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The Equality of Hochschild Cohomology Group and Module Cohomology Group for Semigroup Algebras

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Abstract: Let S be a (not necessarily unital) commutative inverse semigroup with idempotent set E. In this paper, we show that for every $n \in \mathbb{N}_0$, n-th Hochschild cohomology group of semigroup algebra $\ell^1(S)$ with coefficients in $\ell^\infty(S)$ and its n-th $\ell^1(E)$-module cohomology group, are equal. Indeed, we prove that

$$\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) = \mathcal{H}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)),$$

for all $n \geq 0$.

Key Words: Inverse semigroup, Semigroup algebra, Hochschild cohomology group, Module cohomology group.

Contents

1. Introduction

The concept of module amenability for Banach algebras which are Banach module over another Banach algebra with compatible actions, was introduced by Amini in [1]. Immediately after that Amini and Bagha in [2] introduced and studied the concept of weak module amenability for Banach algebras. As an example they showed that the semigroup algebra $\ell^1(S)$ of a commutative inverse semigroup S is always weakly module amenable as a module over the semigroup algebra $\ell^1(E)$ of its subsemigroup E of idempotents, when $\ell^1(S)$ is a Banach $\ell^1(E)$-module with actions

$$\delta_s \cdot \delta_e = \delta_e \cdot \delta_s = \delta_e * \delta_s = \delta_{se} \qquad (s \in S, e \in E), \tag{1.1}$$

where δ_s and δ_e are the point mass at $s \in S$ and $e \in E$, respectively. The author along with Pourabbas in [11] and [12], after introducing the concept of module cohomology group for Banach algebras extended this result and showed that the first and second module cohomology groups of $\ell^1(S)$ with coefficients in $\ell^\infty(S)$ $(\ell^1(S)^{(2n-1)}$ $(n \in \mathbb{N}))$, are zero and Banach space, respectively.

In this paper, for every $n \in \mathbb{N}_0$, we show that n-th Hochschild cohomology group of semigroup algebra $\ell^1(S)$ with coefficient in $\ell^\infty(S)$ and its n-th $\ell^1(E)$-module cohomology group are equal, when S is a commutative inverse semigroup with idempotent set E. Indeed we prove that

$$\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{H}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \qquad (n \in \mathbb{N}_0).$$

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2. Preliminaries

In this section, we remind the concept of n-th Hochschild cohomology group and n-th module cohomology group which are introduced by Johnson in [7] and the author of the current article along with Pourabbas in [12], respectively.

Let A be a Banach algebra and X be a Banach A-bimodule, then so is the dual space X^*, where the actions of A on X^* are defined by

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*). \tag{2.1}$$

The cohomology complex is

$$\mathcal{C}(A,X): \qquad 0 \longrightarrow X \xrightarrow{\delta^0} \mathcal{C}^1(A,X) \xrightarrow{\delta^1} \mathcal{C}^2(A,X) \xrightarrow{\delta^2} \cdots, \tag{2.2}$$

when the map $\delta^0 : X \longrightarrow \mathcal{C}^1(A,X)$ is given by $\delta^0(x)(a) = a \cdot x - x \cdot a$ and for $n \in \mathbb{N}$, the n-coboundary operators $\delta^n : \mathcal{C}^n(A,X) \longrightarrow \mathcal{C}^{n+1}(A,X)$ is given by

$$\begin{aligned} \delta^n \phi(a_1, \ldots, a_{n+1}) =& a_1 \cdot \phi(a_2, \ldots, a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots a_{n+1}) \\ &+ (-1)^{n+1} \phi(a_1, \ldots, a_n) \cdot a_{n+1}, \end{aligned} \tag{2.3}$$

where $\mathcal{C}^n(A,X)$ is the set of all bounded n-linear maps from A to X that are called n-cochains, $\phi \in \mathcal{C}^n(A,X)$ and $a_1, a_2, \ldots, a_{n+1} \in A$. It is easy to see that $\delta^{n+1} \circ \delta^n = 0$ for every $n \in \mathbb{Z}^+$. The space $\ker \delta^n$ of all bounded n-cocycles is denoted by $\mathcal{Z}^n(A,X)$ and the space $\operatorname{Im} \delta^{n-1}$ of all bounded n-coboundaries is denoted by $\mathcal{B}^n(A,X)$. We also recall that $\mathcal{B}^n(A,X)$ is included in $\mathcal{Z}^n(A,X)$ and the n-th Hochschild cohomology group $\mathcal{H}^n(A,X)$ is defined by the quotient

$$\mathcal{H}^n A, X) = \frac{\mathcal{Z}^n(A,X)}{\mathcal{B}^n(A,X)}.$$

Let $\mathfrak{A}$ and A be (not necessarily unital) Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is,

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad a(\alpha \cdot b) = (a \cdot \alpha)b \quad (\alpha \in \mathfrak{A},\ a, b \in A), \tag{2.4}$$

and the same for the other side action.

Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is,

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot \alpha) \cdot x = a \cdot (\alpha \cdot x), \quad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a), \tag{2.5}$$

where $\alpha \in \mathfrak{A}$, $a \in A$ and $x \in X$ and the same for the other side action. Then X is called a Banach A-$\mathfrak{A}$-module. X is called a commutative Banach A-$\mathfrak{A}$-module whenever $\alpha \cdot x = x \cdot \alpha$ for every $\alpha \in \mathfrak{A}$ and $x \in X$.

Let X be a Banach space with the dual space X^*. Suppose X is a commutative Banach A-$\mathfrak{A}$-module, then so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined as (2.1). In particular, if A is a commutative Banach $\mathfrak{A}$-bimodule, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space A^* is also a commutative Banach A-$\mathfrak{A}$-module.

An n-$\mathfrak{A}$-module map is a bounded mapping $\phi : A^n = \underbrace{A \times A \times \ldots \times A}_{n} \rightarrow X$ with the following properties:

$$\begin{aligned} \phi(a_1, a_2, \ldots, a_{i-1}, b \pm c, a_{i+1}, \ldots, a_n) &= \phi(a_1, a_2, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \\ &\quad \pm \phi(a_1, a_2, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n), \\ \phi(\alpha \cdot a_1, a_2, \ldots, a_n) &= \alpha \cdot \phi(a_1, a_2, \ldots, a_n), \\ \phi(a_1, a_2, \ldots, a_n \cdot \alpha) &= \phi(a_1, a_2, \ldots, a_n) \cdot \alpha, \\ &\text{and} \\ \phi(a_1, a_2, \ldots, a_i \cdot \alpha, a_{i+1}, \ldots, a_n) &= \phi(a_1, a_2, \ldots, a_i, \alpha \cdot a_{i+1}, \ldots, a_n), \end{aligned} \tag{2.6}$$

where $a_1,\dots,a_n,b,c\in A$ and $\alpha\in\mathfrak{A}$. From now on, we remove the dot (sing "$\cdot$") for simplicity. Note that, in case of $\mathfrak{A}$ is not necessarily unital ϕ is not necessarily n-linear, but still its boundedness implies its norm continuity (since ϕ preserves subtraction). We use the notation $\mathcal{C}^n_{\mathfrak{A}}(A,X)$ for the set of all bounded (continuous) n-$\mathfrak{A}$-module maps from A to X that are called n-$\mathfrak{A}$-module cochains.

The $\mathfrak{A}$-module cohomology complex is

$$\mathcal{C}_{\mathfrak{A}}(A,X): \qquad 0\longrightarrow X\xrightarrow{\delta^0_{\mathfrak{A}}}\mathcal{C}^1_{\mathfrak{A}}(A,X)\xrightarrow{\delta^1_{\mathfrak{A}}}\mathcal{C}^2_{\mathfrak{A}}(A,X)\xrightarrow{\delta^2_{\mathfrak{A}}}\cdots, \tag{2.7}$$

where the n-coboundary operators $\delta^n_{\mathfrak{A}}$ is given as (2.3) (for more details see [11] and [12]). The space $\ker\delta^n_{\mathfrak{A}}$ of all bounded n-$\mathfrak{A}$-module cocycles is denoted by $\mathcal{Z}^n_{\mathfrak{A}}(A,X)$ and the space $\operatorname{Im}\delta^{n-1}_{\mathfrak{A}}$ of all bounded n-$\mathfrak{A}$-module coboundaries is denoted by $\mathcal{B}^n_{\mathfrak{A}}(A,X)$. From now on, $\delta^n_{\mathfrak{A}}$ is displayed with the same δ^n for simplicity. We know that $\mathcal{B}^n_{\mathfrak{A}}(A,X)$ is included in $\mathcal{Z}^n_{\mathfrak{A}}(A,X)$. The n-th $\mathfrak{A}$-module cohomology group $\mathcal{H}^n_{\mathfrak{A}}(A,X)$ is defined by the quotient

$$\mathcal{H}^n_{\mathfrak{A}}A,X)=\frac{\mathcal{Z}^n_{\mathfrak{A}}(A,X)}{\mathcal{B}^n_{\mathfrak{A}}(A,X)}.$$

Remark 2.1. *In the above definitions all module maps are additive $\mathfrak{A}$-n-linear, that is, comparing with a n-linear map the coefficient α is coming from $\mathfrak{A}$ instead of $\mathbb{C}$ (see (2.6)). So in general case, since n-$\mathfrak{A}$-module maps are not necessarily n-linear, the $\mathfrak{A}$-module complex $\mathcal{C}_{\mathfrak{A}}(A,X)$ is not subcomplex of cohomology complex $\mathcal{C}(A,X)$. But if we consider $\mathfrak{A}=\mathbb{C}$ and module actions are scaler multiplication, the all additive maps will be linear which means that, $\mathcal{C}^n_{\mathfrak{A}}(A,X)=\mathcal{C}^n(A,X)$, for every $n\in\mathbb{N}_0$. So the module cohomology is just the Hochschild cohomology. That is, $\mathcal{H}^n_{\mathbb{C}}(A,X)=\mathcal{H}^n(A,X)$.*

Definition 2.2. *The Banach algebra A is called $\mathfrak{A}$-module amenable if $\mathcal{H}^1_{\mathfrak{A}}(A,X^*)=0$ for every commutative Banach $\mathfrak{A}$-A-module X. Also A is called weak $\mathfrak{A}$-module amenable (Resp. (n)-weak $\mathfrak{A}$-module amenable) if A is a commutative Banach $\mathfrak{A}$-A-module and $\mathcal{H}^1_{\mathfrak{A}}(A,A^*)=0$ $\big($Resp. $\mathcal{H}^1_{\mathfrak{A}}(A,A^{(n)})=0\big)$.*

Definition 2.3. *The Banach algebra A is called amenable if $\mathcal{H}^1(A,X^*)=0$ for every Banach A-bimodule X and is called weak amenable (Resp. (n)-weak amenable) if $\mathcal{H}^1(A,A^*)=0$ $\big($Resp. $\mathcal{H}^1(A,A^{(n)})=0\big)$.*

3. n-$\ell^1(E)$-module cocycles from $\ell^1(S)$ to $\ell^\infty(S)$

Throughout this paper, we assume S is a commutative inverse semigroup with idempotent set E and semigroup algebra $\ell^1(S)$ is a Banach $\ell^1(E)$-module with actions (1.1). Also it is assumed that $n\in\mathbb{N}$, unless otherwise stated.

Theorem 3.1 (Theorem 4.1 of [8])**.** *Let B be an amenable closed subalgebra of Banach algebra A, X be a dual A-bimodule and $\phi\in\mathcal{Z}^n(A,X)$. Then there is a $\psi\in\mathcal{C}^{n-1}(A,X)$ such that*

$$(\phi-\delta^{n-1}\psi)(a_1,a_2,\dots,a_n)=0,$$

if any one of $a_1,a_2,\dots,a_n\in B$.

Lykova in Theorem 2.6 of [10] by the help of Theorem 3.1, establish a connection between the Hochschild cohomology group and the relative cohomology group of a Banach algebra A for dual A-bimodules X, and showed that

$$\mathcal{H}^n(A,X)=\mathcal{H}^n_B(A,X) \qquad\qquad (n\in\mathbb{N}_0),$$

where B is an amenable closed subalgebra of A.

In Theorem 4.1 of [8], the authors present a method of adjusting cocycles (i.e. perturbing them by coboundaries) via averaging techniques. While some of the results are stated in terms of continuous cohomology with coefficients in a dual Banach module, they hold in greater generality. We have replaced the condition that B be amenable with the weaker condition $\mathcal{H}^1(B,\mathcal{C}^{n-1}(A,X))=0$. An examination of the proof of that Theorem 4.1 of [8], shows that this is the only place where the amenability of B was used. Therefore, in the case that $A=\ell^1(S)$, $X=\ell^\infty(S)$ and $B=\ell^1(E)$, since $\ell^1(E)$ is commutative and weak amenable closed subalgebra of $\ell^1(S)$ so $\mathcal{H}^1(\ell^1(E), \mathcal{Z}^{n-1}(\ell^1(S), \ell^\infty(S))) = 0$ by Theorem 2.8.63 of [4], where $\mathcal{Z}^{n-1}(\ell^1(S), \ell^\infty(S))$ is commutative closed $\ell^1(E)$-submodule of $\mathcal{C}^{n-1}(\ell^1(S), \ell^\infty(S))$ with the actions (8) and (10) in [8].

In this section, in the case that $A = \ell^1(S)$, $X = \ell^\infty(S)$ and $B = \ell^1(E)$, for a commutative inverse semigroup S with idempotent set E, first we show that the concepts relative cohomology group introduced by Lykova in [10] and module cohomology group introduced by the author of the current article and Pourabbas in [11] and [12], are equal. Then, we use some ideas of [10] and prove

$$\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) = \mathcal{H}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \qquad (n \in \mathbb{N}_0),$$

while $\ell^1(E)$ is not necessary amenable Banach algebra.

Lemma 3.2. $\mathcal{C}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \subseteq \mathcal{C}^n(\ell^1(S), \ell^\infty(S))$.

Proof. Let $s_1, s_2, ..., s_n \in S, \lambda \in \mathbb{C}$ and $\phi \in \mathcal{C}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$. For every $1 \le i \le n$, since $\delta_{s_i s_i^*}, \lambda\delta_{s_i s_i^*} \in \ell^1(E)$, we have

$$\begin{aligned}\phi(\delta_{s_1}, ..., \lambda\delta_{s_i}, ..., \delta_{s_n}) &= \phi(\delta_{s_1}, ..., \lambda\delta_{s_i s_i^*}\delta_{s_i}, ..., \delta_{s_n})\\ &= \lambda\delta_{s_i s_i^*}\phi(\delta_{s_1}, ..., \delta_{s_i}, ..., \delta_{s_n})\\ &= \lambda\phi(\delta_{s_1}, ..., \delta_{s_i s_i^* s_i}, ..., \delta_{s_n})\\ &= \lambda\phi(\delta_{s_1}, ..., \delta_{s_i}, ..., \delta_{s_n}).\end{aligned}$$

But since the set of point mass $\{\delta_s : s \in S\}$ is dens in $\ell^1(S)$, thus the result directly follows from continuity ϕ. □

Corollary 3.3. *Previous Lemma shows that for $A = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$ where S be a commutative inverse semigroup with idempotent set E, the concept of relative cohomology group introduced by Lykova in [10] is equivalent to the concept of module cohomology group introduced by the author of the current article and Pourabbas in [11] ([12]).*

Before proceeding further we set up our notations. Let $\phi \in \mathcal{C}^n(\ell^1(S), \ell^\infty(S))$ $(n \in \mathbb{N})$. Suppose $1 \le k \le n$, we say that ϕ is zero on $\ell^1(E)$ of degree k, if $\phi(a_1, a_2, ..., a_n) = 0$ if any one of $a_1, a_2, ..., a_k$ lies in $\ell^1(E)$ and we denote it with $\phi \approx_k 0$. If $\phi \approx_n 0$ we write $\phi \approx 0$. But ϕ is a continuous map and the sets of point masses $\{\delta_s : s \in S\}$ and $\{\delta_e : e \in E\}$ are dens in $\ell^1(S)$ and $\ell^1(E)$, respectively. This fact leads to the following:

$$\phi \approx_k 0 \iff \phi(\delta_{s_1}, \delta_{s_2}, ..., \delta_{s_k}) \text{ if any one of } s_1, s_2, ..., s_k \text{ lies in } E. \tag{3.1}$$

for every $k \in \{1, 2, \dots, n\}$.

The following Lemma is special case of Lemma 2.2 in [10].

Lemma 3.4. *Let $\phi \in \mathcal{C}^n(\ell^1(S), \ell^\infty(S))$ such that $(\delta^n\phi) \approx 0$ and $\phi \approx 0$. Then $\phi \in \mathcal{C}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$.*

According to the preliminary discussion of this section, as a Proposition we obtain:

Proposition 3.5. *Let $\phi \in \mathcal{C}^n(\ell^1(S), \ell^\infty(S))$ such that $(\delta^n\phi) \approx 0$. Then there exists*

$$\psi \in \mathcal{C}^{n-1}(\ell^1(S), \ell^\infty(S))$$

such that $(\phi - \delta^{n-1}\psi) \approx 0$.

Corollary 3.6. *Let $\phi \in \mathcal{Z}^n(\ell^1(S), \ell^\infty(S))$. Then there exists $\psi \in \mathcal{C}^{n-1}(\ell^1(S), \ell^\infty(S))$ such that $(\phi - \delta^{n-1}\psi) \approx 0$. Moreover $(\phi - \delta^{n-1}\psi) \in \mathcal{Z}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$.*

Proof. Using the Lemma 3.4 and Proposition 3.5, the proof is clear. □

Proposition 3.7. *Let $\phi \in \mathcal{C}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ such that $(\delta^n\phi) \approx 0$. Then there exists*

$$\psi \in \mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$$

such that $(\phi - \delta^{n-1}\psi) \approx 0$.

Proof. For $n = 1$, by assumption, for each $e \in E$, since $\delta_e \in \ell^1(E)$, we have

$$0 = (\delta^1\phi)(\delta_e, \delta_e) = \delta_e\phi(\delta_e) - \phi(\delta_{e^2}) + \phi(\delta_e)\delta_e = \phi(\delta_e),$$

and so $\phi \approx 0$. Hence if we take $\psi = 0$, then $(\phi - \delta^0\psi) \approx 0$.

For $n > 1$, we construct, inductively on k, $\psi_1, \psi_2, ..., \psi_k$ in $\mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ such that

$$(\phi - \delta^{n-1}\psi_k) \approx_k 0,$$

for $1 \leq k \leq n$. The conclusion of the Proposition then follows, with $\psi = \psi_n$. To construct ψ_1, we define $\psi_1 \in \mathcal{C}^{n-1}(\ell^1(S), \ell^\infty(S))$ by

$$\psi_1(\delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) := \phi(\delta_{\boldsymbol{e_0}}, \delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}),$$

where $\boldsymbol{e_0} = (s_1s_2...s_{n-1})(s_1s_2...s_{n-1})^*$. It is routine to check that $\psi_1 \in \mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$.

By assumption, for $s_1, s_2, ..., s_{n-1} \in S$ and fix $e \in E$, we have

$$\begin{aligned}
0 &= \delta^n\phi(\delta_{\boldsymbol{e_0}}, \delta_e, \delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&= \delta_{\boldsymbol{e_0}}\phi(\delta_e, \delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad - \phi(\delta_{\boldsymbol{e_0}}\delta_e, \delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad\quad + \phi(\delta_{\boldsymbol{e_0}}, \delta_e\delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad\quad\quad + \sum_{j=1}^{n-2}(-1)^j\phi(\delta_{\boldsymbol{e_0}}, \delta_e, \delta_{s_1}, ..., \delta_{s_js_{j+1}}, ..., \delta_{s_{n-1}}) \\
&\quad\quad\quad\quad + (-1)^{n-1}\phi(\delta_{\boldsymbol{e_0}}, \delta_e, \delta_{s_1}, ..., \delta_{s_{n-2}})\delta_{s_{n-1}} \\
&= \phi(\delta_{\boldsymbol{e_0}}, \delta_e\delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad + \sum_{j=1}^{n-2}(-1)^j\phi(\delta_{\boldsymbol{e_0}}, \delta_e, \delta_{s_1}, ..., \delta_{s_js_{j+1}}, ..., \delta_{s_{n-1}}) \\
&\quad\quad + (-1)^{n-1}\phi(\delta_{\boldsymbol{e_0}}, \delta_e, \delta_{s_1}, ..., \delta_{s_{n-2}})\delta_{s_{n-1}}.
\end{aligned} \tag{3.2}$$

Thus

$$\begin{aligned}
&\delta^{n-1}\psi_1(\delta_e, \delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad = \delta_e\psi_1(\delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad\quad - \psi_1(\delta_e\delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad\quad\quad - \sum_{j=1}^{n-2}(-1)^j\psi_1(\delta_e, \delta_{s_1}, ..., \delta_{s_js_{j+1}}, ..., \delta_{s_{n-1}}) \\
&\quad\quad\quad\quad - (-1)^{n-1}\psi_1(\delta_e, \delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-2}})\delta_{s_{n-1}} \\
&\quad = \delta_e\phi(\delta_{\boldsymbol{e_0}}, \delta_{s_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad\quad - \phi(\delta_{e\boldsymbol{e_0}}, \delta_{es_1}, \delta_{s_2}, ..., \delta_{s_{n-1}}) \\
&\quad\quad\quad - \sum_{j=1}^{n-2}(-1)^j\phi(\delta_{e\boldsymbol{e_0}}, \delta_e, \delta_{s_1}, ..., \delta_{s_js_{j+1}}, ..., \delta_{s_{n-1}}) \\
&\quad\quad\quad\quad - (-1)^{n-1}\phi(\delta_{e\boldsymbol{e_0}}, \delta_e, \delta_{s_2}, \delta_{s_3}, ..., \delta_{s_{n-2}})\delta_{s_{n-1}}.
\end{aligned}$$

Now the sum of the last third terms vanish by (3.2) and we get

$$\begin{aligned}\delta^{n-1}\psi_1(\delta_e,\delta_{s_1},\delta_{s_2},...,\delta_{s_{n-1}}) &= \delta_e\phi(\delta_{\boldsymbol{e_0}},\delta_{s_1},\delta_{s_2},...,\delta_{s_{n-1}})\\ &= \phi(\delta_e\delta_{\boldsymbol{e_0}},\delta_{s_1},\delta_{s_2},...,\delta_{s_{n-1}})\\ &= \phi(\delta_e,\delta_{s_1s_1^*}\delta_{s_1},\delta_{s_2s_2^*}\delta_{s_2},...,\delta_{s_{n-1}s_{n-1}^*}\delta_{s_{n-1}})\\ &= \phi(\delta_e,\delta_{s_1},\delta_{s_2},...,\delta_{s_{n-1}}),\end{aligned}$$

therefore

$$(\phi-\delta^{n-1}\psi_1)(\delta_e,\delta_{s_2},\delta_{s_3},...,\delta_{s_n})=0.$$

This shows that $(\phi-\delta^{n-1}\psi_1)\approx_1 0$.

Suppose now that $1\le k<n$, and a suitable cochain $\psi_k\in\mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$ has been constructed. With define $\sigma:=\phi-\delta^{n-1}\psi_k\in\mathcal{C}^{n}_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$ we have $\sigma\approx_k 0$. In order to continue the inductive process (and so complete the proof of the Proposition), it suffices to construct ψ' in $\mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$ such that $[\sigma-\delta^{n-1}\psi']\approx_{k+1} 0$. For then we have $\phi-\delta^{n-1}(\psi_k+\psi')=\sigma-\delta^{n-1}\psi'$, and we may take $\psi_{k+1}=\psi_k+\psi'$. Now To construct ψ', we define $\omega\in\mathcal{C}^{n-1}(\ell^1(S),\ell^\infty(S))$ by

$$\omega(\delta_{s_1},\delta_{s_2},...,\delta_{s_{n-1}}):=\sigma(\delta_{s_1},\delta_{s_2},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_{s_{k+1}},...,\delta_{s_{n-1}}),\tag{3.3}$$

where $\boldsymbol{e_0}=(s_1s_2...s_{n-1})(s_1s_2...s_{n-1})^*$. It can checked that $\omega\in\mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$ and $\omega\approx_k 0$. Since $\delta^n\phi=\delta^n\sigma$, so by using the coboundary formula (2.3), for each $s_1,s_2,...,s_{n-1}$ and fix $e\in E$, we have

$$\begin{aligned}0 &= \delta^n\sigma(\delta_{s_1},\delta_{s_2},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\ &= \delta_{s_1}\sigma(\delta_{s_2},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\ &\quad+\sum_{j=1}^{k-1}(-1)^j\sigma(\delta_{s_1},...,\delta_{s_js_{j+1}},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\ &\quad+(-1)^k\sigma(\delta_{s_1},...,\delta_{s_k}\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\ &\quad+(-1)^{k+1}\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{\boldsymbol{e_0}}\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\ &\quad+(-1)^{k+2}\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\ &\quad+\sum_{j=k+1}^{n-2}(-1)^{j+2}\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_js_{j+1}},...,\delta_{s_{n-1}})\\ &\quad+(-1)^{n+1}\sigma(\delta_{s_1},\delta_{s_2},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-2}})\delta_{s_{n-1}}.\end{aligned}$$

Now since $\sigma\approx_k 0$, the first and second terms vanish, and since σ is n-$\ell^1(E)$-module map, the third and fourth cancel. Thus

$$\begin{aligned}0 &= (-1)^{k+2}\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\ &\quad+\sum_{j=k+1}^{n-2}(-1)^{j+2}\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_js_{j+1}},...,\delta_{s_{n-1}})\\ &\quad+(-1)^{n+1}\sigma(\delta_{s_1},\delta_{s_2},...,\delta_{s_k},\delta_{\boldsymbol{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-2}})\delta_{s_{n-1}}.\end{aligned}\tag{3.4}$$

On the other hand, by the coboundary formula (2.3), we have

$$\begin{aligned}
&\delta^{n-1}\omega(\delta_{s_1},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\quad=\delta_{s_1}\omega(\delta_{s_2},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\quad+\sum_{j=1}^{k-1}(-1)^j\omega(\delta_{s_1},...,\delta_{s_js_{j+1}},...,\delta_{s_k},\delta_e\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\qquad+(-1)^k\omega(\delta_{s_1},...,\delta_{s_ke},\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\qquad+(-1)^{k+1}\omega(\delta_{s_1},...,\delta_{s_k},\delta_{es_{k+1}},...,\delta_{s_{n-1}})\\
&\qquad+\sum_{j=k+1}^{n-2}(-1)^{j+1}\omega(\delta_{s_1},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_js_{j+1}},...,\delta_{s_{n-1}})\\
&\qquad+(-1)^n\omega(\delta_{s_1},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-2}})\delta_{s_{n-1}}.
\end{aligned}$$

Since $\omega\approx_k 0$, the first and second terms vanish. Therefore, we have

$$\begin{aligned}
&\delta^{n-1}\omega(\delta_{s_1},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\quad=(-1)^k\sigma(\delta_{s_1},...,\delta_{s_ke},\delta_{e\mathbf{e_0}},\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\quad+(-1)^{k+1}\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{e\mathbf{e_0}},\delta_{es_{k+1}},...,\delta_{s_{n-1}})\\
&\quad+\sum_{j=k+1}^{n-2}(-1)^{j+1}\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{e\mathbf{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_js_{j+1}},...,\delta_{s_{n-1}})\\
&\quad+(-1)^n\sigma(\delta_{s_1},...,\delta_{s_k},\delta_{e\mathbf{e_0}},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-2}})\delta_{s_{n-1}}.
\end{aligned}$$

Now the sum of the last third terms vanish by (3.4). Thus

$$\begin{aligned}
&\delta^{n-1}\omega(\delta_{s_1},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\quad=(-1)^k\sigma(\delta_{s_1},...,\delta_{s_k}\delta_e,\delta_e\delta_{\mathbf{e_0}},\delta_{s_{k+1}},...,\delta_{s_{n-1}})\\
&\quad=(-1)^k\sigma(\delta_{s_1s_1^*}\delta_{s_1},...,\delta_{s_ks_k^*}\delta_{s_k},\delta_e,\delta_{s_{k+1}s_{k+1}^*}\delta_{s_{k+1}},...,\delta_{s_{n-1}s_{n-1}^*}\delta_{s_{n-1}})\\
&\quad=(-1)^k\sigma(\delta_{s_1},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}}),
\end{aligned}$$

and hence

$$[\sigma-(-1)^k\delta^{n-1}\omega](\delta_{s_1},...,\delta_{s_k},\delta_e,\delta_{s_{k+1}},...,\delta_{s_{n-1}})=0.$$

This shows that, if $\psi'=(-1)^k\omega$, then $\sigma-\delta^{n-1}\psi'(\delta_{s_1},\delta_{s_2},\dots,\delta_{s_n})$ vanishes when $(k+1)$-th argument lies in $\{\delta_e : e\in E\}$. Thus we can simply show that

$$[\sigma-\delta^{n-1}\psi']\approx_{k+1}0,$$

and the proof is complete. □

Proposition 3.8. *Suppose* $\phi\in\mathcal{C}^n_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))\bigcap\mathcal{B}^n(\ell^1(S),\ell^\infty(S))$. *Then* $\phi\in\mathcal{B}^n_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$.

Proof. For $n=1$, since S is commutative, we have

$$\mathcal{C}^0_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))=\ell^\infty(S)=\mathcal{C}^0(\ell^1(S),\ell^\infty(S)),$$

and therefore,

$$\mathcal{B}^1_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))=\mathcal{B}^1(\ell^1(S),\ell^\infty(S)).$$

For $n\geq 2$, by Proposition 3.5, there exists $\psi\in\mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$ such that

$$(\phi-\delta^{n-1}\psi)\approx 0. \tag{3.5}$$

Now we define

$$\phi' := \phi - \delta^{n-1}\psi.$$

Since $\phi' \approx 0$ by (3.5) and $\delta^n\phi' = \delta^n\phi \approx 0$ so $\phi' \in \mathcal{C}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ by Lemma 3.4.

On the other hand, by assumption, there exists $\psi' \in \mathcal{C}^{n-1}(\ell^1(S), \ell^\infty(S))$ such that $\phi = \delta^{n-1}\psi'$. We have

$$\phi' = \phi - \delta^{n-1}\psi = \delta^{n-1}\psi' - \delta^{n-1}\psi = \delta^{n-1}(\psi' - \psi).$$

Further, we define $\phi'' := \psi' - \psi$. The map ϕ'' satisfies the assumption of Proposition 3.5, so there exists $\psi'' \in \mathcal{C}^{n-2}(\ell^1(S), \ell^\infty(S))$ such that

$$(\phi'' - \delta^{n-2}\psi'') \approx 0. \tag{3.6}$$

Therefore

$$\phi' = \delta^{n-1}(\psi' - \psi) = \delta^{n-1}\phi'' = \delta^{n-1}(\phi'' - \delta^{n-2}\psi'' + \delta^{n-2}\psi'') = \delta^{n-1}\bar{\psi},$$

where $\bar{\psi} := \phi'' - \delta^{n-2}\psi''$. But $\bar{\psi} \approx 0$ by (3.6) and $\delta^{n-1}\bar{\psi} = \phi' \approx 0$ by (3.5), thus $\bar{\psi} \in \mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ by Lemma 3.4. Finally

$$\phi = \phi' + \delta^{n-1}\psi = \delta^{n-1}\bar{\psi} + \delta^{n-1}\psi = \delta^{n-1}(\bar{\psi} + \psi),$$

where $\bar{\psi} + \psi \in \mathcal{C}^{n-1}_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$. This implies $\phi \in \mathcal{B}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$, and the proof is complete. □

4. Module Cohomology Group of Inverse Semigroup Algebras

In the final section, we get the our main results and we establish a connection between n-th Hochschild cohomology group of semigroup algebra $\ell^1(S)$ with coefficients in $\ell^\infty(S)$ and its n-th module cohomology group, for all $n \geq 0$.

Theorem 4.1. *Let S be a commutative inverse semigroup with idempotent set E. Then*

$$\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) = \mathcal{H}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \qquad (n \in \mathbb{N}_0).$$

Proof. For $n = 0$, we have

$$\mathcal{H}^0(\ell^1(S), \ell^\infty(S)) = \mathcal{H}^0_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = \ell^\infty(S).$$

For fix $n \geq 1$, we define morphism

$$\begin{aligned} \Gamma: \qquad & \mathcal{H}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \to \mathcal{H}^n(\ell^1(S), \ell^\infty(S)) \\ & \phi + \mathcal{B}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \mapsto \phi + \mathcal{B}^n(\ell^1(S), \ell^\infty(S)). \end{aligned}$$

where $\phi \in \mathcal{Z}^n_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$. In this case, Γ is well define by Lemma 3.2, surjective by Corollary 3.6 and injective by Proposition 3.8. Hence, the result follows from Lemma 0.5.9 of [9] and Γ is topological isomorphism. □

Finally, we know that $\overline{\ell^\infty(S)}^{w^*} = \ell^\infty(S)^{**}$ and every n-$\ell^1(E)$-module maps from $\ell^1(S)$ to $\ell^\infty(S)$ are continuous and n-linear, by Lemma 3.2. This fact leads to the following result:

Corollary 4.2. *Let S be a commutative inverse semigroup with idempotent set E. Then*

$$\mathcal{H}^n(\ell^1(S), \ell^1(S)^{(2k+1)}) = \mathcal{H}^n_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2k+1)}) \qquad (n, k \in \mathbb{N}_0).$$

Bowling and Duncan in [3] and Gourdeau, Pourabbas and White in [6] show that, the first cohomology group and second cohomology group of $\ell^1(S)$ with coefficients in $\ell^\infty(S)$ are zero and Banach space, respectively, for every Clifford semigroup (and so commutative inverse semigroup) S. Indeed, their results are along with our findings, not only confirms the correctness of Theorem 3.1 of [2], Theorem 2.2 of [11] and Theorem 2.3 of [12], but they improve.

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