)\|(\varphi,\psi)\|_U\\
&\leq \widetilde{R}\|(\varphi,\psi)\|_U,
\end{aligned}$$

for all $(\varphi,\psi)\in \mathrm{U}$, and hence

$$H\Big(\bar{B}^{\mathrm{V}}(R)\times[0,1]\Big)\subset \bar{B}^{\mathrm{V}^*}(\widetilde{R}). \tag{3.2}$$

3.3. H is continuous.

Let $(u_n,v_n,t_n)\in \bar{B}^{\mathrm{V}}(R)\times[0,1]$ converge to (u,v,t) in $V\times[0,1]$, i.e in $U\times[0,1]$. Since $(H(u_n,v_n,t_n))$ is bounded because of (3.2), to prove that

$$H(u_n,v_n,t_n)\to H(u,v,t),$$

it is sufficient to show that $H(u,v,t)$ is the unique cluster point of $(H(u_n,v_n,t_n))$. Let $M\in \mathrm{V}^*$ be such a cluster point, still we denote by $(t_n),(u_n)$ and (v_n) a subsequence of $(t_n),(u_n)$ and (v_n) respectively such that

$$H(u_n,v_n,t_n)\to M \text{ in } \mathrm{V}^*.$$

Since $(u_n,v_n)\to(u,v)$ in U, it follows that $(u_n,v_n)\to(u,v)$ in $\widetilde{V}$, and hence, going if necessary to a subsequence, we may assume that $(u_n,v_n)\to(u,v)$ a.e in Ω. On the other hand, $(\partial_i u_n,\partial_i v_n)\to(\partial_i u,\partial_i v)$ in $\widetilde{V}$, therefore $(\nabla u_n,\nabla v_n)\to(\nabla u,\nabla v)$ a.e in Ω. This implies that

$$g_1(t_n|\nabla v_n|)\to g_1(t|\nabla v|) \quad \text{ a.e in } \Omega,$$

$$g_2(t_n|\nabla u_n|) \to g_2(t|\nabla u|) \quad \text{a.e in } \Omega,$$

and hence, for any $(\varphi,\psi) \in V$,

$$g_1(t_n|\nabla v_n|)\nabla\varphi \to g_1(t|\nabla v|)\nabla\varphi \text{ in } L^2(\Omega),$$

$$g_2(t_n|\nabla u_n|)\nabla\psi \to g_2(t|\nabla u|)\nabla\psi \text{ in } L^2(\Omega).$$

We conclude that

$$\begin{aligned}
&\langle H(u_n,v_n,t_n),(\varphi,\psi)\rangle_U \\
&= \int_\Omega u_n h_1(x)\varphi\,\mathrm{d}x + \int_\Omega v_n h_2(x)\psi\,\mathrm{d}x + \int_\Omega \Big(g_1(t_n|\nabla v_n|) + \frac{1}{\lambda_1^2}\Big)\nabla u_n \nabla\varphi\,\mathrm{d}x \\
&+ \int_\Omega \Big(g_2(t_n|\nabla u_n|) + \frac{1}{\lambda_2^2}\Big)\nabla v_n \nabla\psi\,\mathrm{d}x \\
&\to \int_\Omega u h_1(x)\varphi\,\mathrm{d}x + \int_\Omega v h_2(x)\psi\,\mathrm{d}x + \int_\Omega \Big(g_1(t|\nabla v|) + \frac{1}{\lambda_1^2}\Big)\nabla u \nabla\varphi\,\mathrm{d}x \\
&+ \int_\Omega \Big(g_2(t|\nabla u|) + \frac{1}{\lambda_2^2}\Big)\nabla v \nabla\psi\,\mathrm{d}x = \langle H(u,v,t),(\varphi,\psi)\rangle_U.
\end{aligned}$$

Thus $M = H(u,v,t)$. All those properties allow us to apply the homotopy invariance property to

$$\deg_B\Big(H(\cdot,\cdot,1),B(R),0\Big) = \deg_B\Big(H(\cdot,\cdot,0),B(R),0\Big). \tag{3.3}$$

But $H(u,v,0)=0$ is equivalant to the problem

$$\begin{aligned}
&(1+\frac{1}{\lambda_1^2})\int_\Omega \nabla u\nabla\varphi\,\mathrm{d}x + (1+\frac{1}{\lambda_2^2})\int_\Omega \nabla v\nabla\psi\,\mathrm{d}x \\
&= \int_\Omega f_1(x)\varphi\,\mathrm{d}x + \int_\Omega f_2(x)\psi\,\mathrm{d}x - \int_\Omega u h_1(x)\varphi\,\mathrm{d}x - \int_\Omega v h_2(x)\psi\,\mathrm{d}x,
\end{aligned}$$

for all $(\varphi,\psi) \in \mathrm{V}$, whose solution is unique because of the boundedness of the set of its possible solutions. Consequently,

$$\deg_B\Big(H(\cdot,\cdot,0),B(R),0\Big) = \pm 1,$$

and from (3.3) and the existence property of the degree, there exists $(u,v) \in B^{\mathrm{V}}(R)$ which satisfies

$$\left\{\begin{aligned}
&\int_\Omega \Big(g_1(|\nabla v|) + \frac{1}{\lambda_1^2}\Big)\nabla u\nabla\varphi\,\mathrm{d}x + \int_\Omega \Big(g_2(|\nabla u|) + \frac{1}{\lambda_2^2}\Big)\nabla v\nabla\psi\,\mathrm{d}x \\
&= \int_\Omega f_1(x)\varphi\,\mathrm{d}x + \int_\Omega f_2(x)\psi\,\mathrm{d}x - \int_\Omega u h_1(x)\varphi\,\mathrm{d}x - \int_\Omega v h_2(x)\psi\,\mathrm{d}x, \\
&\qquad\qquad \|(u,v)\|_U \le \frac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_{\widetilde{V}},
\end{aligned}\right. \tag{3.4}$$

for all $(\varphi,\psi) \in V$.

3.4. Passing to the limit.

We now show the passage to the limit.
Consider the function $a_i : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$a_i(\xi_i) = \Big(g_i(\xi_i) + \frac{1}{\lambda_i^2}\Big)\xi_i \quad \text{for any } \xi_i \in \mathbb{R}^N \text{ and } i = 1, 2.$$

To prove the passage to the limit, we need the following lemma:

Lemma 3.1. *[1] Let $0 < \lambda_i \leq 1$, for any $\xi_i, \eta_i \in \mathbb{R}^N$ such that $\xi_i \neq \eta_i$ we have*

$$\big(a_i(\xi_i) - a_i(\eta_i)\big)(\xi_i - \eta_i) > 0 \textit{ for } i = 1, 2.$$

The proof of the above lemma can be found in [1].

Lemma 3.2. *If $a \in C(\mathbb{R}^N, \mathbb{R}^N)$, $a(\xi) \leq (1 + \frac{1}{\lambda^2})\xi$ for all $\xi \in \mathbb{R}^N$ and if $u_n \to u$ in $H^1(\Omega)$ then $a(\nabla u_n) \to a(\nabla u)$ in $L^2(\Omega)$.*

Lemma (3.2) is proved by the dominated convergence theorem of Lebesgue.
Now, it is well known that one can write $U = \overline{\bigcup_{n\geq 1} V_n}$ where $V_n \subset V_{n+1} (n \geq 1)$ and V_n has dimension n. Consequently, given any $(\varphi, \psi) \in U$, there exists a sequence (φ_n, ψ_n) with $(\varphi_n, \psi_n) \in V_n$ which converges to (φ, ψ). On the other hand, by (3.4) applied to $V = V_n$, there exists, for each $n \geq 1$, some $(u_n, v_n) \in V_n$ such that

$$\begin{aligned}
&\int_\Omega a_1(\nabla u_n)\nabla\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega a_2(\nabla v_n)\nabla\widetilde{\psi}\,\mathrm{d}x \\
&= \int_\Omega f_1(x)\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega f_2(x)\widetilde{\psi}\,\mathrm{d}x - \int_\Omega u_n h_1(x)\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega v_n h_2(x)\widetilde{\psi}\,\mathrm{d}x, \\
&\|(u_n, v_n)\|_U \leq \frac{2}{\min(c_1, c_2)}\|(f_1, f_2)\|_{\widetilde{V}},
\end{aligned}$$

for all $(\widetilde{\varphi}, \widetilde{\psi}) \in V_n$. In particular, taking $(\widetilde{\varphi}, \widetilde{\psi}) = (\varphi_n, \psi_n)$ introduced above,

$$\begin{aligned}
&\int_\Omega a_1(\nabla u_n)\nabla\varphi_n\,\mathrm{d}x + \int_\Omega a_2(\nabla v_n)\nabla\psi_n\,\mathrm{d}x \\
&= \int_\Omega f_1(x)\varphi_n\,\mathrm{d}x + \int_\Omega f_2(x)\psi_n\,\mathrm{d}x - \int_\Omega u_n h_1(x)\varphi_n\,\mathrm{d}x - \int_\Omega v_n h_2(x)\psi_n\,\mathrm{d}x, \\
&\|(u_n, v_n)\|_U \leq \frac{2}{\min(c_1, c_2)}\|(f_1, f_2)\|_{\widetilde{V}},
\end{aligned} \tag{3.5}$$

for all $n \geq 1$. The estimate in (3.5) implies that, going if necessary to subsequences, we can assume that there exists $(u, v) \in U$ such that $u_n \to u$ weakly in U, $u_n \to u$ strongly in $\widetilde{V}$ and $u_n \to u$ a.e. in Ω. As $\big(a_1(\nabla u_n)\big)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, then there exists $\zeta_1 \in L^2(\Omega)$ such that

$$a_1(\nabla u_n) \to \zeta_1 \text{ weakly in } L^2(\Omega).$$

Similarly, we obtain

$$a_2(\nabla v_n) \to \zeta_2 \text{ weakly in } L^2(\Omega),$$

and $(\nabla\varphi_n, \nabla\psi_n) \to (\nabla\varphi, \nabla\psi)$ strongly in $\widetilde{V}$, one can let $n \to \infty$ in (3.5) to obtain

$$\begin{aligned}
&\int_\Omega \zeta_1\nabla\varphi\,\mathrm{d}x + \int_\Omega \zeta_2\nabla\psi\,\mathrm{d}x \\
&= \int_\Omega f_1(x)\varphi\,\mathrm{d}x + \int_\Omega f_2(x)\psi\,\mathrm{d}x - \int_\Omega u h_1(x)\varphi\,\mathrm{d}x - \int_\Omega v h_2(x)\psi\,\mathrm{d}x.
\end{aligned} \tag{3.6}$$

It remains to show that

$$\int_\Omega \zeta_1 \nabla\varphi \,\mathrm{d}x = \int_\Omega a_1(\nabla u)\nabla\varphi \,\mathrm{d}x, \tag{3.7}$$

and

$$\int_\Omega \zeta_2 \nabla\psi \,\mathrm{d}x = \int_\Omega a_2(\nabla v)\nabla\psi \,\mathrm{d}x. \tag{3.8}$$

To prove the two equalities, we use the trick of Minty [7]; we begin by studying the limit of

$$\int_\Omega a_1(\nabla u_n)\nabla u_n \,\mathrm{d}x,$$

and

$$\int_\Omega a_2(\nabla v_n)\nabla v_n \,\mathrm{d}x.$$

Indeed

$$\int_\Omega a_1(\nabla u_n)\nabla u_n \,\mathrm{d}x = \int_\Omega f_1(x)u_n \,\mathrm{d}x - \int_\Omega u_n^2 h_1(x)\,\mathrm{d}x \to \int_\Omega f_1(x)u \,\mathrm{d}x - \int_\Omega u^2 h_1(x)\,\mathrm{d}x,$$

$$\int_\Omega a_2(\nabla v_n)\nabla v_n \,\mathrm{d}x = \int_\Omega f_2(x)v_n \,\mathrm{d}x - \int_\Omega v_n^2 h_2(x)\,\mathrm{d}x \to \int_\Omega f_2(x)v \,\mathrm{d}x - \int_\Omega v^2 h_2(x)\,\mathrm{d}x,$$

because $(u_n, v_n) \to (u, v)$ weakly in U. But we know that (u, v) satisfies (3.6), and hence

$$\int_\Omega f_1(x)u \,\mathrm{d}x - \int_\Omega u^2 h_1(x)\,\mathrm{d}x = \int_\Omega \zeta_1 \nabla u \,\mathrm{d}x,$$

and

$$\int_\Omega f_2(x)v \,\mathrm{d}x - \int_\Omega v^2 h_2(x)\,\mathrm{d}x = \int_\Omega \zeta_2 \nabla v \,\mathrm{d}x.$$

Therefore

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega a_1(\nabla u_n)\nabla u_n \,\mathrm{d}x &= \int_\Omega f_1(x)u \,\mathrm{d}x - \int_\Omega u^2 h_1(x)\,\mathrm{d}x \\ &= \int_\Omega \zeta_1 \nabla u \,\mathrm{d}x,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega a_2(\nabla v_n)\nabla v_n \,\mathrm{d}x &= \int_\Omega f_2(x)v \,\mathrm{d}x - \int_\Omega v^2 h_2(x)\,\mathrm{d}x \\ &= \int_\Omega \zeta_2 \nabla v \,\mathrm{d}x.\end{aligned} \tag{3.10}$$

Let $(\varphi, \psi) \in U$, it exists $(\varphi_n, \psi_n)_{n\in\mathbb{N}}$ such that $(\varphi_n, \psi_n) \in V_n$ for all $n \in \mathbb{N}$ and $(\varphi_n, \psi_n) \to (\varphi, \psi)$ in U when $n \to +\infty$. Thanks to Lemma 3.1, we will pass to the limit in the two terms

$$\int_\Omega a_1(\nabla u_n)\nabla\varphi_n \,\mathrm{d}x,$$

and

$$\int_\Omega a_2(\nabla v_n)\nabla\psi_n \,\mathrm{d}x.$$

Indeed, for the first equation

$$\begin{aligned}&0 \le \int_\Omega (a_1(\nabla u_n) - a_1(\nabla\varphi_n))(\nabla u_n - \nabla\varphi_n)\,\mathrm{d}x = \\ &\int_\Omega a_1(\nabla u_n)\nabla u_n \,\mathrm{d}x - \int_\Omega a_1(\nabla u_n)\nabla\varphi_n \,\mathrm{d}x - \int_\Omega a_1(\nabla\varphi_n)\nabla u_n \,\mathrm{d}x + \int_\Omega a_1(\nabla\varphi_n)\nabla\varphi_n \,\mathrm{d}x \\ &= \quad F_{1,n} - F_{2,n} - F_{3,n} + F_{4,n},\end{aligned}$$

we saw in (3.9) that $F_{1,n} \to \int_\Omega \zeta_1 \nabla u \,\mathrm{d}x$ when $n \to \infty$. We have

$$\lim_{n\to+\infty} F_{2,n} = \int_\Omega \zeta_1 \nabla \varphi \,\mathrm{d}x.$$

Similarly

$$\lim_{n\to+\infty} F_{3,n} = \int_\Omega a_1(\nabla \varphi)\nabla u \,\mathrm{d}x.$$

Finally, we also have

$$\lim_{n\to+\infty} F_{4,n} = \int_\Omega a_1(\nabla \varphi)\nabla \varphi \,\mathrm{d}x,$$

when $n \to +\infty$. The passage to the limit therefore gives:

$$\int_\Omega (\zeta_1 - a_1(\nabla\varphi))(\nabla u - \nabla\varphi)\,\mathrm{d}x \geq 0 \text{ for all } \varphi \in H^1(\Omega).$$

Similarly, we obtain

$$\int_\Omega (\zeta_2 - a_2(\nabla\psi))(\nabla u - \nabla\psi)\,\mathrm{d}x \geq 0 \text{ for all } \psi \in H^1(\Omega).$$

We now choose judicious test functions φ and ψ. We take

$$\varphi = u + \frac{1}{n}\varphi^*, \text{ with } \varphi^* \in H^1(\Omega) \text{ and } n \in \mathbb{N}^*,$$

and

$$\psi = v + \frac{1}{n}\psi^*, \text{ with } \psi^* \in H^1(\Omega) \text{ and } n \in \mathbb{N}^*.$$

We thus obtain:

$$-\frac{1}{n}\int_\Omega \big(\zeta_1 - a_1(\nabla u + \frac{1}{n}\nabla\varphi^*)\big)\nabla\varphi^*\,\mathrm{d}x \geq 0,$$

and

$$-\frac{1}{n}\int_\Omega \big(\zeta_2 - a_2(\nabla v + \frac{1}{n}\nabla\psi^*)\big)\nabla\psi^*\,\mathrm{d}x \geq 0,$$

then

$$\int_\Omega \big(\zeta_1 - a_1(\nabla u + \frac{1}{n}\nabla\varphi^*)\big)\nabla\varphi^*\,\mathrm{d}x \leq 0,$$

and

$$\int_\Omega \big(\zeta_2 - a_2(\nabla v + \frac{1}{n}\nabla\psi^*)\big)\nabla\psi^*\,\mathrm{d}x \leq 0.$$

But

$$u + \frac{1}{n}\varphi^* \to u \text{ in } H^1(\Omega),$$
$$v + \frac{1}{n}\psi^* \to v \text{ in } H^1(\Omega),$$

thanks to Lemma 3.2, we obtain

$$a_1\big(\nabla u + \frac{1}{n}\nabla\varphi^*\big) \to a_1(\nabla u) \text{ in } L^2(\Omega),$$

and

$$a_2\big(\nabla v + \frac{1}{n}\nabla\psi^*\big) \to a_2(\nabla v) \text{ in } L^2(\Omega).$$

Passing to the limit when $n \to +\infty$, we then obtain

$$\int_\Omega (\zeta_1 - a_1(\nabla u))\nabla\varphi^* \,\mathrm{d}x \leq 0, \quad \forall\varphi^* \in H^1(\Omega),$$

and

$$\int_\Omega (\zeta_2 - a_2(\nabla v))\nabla\psi^* \,\mathrm{d}x \leq 0, \quad \forall\psi^* \in H^1(\Omega).$$

By linearity (can change φ^* into $-\varphi^*$ and ψ^* into $-\psi^*$), we have

$$\int_\Omega (\zeta_1 - a_1(\nabla u))\nabla\varphi^* \,\mathrm{d}x = 0, \quad \forall\varphi^* \in H^1(\Omega),$$

and

$$\int_\Omega (\zeta_2 - a_2(\nabla v))\nabla\psi^* \,\mathrm{d}x = 0, \quad \forall\psi^* \in H^1(\Omega).$$

We deduce that

$$\int_\Omega \zeta_1 \nabla\varphi^* \,\mathrm{d}x = \int_\Omega a_1(\nabla u)\nabla\varphi^* \,\mathrm{d}x, \quad \forall\varphi^* \in H^1(\Omega),$$
$$\int_\Omega \zeta_2 \nabla\psi^* \,\mathrm{d}x = \int_\Omega a_2(\nabla v)\nabla\psi^* \,\mathrm{d}x, \quad \forall\psi^* \in H^1(\Omega).$$

Hence we have showed that (u, v) is a solution of (1.1).

References

1. A. Atlas, F. Karami, D. Meskine, *The Perona-Malik inequality and application to image denoising*, Nonlinear Analysis: Real World Applications, **18**, 57-68, (2014).
2. M. Biegert, M. Warma, *The heat equation with nonlinear generalized Robin boundary conditions*, J. Differential Equations, **247**, 1949-1979, (2009).
3. Th. Gallouet, O. Kavian, *Résultats d'existence et de non-existence pour certains problèmes demi-linèares à l'infini*, Ann. Fac. Sci. de Toulouse Math, **5**(3), 201-246, (1981).
4. P. Guidotti, *A bakward-forward regularization of the Perona-Malik equation*, J.Differential Equations **252**(4), 3226-3244, (2012).
5. H. Lakehal, B. Khodja, W. Gharbi, *Existence results of nontrivial solutions for a semi linear elliptic system at resonance*, Journal of Advanced Research in Dynamical and Control Systems, **5**(3), 1-12, (2013).
6. H. Lakhal, B. Khodja, *Elliptic systems at resonance for jumping non- linearities*, Electronic Journal of Differential Equations, ,N 70, 1-13, (2016).
7. S. Lecheheb, M. Maouni, H. Lakhal, *Existence of the solution of a quasilinear equation and its application to image denoising*, International Journal of Computer Science, Computer Science, Communication and Information Technology (CSCIT) **7** (2), 1-6, 2019.
8. S. Lecheheb, M. Maouni, H. Lakhal, *Image restoration using nonlinear elliptic Equaion*, International Journal of Computer Science, Computer Science, Communication and Information Technology (CSCIT) **6** (2), 32-37, 2019.
9. S. Lecheheb, H. Lakhal, M. Maouni, K. Slimani, *Study of a system of Diffusion-Convection-Reaction*, International Journal of Partial Differential Equations and Applications. **4**(2), 32-37, 2016.
10. J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, (1969).
11. A. Moussaoui and · J. Vélin, *Existence and a priori estimates of solutions for quasilinear singular elliptic systems with variable exponents*, Journal of Elliptic and Parabolic Equations, **4** (2), 417-440, (2018)
12. A. Moussaoui, B. Khodja, *Existence results for a class of semilinear elliptic systems*, Journal of Partial Differential Equations, **22**, 111-126, (2009).
13. V Kamalaveni, R Anitha Rajalakshmi, K A Narayanankutty, *Image Denoising using Variations of Perona-Malik Model with different Edge Stopping Functions*, Procedia Computer Science **58**, 673-682, (2015).
14. Y.Q. Wang, Jichang Guo, Wufan Chen, Wenxue Zhang, *Image denoising using Modified Perona-Malik Model based on Directional Laplacian*, Signal Processing, **93**, 2548-2558, (2013).

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