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Existence of a Renormalized Solution of Nonlinear Parabolic Equations With General Measure Data

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ABSTRACT: In this paper we prove the existence of a renormalized solution for nonlinear parabolic equations of the type:

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u)\Big) = \mu \qquad \text{in } \Omega \times (0,T),$$

where the right hand side is a general measure, $b(x,u)$ is an unbounded function of u and $-\operatorname{div}(a(x,t,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u.

Key Words: Nonlinear parabolic equations, Renormalized solutions, General measure.

Contents

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N \geq 1)$, T is a positive real number, and let $Q := \Omega \times (0,T)$, $p > 1$. We will consider the following nonlinear parabolic problem

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u)\Big) = \mu \;\; \text{in } Q, \tag{1.1}$$

$$b(x,u)(t=0) = b(x,u_0) \;\; \text{in } \Omega, \tag{1.2}$$

$$u = 0 \;\; \text{on } \partial\Omega \times (0,T). \tag{1.3}$$

In Problem (1.1)-(1.3) the framework is the following: the data μ is a bounded Radon measure on Q, the function $b(x,u_0)$ belongs to $L^1(\Omega)$.
The operator $-\operatorname{div}(a(x,t,\nabla u))$ is a Leray–Lions operator which is coercive and which grows like $|\nabla u|^{p-1}$ with respect to ∇u, (see assumptions (3.4), (3.5) and (3.6) of Section 3).

In the case where $b(x,u) = u$, and the right hand side is a bounded measure, the existence of a distributional solution was proved in [16], but due the lack of regularity of the solutions, the distributional formulation is not strong enough to provide uniqueness (see [41] and [35] for a counter example in the elliptic case). To overcome this difficulty the notion of renormalized solutions firstly introduced by DiPerna and Lions [23] for the study of Boltzmann equation was adapted to the parabolic and elliptic equations with L^1 data (see [4,6,7,30,31,34]). The equivalent notion of entropy solution has been developed independently by [3] for the study of nonlinear elliptic problems and by [37] in the parabolic case. Both renormalized and entropy solutions provide a convenient framework to deal with elliptic or parabolic equations with L^1 data. A large number of papers was then devoted to the study the existence of renormalized (or entropy) solution of parabolic problems with rough data under various

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assumptions and different contexts: in addition to the references already mentioned see, among others, [1,2,9,10,11,12,13,18,19,32,40].

Concerning the datum μ, the existence and uniqueness of renormalized solution of (1.1)-(1.3) have proved in [39] in the case where $b(x,u) = u$, $u_0 \in L^1(\Omega)$ and for every measure μ which does not charge the sets of zero p-capacity, the so-called diffuse measures or soft measures (see Section 2 for the definitions of p-capacity and diffuse measure). The importance of the measures not charging sets of null p-capacity was first observed in the stationary case in [15], and developed in the evolution case in [39].

For $\mu \in \mathcal{M}_0(Q)$, $b(x,u) = b(u)$ and $u_0 \in L^1(\Omega)$, the existence and uniqueness of renormalized solution have been proved in [12]. In the case where $\mu \in \mathcal{M}_0(Q)$ and with the parabolic term on $b(x,u)$, the existence of renormalized solution of problem (1.1)-(1.3) was proved in [38]. For $\mu \in \mathcal{M}(Q)$ (the space of all bounded Radon measures on Q), $b(x,u) = u$ and $u_0 \in L^1(\Omega)$, the existence of renormalized solution was proved in [21] for elliptic case and [33] for parabolic case. Our goal in this paper is to extend the approach in [7] (see also [12], [38]) to the general measure data.

The paper is organized as follows as follows. In Section 2 we give some preliminaries and, in particular, we provide the definition of parabolic capacity and some basic properties.
Section 3 is devoted to specify the assumptions on b, a, u_0 and μ and to give the definition of renormalized solution of (1.1)-(1.3) and see how the definition of renormalized solution does not depend on the decomposition (not uniquely determined) of the regular part of μ we mentioned above and to the statement of standard approximation argument we will use later. In Section 4 we establish (Theorem 4.1) the existence of such a solution.

2. Preliminaries on parabolic capacity

We recall the notion of p-capacity associated to our problem. Let $Q = \Omega \times (0,T)$ for any fixed $T > 0$ and let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|.\|_{W_0^{1,p}(\Omega)} + \|.\|_{L^2(\Omega)}$ and

$$W = \Big\{ u \in L^p(0,T;V), u_t \in L^{p'}(0,T;V') \Big\},$$

endowed with its natural norm $\|.\|_{L^p(0,T;V)} + \|.\|_{L^{p'}(0,T;V')}$, remark that W is continuously embedded in $C([0,T], L^2(\Omega))$, and if $1 < p < \infty$, then $C_c^\infty(\Omega \times [0,T])$ is dense in W.

Let $U \subseteq Q$ be an open set, we define the parabolic p-capacity of U as

$$cap_p(U) = \inf \Big\{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \Big\},$$

where as usual we set $inf\{\emptyset\} = +\infty$, then for any Borel set $B \subseteq Q$ we define

$$cap_p(B) = \inf \Big\{ cap_p(U) : U \text{ open set of } Q, B \subseteq U \Big\}.$$

We define the space $\mathcal{S}$ by

$$\mathcal{S} = \Big\{ u \in L^p(0,T;W_0^{1,p}(\Omega)), u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q) \Big\},$$

endowed with its natural norm $\|.\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|.\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$.
We will denote by $\mathcal{M}(Q)$ the set of all Radon measures with bounded total variation on Q, while $\mathcal{M}_0(Q)$ is the set of all measures with bounded total variation over Q that do not charge the sets of zero p-capacity, that is if $\mu \in \mathcal{M}_0(Q)$, then $\mu(E) = 0$, for all $E \subseteq Q$ such that $cap_p(E) = 0$.

Theorem 2.1. *Let μ be a bounded measure in Q. If $\mu \in \mathcal{M}_0(Q)$ then there exists (f, g_1, g_2) such that $f \in L^1(Q)$, $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T;V)$ and*

$$\int_Q \phi \, d\mu = \int_Q f\phi \, dx \, dt + \int_0^T \langle g_1, \phi \rangle \, dt - \int_0^T \langle \phi_t, g_2 \rangle \, dt \quad \phi \in C_c^\infty(\Omega \times [0,T]).$$

Such a triplet (f, g_1, g_2) will called the decomposition of μ.

Proof. See [39]. □

By a well known decomposition result (see for instance [27]), every μ in $\mathcal{M}(Q)$, can be written as a sum (uniquely determined) of its absolutely continuous part μ_0 with respect to p-capacity and its singular part μ_s concentrated on a set E of zero p-capacity; we will say that $\mu_s \perp cap_p$. Hence, if $\mu \in \mathcal{M}(Q)$, thanks to Theorem 2.1, we have

$$\mu = f - \operatorname{div}(G) + g_t + \mu_s^+ - \mu_s^-,$$

in the sense of distributions, for $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^p(0,T;V)$, where μ_s^+ and μ_s^- are respectively the positive and the negative part of μ_s; note that the decomposition of the absolutely continuous part of μ according to Theorem 2.1 is not uniquely determined. Let us state the following result that will be very useful in the sequel; its proof relies on an easy application of Egorov and Dunford-Pettis theorems.

Proposition 2.1. *Let ρ_ε be a sequence of $L^1(Q)$ functions that converges to ρ weakly in $L^1(Q)$, and let σ_ε be sequence of functions in $L^\infty(Q)$ that is bounded in $L^\infty(Q)$ and converges to σ almost everywhere on Q. Then*

$$\lim_{\varepsilon\to 0}\int_Q \rho_\varepsilon\ \sigma_\varepsilon\ dxdt = \int_Q \rho\ \sigma\ dxdt.$$

Here are some notations we will use throughout this paper. For any non negative real number k we denote by $T_k(r) = min(k, max(r,-k))$ the truncation function at level k. By $\langle .,. \rangle$ we mean the duality between suitable spaces in which functions are involved, in particular we will consider both the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and the duality between $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $W^{-1,p'}(\Omega)+L^1(\Omega)$.

3. Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$

$$b,\ \frac{\partial b}{\partial s} : \Omega\times\mathbb{R}\to\mathbb{R} \text{ and } \nabla_x b : \Omega\times\mathbb{R}\to\mathbb{R}^N, \text{are Carathéodory functions} \tag{3.1}$$

such that for almost every $x \in \Omega$, $b(x,.)$ is a strictly increasing C^1-function with $b(x,0) = 0$. For all $s \in \mathbb{R}$, $b(.,s)$ is in $W^{1,p}(\Omega)$, and there exist $\gamma,\ \Lambda > 0$ such that

$$\gamma \leq \frac{\partial b(x,s)}{\partial s} \leq \Lambda, \tag{3.2}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$. There exists a function B in $L^p(\Omega)$ such that

$$\Big|\nabla_x b(x,s)\Big| \leq B(x), \tag{3.3}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$.

$$a : Q\times\mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function} \tag{3.4}$$

$$a(x,t,\xi).\xi \geq \alpha|\xi|^p, \tag{3.5}$$

for almost every $(x,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ given real number.

$$|a(x,t,\xi)| \leq \beta(L(x,t) + |\xi|^{p-1}), \tag{3.6}$$

for almost every $(x,t)\in Q$, for every $\xi\in\mathbb{R}^N$, where $\beta>0$ given real number, L is a non negative function in $L^{p'}(Q)$.

$$[a(x,t,\xi)-a(x,t,\xi')][\xi-\xi']>0, \tag{3.7}$$

for any $(\xi,\xi')\in\mathbb{R}^{2N}$ with $\xi\neq\xi'$ and for almost every $(x,t)\in Q$.

$$\mu\in\mathcal{M}(Q), \tag{3.8}$$

$$u_0\in L^1(\Omega). \tag{3.9}$$

To simplify notation, let us also define $v=b(x,u)-g$, the definition of a renormalized solution for Problem (1.1)-(1.3) is given below.

Definition 3.1. *A measurable function u defined on Q is a renormalized solution of Problem (1.1)-(1.3) if*

$$T_k(v)\in L^p(0,T;W_0^{1,p}(\Omega))\ \forall k\geq 0\ ,\ \ v\in L^\infty(0,T;L^1(\Omega)), \tag{3.10}$$

and, for every function S in $W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has a compact support and $S(0)=0$, we have

$$S(v)_t-\operatorname{div}\Big(S'(v)a(x,t,\nabla u)\Big)+\ S''(v)a(x,t,\nabla u)\nabla v \tag{3.11}$$

$$=fS'(v)-\operatorname{div}\Big(GS'(v)\Big)+GS''(v)\nabla v\quad in\ D'(Q),$$

$$S(v)(t=0)=S(b(x,u_0))\quad in\ L^1(\Omega). \tag{3.12}$$

For every $\psi\in C(\overline{Q})$, we have

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{(x,t)\in Q\ ;\ n\leq v<2n\}}a(x,t,\nabla u)\nabla v\,\psi\,dx\,dt=\int_Q\psi\,d\mu_s^+, \tag{3.13}$$

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{(x,t)\in Q\ ;\ -2n<v\leq -n\}}a(x,t,\nabla u)\nabla v\,\psi\,dx\,dt=\int_Q\psi\,d\mu_s^-. \tag{3.14}$$

Remark 3.2. *Note that, all terms in (3.11) are well defined, indeed, let $k>0$ such that $supp(S')\subset[-k,k]$, we have $\nabla S(v)=S'(T_k(v))\nabla T_k(v)\in(L^p(Q))^N$, then $S(v)\in L^p(0,T;W_0^{1,p}(\Omega))$. The term $S'(v)a(x,t,\nabla u)$ identifies with*

$$S'(T_k(v))a\Big(x,t,\Big(\frac{\partial b(x,u)}{\partial s}\Big)^{-1}\Big(\nabla T_k(v)+\nabla g-\nabla_x b(x,u)\Big)\Big)$$

a.e. in Q. Using assumptions (3.2) and (3.6) we have

$$\Big|S'(v)a(x,t,\nabla u)\Big| \tag{3.15}$$

$$\leq\beta\ \|S'\|_{L^\infty(\mathbb{R})}\Big[L(x,t)+\gamma^{-(p-1)}\Big|\nabla T_k(v)+\nabla g-\nabla_x b(x,u)\Big|^{p-1}\Big]\ a.e.\ in\ Q,$$

and by (3.3) and (3.10) we deduce that $S'(v)a(x,t,\nabla u)\in(L^{p'}(Q))^N$.

The term $S''(v)a(x,t,\nabla u)\nabla v$ identifies with

$$S''(T_k(v))a\Big(x,t,\Big(\frac{\partial b(x,u)}{\partial s}\Big)^{-1}\Big(\nabla T_k(v)+\nabla g-\nabla_x b(x,u)\Big)\Big)\nabla T_k(v)\ a.e.\ in\ Q.$$

In view of (3.2), (3.3), (3.10), (3.15) and by Hölder inequality we conclude that $S''(v)a(x,t,\nabla u)\nabla v$ in $L^1(Q)$.

Finally, $fS'(v)$, $S''(v)G\nabla v$ are in $L^1(Q)$ and $GS'(v)$ is in $(L^{p'}(Q))^N$. We also have $S(v)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and $S(v) \in L^p(0,T;W_0^{1,p}(\Omega))$, which implies that $S(v) \in C([0,T],L^1(\Omega))$ (see [34]) and (3.12) makes a weak sense.

Remark that, since $\mu_0 \in \mathcal{M}_0(Q)$ and it is defined on the σ-algebra of the borelian of the open set Q, then μ_0 does not charge set at $t=0$, which implies, in the weak sense, that $g(x,0)=0$ for any g such that $(f, div(G), g)$ is a decomposition of μ_0, this explains (3.12), (see [39]).

Furthermore, since $S(v)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, we can use, as test functions in (3.11) not only functions in $C_c^\infty(Q)$ but also functions in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

Now we give the following property of renormalized solutions; throughout the paper C will indicate any positive constant whose value may change from line to line.

Proposition 3.1. *Let $v = b(x,u) - g$ be a renormalized solution of problem (1.1)-(1.3). Then, for every $k>0$, we have*

$$\int_Q |\nabla T_k(v)|^p \, dxdt \leq C(k+1), \tag{3.16}$$

where C is a positive constant not depending on k.

Proof. Using assumptions (3.2) and (3.3), following the same arguments as in [33], yield (3.16). □

Here, we give two results which show that the renormalized solution does not depend on the decomposition of the regular part of μ.

Lemma 3.3. *Let $\mu_0 \in \mathcal{M}_0(Q)$, and let (f, g_1, g_2) and $(\overline{f}, \overline{g_1}, \overline{g_2})$ to be two different decomposition of μ according to Theorem 2.1. Then we have $(g_2 - \overline{g_2})_t = \overline{f} - f + \overline{g_1} - g_1$ in distribution sense, $g_2 - \overline{g_2} \in C([0,T],L^1(\Omega))$ and $(g_2 - \overline{g_2})(0) = 0$.*

Proof. See [39], Lemma 2.29. □

The following result shows that the definition of renormalized solution does not depend on the decomposition of the absolutely continuous part of μ under the condition of bounded perturbations of time derivative part of μ_0, and thanks to the estimate (3.16).

Proposition 3.2. *Let u be a renormalized solution of problem (1.1)-(1.3). Then, u satisfies the definition 3.1 for every decomposition (f, g_1, g_2) such that $g_2 - \overline{g_2} \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.*

Proof. See [33], Proposition 3 and Remark 6. □

4. Existence result

This section is devoted to establish the following existence theorem.

Theorem 4.1. *Under assumptions (3.1)-(3.9), there exists at least a renormalized solution u of the Problem (1.1)-(1.3).*

Proof. We will obtain the existence result by an approximation process, we approximate the measure $\mu \in \mathcal{M}(Q)$ by a sequence defined by

$$\mu^\varepsilon = f^\varepsilon - \operatorname{div}(G^\varepsilon) + \frac{\partial g^\varepsilon}{\partial t} + \lambda_+^\varepsilon - \lambda_-^\varepsilon \tag{4.1}$$

where $f^\varepsilon \in C_c^\infty(Q)$ is a sequence of functions which converges to f weakly in $L^1(Q)$, $G^\varepsilon \in (C_c^\infty(Q))^N$ is a sequence of functions which converges to G strongly in $(L^{p'}(Q))^N$, $g^\varepsilon \in C_c^\infty(Q)$ is a sequence of functions which converges to g strongly in $L^p(0,T;W_0^{1,p}(\Omega))$, and $\lambda_+^\varepsilon \in C_c^\infty(Q)$ (respectively λ_-^ε) is a sequence of non negatives functions that converges to μ_s^+ (respectively μ_s^-) in the narrow topology of measures. Moreover let $u_0^\varepsilon \in C_c^\infty(\Omega)$ such that

$$u_0^\varepsilon \in C_c^\infty(\Omega) : b(x,u_0^\varepsilon) \to b(x,u_0) \textit{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.2}$$

We also assume

$$\|\mu^\varepsilon\|_{L^1(Q)} \le C|\mu|_{\mathcal{M}(Q)} \text{ and } \|b(x,u_0^\varepsilon)\|_{L^1(Q)} \le C\|b(x,u_0)\|_{L^1(Q)}.$$

Let us now consider the following regularized problem:

$$u^\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega)), \tag{4.3}$$

$$\int_0^T \langle \frac{\partial v^\varepsilon}{\partial t}, \varphi\rangle\, dt + \int_Q a(x,t,\nabla u^\varepsilon)\nabla\varphi\, dxdt = \int_Q f^\varepsilon \varphi\, dxdt + \int_Q G^\varepsilon \nabla\varphi\, dxdt + \int_Q \varphi\, d\lambda_+^\varepsilon - \int_Q \varphi\, d\lambda_-^\varepsilon \tag{4.4}$$

$$\forall \varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q),$$

$$b(x,u^\varepsilon)(t=0) \;=\; b(x,u_0^\varepsilon) \text{ in } \Omega, \tag{4.5}$$

where $v^\varepsilon = b(x,u^\varepsilon) - g^\varepsilon$.

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ of (4.3)-(4.5) is an easy task (see [5] and [28]).
Now we prove the following proposition which gives some compactness results.

Proposition 4.1. *Let u^ε and v^ε be defined as before. Then*

$$\|u^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{4.6}$$

$$\int_Q |\nabla T_k(u^\varepsilon)|^p\, dxdt \le Ck, \tag{4.7}$$

$$u^\varepsilon \textit{ is bounded in } L^q(0,T;W_0^{1,q}(\Omega)) \quad \forall\ 1 < q < p - \frac{N}{N+1}, \tag{4.8}$$

$$\|v^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{4.9}$$

$$\int_Q |\nabla T_k(v^\varepsilon)|^p\, dxdt \le C(k+1), \tag{4.10}$$

and, up to a subsequence, for any $k > 0$ we have

$$u^\varepsilon \to u \textit{ a.e. on } Q \textit{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \textit{ and strongly in } L^1(Q), \tag{4.11}$$

$$v^\varepsilon \to v \textit{ a.e. on } Q \textit{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \textit{ and strongly in } L^1(Q), \tag{4.12}$$

$$T_k(u^\varepsilon) \rightharpoonup T_k(u) \textit{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \textit{ and a.e. on } Q, \tag{4.13}$$

$$a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}} \rightharpoonup \sigma_k \textit{ in } (L^{p'}(Q))^N, \tag{4.14}$$

$$T_k(v^\varepsilon) \rightharpoonup T_k(v) \textit{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \textit{ and a.e. on } Q, \tag{4.15}$$

$$\nabla_x b(x,u^\varepsilon) \to \nabla_x b(x,u) \textit{ strongly in } (L^p(Q))^N. \tag{4.16}$$

Proof. We prove (4.6) and (4.7), using $T_k(u^\varepsilon)$ as a test function in (4.4) and we integrate in $]0,t[$ we get

$$\int_\Omega B_k(x,u^\varepsilon)(t)\,dx + \int_0^t\int_\Omega a(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)\,dx\,ds = \int_0^t\int_\Omega \mu^\varepsilon T_k(u^\varepsilon)\,dx\,ds + \int_\Omega B_k(x,u_0^\varepsilon)\,dx, \quad (4.17)$$

for almost every $t \in (0,T)$, and where $B_k(x,s) = \int_0^s T_k(r)\dfrac{\partial b(x,r)}{\partial r}\,dr$.

Using assumption (3.5) and since $B_k(x,u^\varepsilon) \geq 0$, (4.7) derives from (4.17). By (3.2) we have

$$B_1(x,s) \geq \gamma \int_0^s T_1(r)\,dr \quad \forall s \in \mathbb{R},$$

and since $\int_0^s T_1(r)\,dr \geq |s| - 1\ \forall s \in \mathbb{R}$, we obtain

$$\int_\Omega |u^\varepsilon(t)|\,dx \leq \frac{1}{\gamma}(\|b(x,u_0^\varepsilon)\|_{L^1(\Omega)} + |\mu|_{\mathcal{M}(Q)}) + \ meas(\Omega).$$

Hence u^ε is bounded in $L^\infty(0,T;L^1(\Omega))$, which yields (4.6). Moreover, the estimate (4.6) and (4.7) imply also that u^ε is bounded in $L^q(0,T;W_0^{1,q}(\Omega))\ \forall\ 1 < q < p - \frac{N}{N+1}$, according to the results in [14,16,26,24].

Taking $T_k(v^\varepsilon)$ as test function in (4.4) and we integrate in $]0,t[$, by assumptions (3.3), (3.5), (3.6), and by means of Young's inequality one obtains

$$\begin{aligned}\int_\Omega \overline{T_k}(v^\varepsilon)(t)dx + \frac{\alpha}{2}\int_{\{|v^\varepsilon|\leq k\}} \frac{\partial b(x,u^\varepsilon)}{\partial s}|\nabla u^\varepsilon|^p dxdt \qquad (4.18)\\ \leq C(\|G^\varepsilon\|_{L^{p'}(Q)}^{p'} + \|L\|_{L^{p'}(Q)}^{p'} + \|\nabla g^\varepsilon\|_{L^p(Q)}^p + \|B\|_{L^p(\Omega)}^p)\\ +k(\|f^\varepsilon\|_{L^1(Q)} + \|b(x,u_0^\varepsilon)\|_{L^1(\Omega)} + \|\lambda_-^\varepsilon\|_{L^1(Q)} + \|\lambda_+^\varepsilon\|_{L^1(Q)}),\end{aligned}$$

where $\overline{T_k}(s) = \int_0^s T_k(r)\,dr\ \forall s \in \mathbb{R}$. So that (4.9) and (4.10) hold true.

Now, by (4.38) and since λ_+^ε, λ_-^ε are bounded in $L^1(Q)$, one obtains that $\frac{\partial b(x,u^\varepsilon)}{\partial t}$ is bounded in $L^1(0,T;W^{-1,1}(\Omega))$. Moreover, by assumptions (3.2) and (3.3) we have that $b(x,u^\varepsilon)$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for every $1 < q < p - \frac{N}{N+1}$, so that using compactness arguments (see [42]) yield (4.11), and (4.12). Using (4.7) and (4.11) yield (4.13), while (4.14) and (4.15) derives from (4.10) and (4.12). Finally (3.1), (3.3), (4.11) and Lebesgue's convergence theorem give (4.16). □

Let us introduce for $k \geq 0$ fixed, the time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes. More recently, it has been exploited to solve a few nonlinear evolution problems with L^1 or measure data. This specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(v_0^\nu)_\nu$ in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\|v_0^\nu\|_{L^\infty(\Omega)} \leq k$, for all $\nu > 0$, and $v_0^\nu \to T_k(u_0)$ a.e. in Ω with $\frac{1}{\nu}\|v_0^\nu\|_{L^p(\Omega)} \to 0$ as $\nu \to +\infty$.
For fixed $k \geq 0$ and $\nu > 0$, let us consider the unique solution $T_k(u)_\nu \in L^\infty(Q) \cap L^p(0,T,W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial T_k(u)_\nu}{\partial t} + \nu(T_k(u)_\nu - T_k(u)) = 0 \text{ in } D'(Q),$$
$$T_k(u)_\nu(t=0) = v_0^\nu \text{ in } \Omega.$$

The behavior of $T_k(u)_\nu$ as $\nu \to +\infty$ is investigated in [29] (see also [22]) and we just recall here that:

$$T_k(u)_\nu \to T_k(u) \text{ strongly in } L^p(0,T,W_0^{1,p}(\Omega)) \text{ a.e. in } Q \ as\ \nu \to +\infty$$

with $\|T_k(u)_\nu\|_{L^\infty(\Omega)} \leq k$ for any $\nu > 0$, and $\frac{\partial T_k(u)_\nu}{\partial t} \in L^p(0,T,W_0^{1,p}(\Omega))$.
Here and in the rest of paper $\omega(\varepsilon,n,\delta,\mu)$ will indicate any quantity that vanishes as the parameters go to their limit point with in the same order in which they appear, that is, for example

$$\overline{\lim_{\nu\to\infty}}\ \overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\,|\omega(\varepsilon,n,\delta,\nu)| = 0.$$

Now we give the basic result about approximate capacitary potential.

Lemma 4.2. *Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}(Q)$ where μ_s^+ and μ_s^- are concentrated respectively, on two disjoint E^+ and E^- of zero p-capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+\backslash K_\delta^+) \leq \delta, \quad \mu_s^-(E^+\backslash K_\delta^-) \leq \delta, \tag{4.19}$$

and there exist ψ_δ^+, $\psi_\delta^- \in C_0^1(Q)$, such that

$$\psi_\delta^+ \equiv 1 \text{ and } \psi_\delta^- \equiv 1 \text{ respectively on } K_\delta^+ \text{ and } K_\delta^-, \tag{4.20}$$

$$0 \leq \psi_\delta^+, \quad \psi_\delta^- \leq 1, \tag{4.21}$$

$$supp(\psi_\delta^+) \cap supp(\psi_\delta^-) \equiv \emptyset. \tag{4.22}$$

Moreover

$$\|\psi_\delta^+\|_{\mathcal{S}} \leq \delta, \quad \|\psi_\delta^-\|_{\mathcal{S}} \leq \delta, \tag{4.23}$$

and in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}, \tag{4.24}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}. \tag{4.25}$$

Both ψ_δ^+ and ψ_δ^- converges to zero weakly- in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as δ vanishes. Moreover, if λ_+^ε and λ_-^ε are as in (4.1) we have*

$$\int_Q \psi_\delta^- \, d\lambda_+^\varepsilon = \omega(\varepsilon, \delta), \quad \int_Q \psi_\delta^- \, d\mu_s^+ \leq \delta, \tag{4.26}$$

$$\int_Q \psi_\delta^+ \, d\lambda_-^\varepsilon = \omega(\varepsilon, \delta), \quad \int_Q \psi_\delta^+ \, d\mu_s^- \leq \delta, \tag{4.27}$$

$$\int_Q (1 - \psi_\delta^+ \psi_\eta^+) \, d\lambda_+^\varepsilon = \omega(\varepsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^+ \psi_\eta^+) \, d\mu_s^+ \leq \delta + \eta, \tag{4.28}$$

$$\int_Q (1 - \psi_\delta^- \psi_\eta^-) \, d\lambda_-^\varepsilon = \omega(\varepsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^- \psi_\eta^-) \, d\mu_s^- \leq \delta + \eta. \tag{4.29}$$

Proof. See [33], Lemma 5. □

In what follows we will always refer to subsequences of both ψ_δ^+ and ψ_δ^- that satisfy all the convergence results stated in Lemma 4.2.
Now we will prove the following theorem

Theorem 4.3. *Let v^ε and v be as before. Then, for every $k > 0$*

$$T_k(v^\varepsilon) \to T_k(v) \text{ strongly in } L^p(0, T; W_0^{1,p}(\Omega)).$$

Proof. Let us first introduce the following function that we will use in the proof of Theorem 4.3.

$$H_n(s) = \begin{cases} 1 & \text{if } |s| \leq n, \\ \dfrac{2n - s}{n} & \text{if } \; n < s \leq 2n, \\ \dfrac{2n + s}{n} & \text{if } \; -2n < s \leq -n, \\ 0 & \text{if } \; |s| > 2n. \end{cases}$$

Let also introduce another auxiliary function in terms of H_n by $B_n(s) = 1 - H_n(s)$. Our aim is to prove the following asymptotic estimate:

$$\overline{\lim_{\varepsilon \to 0}} \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \, dxdt \leq \int_Q a(x, t, \nabla u) \nabla T_k(v) \, dx \, dt. \tag{4.30}$$

In order to prove (4.30), we shall follow several steps.

⋆ **Step 1**.

For every $\delta, \eta > 0$, let $\psi_\delta^+, \psi_\eta^+$, ψ_δ^- and ψ_η^- as in lemma 4.2 and let E^+ and E^- be the sets where, respectively, μ_s^+ and μ_s^- are concentrated. Setting $\Phi_{\delta,\eta} = \psi_\delta^+\psi_\eta^+ + \psi_\delta^-\psi_\eta^-$, we can write

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon) - T_k(v)_\nu)H_n(v^\varepsilon)\,dx\,dt \tag{4.31}$$

$$= \int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon) - T_k(v)_\nu)H_n(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt$$

$$+ \int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon) - T_k(v)_\nu)H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\,dxdt.$$

Now, if $n > k$, since $a(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le 2n\}})\nabla T_k(v)_\nu$ is weakly compact in $L^1(Q)$ as ε goes to zero, $H_n(v^\varepsilon)$ converges to $H_n(v)$ *-weakly in $L^\infty(Q)$, and almost everywhere in Q, by proposition 2.1 we have

$$\overline{\lim_{\varepsilon\to 0}}\int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon) - T_k(v)_\nu)H_n(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt \tag{4.32}$$

$$= \overline{\lim_{\varepsilon\to 0}}\Big[\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt\Big] - \int_Q \sigma_{2n}\nabla T_k(v)_\nu H_n(v)\Phi_{\delta,\eta}\,dxdt$$

$$= \overline{\lim_{\varepsilon\to 0}}\Big[\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt\Big] - \int_Q \sigma_{2n}\nabla T_k(v)\Phi_{\delta,\eta}\,dxdt + \omega(\nu).$$

Since $\Phi_{\delta,\eta}$ converges to zero $*$- weakly in $L^\infty(Q)$ as δ goes to zero,

$$\int_Q \sigma_{2n}\nabla T_k(v)\Phi_{\delta,\eta}\,dxdt = \omega(\delta).$$

Therefore, if we prove that

$$\overline{\lim_{\eta\to 0}}\;\overline{\lim_{\delta\to 0}}\;\overline{\lim_{\varepsilon\to 0}}\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt \le 0, \tag{4.33}$$

then we can conclude from (4.32) that

$$\overline{\lim_{\eta\to 0}}\;\overline{\lim_{\delta\to 0}}\;\overline{\lim_{\varepsilon\to 0}}\int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon) - T_k(v)_\nu)H_n(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt \le 0. \tag{4.34}$$

⋆ **Step 2**. Near to E.

Before proving (4.33), we first show the following result

Lemma 4.4. *Let u^ε be a solution of (4.3)-(4.5). Let η be a positive real number, and let φ_+^η and φ_-^η be two non negative functions in $C_c^\infty(Q)$ such that*

$$0 \le \varphi_+^\eta \le 1, \qquad 0 \le \varphi_-^\eta \le 1,$$

and

$$0 \le \int_Q \varphi_-^\eta\,d\mu_s^+ \le \eta, \qquad 0 \le \int_Q \varphi_+^\eta\,d\mu_s^- \le \eta, \tag{4.35}$$

we then have

$$\frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}} \frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\varphi_+^\eta\,dx\,dt = \omega(\varepsilon,n,\eta), \tag{4.36}$$

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} \frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\varphi_-^\eta\,dx\,dt = \omega(\varepsilon,n,\eta). \tag{4.37}$$

Proof. Let us prove (4.37); let $\beta_n(s) = B_n(s^+)$, we can choose $\beta_n(v^\varepsilon)\varphi_-^\eta$ as test function in (4.4) and rearranging conveniently all terms we have

$$\begin{aligned}
&\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} \frac{\partial b(x,u^\varepsilon)}{\partial s} a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \varphi_-^\eta\,dx\,dt + \int_Q \beta_n(v^\varepsilon)\varphi_-^\eta\,d\lambda_-^\varepsilon\\
&= \frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon \varphi_-^\eta\,dx\,dt\\
&- \frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla_x b(x,u^\varepsilon)\varphi_-^\eta\,dx\,dt\\
&+ \int_Q \overline{\beta_n}(v^\varepsilon)\frac{d\varphi_-^\eta}{dt}\,dxdt - \int_Q a(x,t,\nabla u^\varepsilon)\nabla\varphi_-^\eta\beta_n(v^\varepsilon)\,dxdt + \int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi_-^\eta\,dxdt\\
&- \int_0^T \langle \mathrm{div}(G^\varepsilon), \beta_n(v^\varepsilon)\varphi_-^\eta\rangle\,dt + \int_Q \beta_n(v^\varepsilon)\varphi_-^\eta\,d\lambda_+^\varepsilon,
\end{aligned}$$

where $\overline{\beta_n}(s) = \int_0^s \beta_n(r)\,dr$. Using the fact that $\int_Q \beta_n(v^\varepsilon)\varphi_-^\eta\,d\lambda_-^\varepsilon \ge 0$ and by assumptions (3.2), (3.3), (3.5), (3.6) and Young's inequality we obtain

$$\begin{aligned}
&\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} \frac{\partial b(x,u^\varepsilon)}{\partial s} a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \varphi_-\,dx\,dt\\
\le\ & \frac{C}{n}\int_Q \left(|\nabla g^\varepsilon|^p + |L|^{p'} + |B|^p\right)dx\,dt\\
&+ \int_Q \overline{\beta_n}(v^\varepsilon)\frac{d\varphi_-^\eta}{dt}\,dx\,dt - \int_Q a(x,t,\nabla u^\varepsilon).\nabla\varphi_-^\eta\beta_n(v^\varepsilon)\,dx\,dt\\
&+ \int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi_-^\eta\,dx\,dt\\
&- \int_0^T \langle \mathrm{div}(G^\varepsilon), \beta_n(v^\varepsilon)\varphi_-^\eta\rangle\,dt + \int_Q \beta_n(v^\varepsilon)\varphi_-^\eta\,d\lambda_+^\varepsilon.
\end{aligned}$$

By (3.6) and (4.8) we have $a(x,t,\nabla u^\varepsilon)$ converges weakly in $(L^{q'}(Q))^N$ as ε goes to 0 for every $q' < 1 + \frac{1}{(N+1)(p-1)}$, since φ_-^η belongs to $C_c^\infty(Q)$ and $\beta_n(v^\varepsilon)$ converges to $\beta_n(v)$ a.e. in Q and $\star$-weakly in $L^\infty(Q)$ as ε goes to zero and $\beta_n(v)$ converges to 0 a.e. in Q and $\star$-weakly in $L^\infty(Q)$ as n goes to $+\infty$, thanks to proposition 2.1, we obtain

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla\varphi_-^\eta\beta_n(v^\varepsilon)\,dx\,dt = \omega(\varepsilon,n).$$

Since $\overline{\beta_n}(v^\varepsilon)$ converges to $\overline{\beta_n}(v)$ in $L^1(Q)$ as ε goes to 0, and $\overline{\beta_n}(v)$ converges to 0 in $L^1(Q)$ as n goes to $+\infty$, we obtain

$$\int_Q \overline{\beta_n}(v^\varepsilon)\frac{d\varphi_-^\eta}{dt}\,dx\,dt = \omega(\varepsilon,n).$$

Moreover, the weak $L^1(Q)$ convergence of f^ε to f and thanks to proposition 2.1 we obtain

$$\int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi_-^\eta\,dxdt = \omega(\varepsilon,n).$$

Due the strong convergence of $\mathrm{div}(G^\varepsilon)$ to $\mathrm{div}(G)$ in $L^{p'}(0,T,W^{-1,p'}(\Omega))$ and the weak convergence in $L^p(0,T,W_0^{1,p}(\Omega))$ of $\beta_n(v^\varepsilon)$ to $\beta_n(v)$ and $\beta_n(v)$ to 0 strongly in $L^p(0,T,W_0^{1,p}(\Omega))$ (this facts is an easy consequence of the estimate on the truncates of u^ε in Proposition 4.1), we obtain

$$\int_0^T \langle \mathrm{div}(G^\varepsilon), \beta_n(v^\varepsilon)\varphi_-^\eta\rangle\,dxdt = \omega(\varepsilon,n).$$

Finally, by (4.35) and since $\beta_n(v^\varepsilon)$ is non negative and bounded and φ_-^η is continuous, we have

$$\int_Q \beta_n(v^\varepsilon)\varphi_-^\eta \, d\lambda_+^\varepsilon \le \int_Q \varphi_-^\eta \, d\mu_s^+ + \omega(\varepsilon) = \omega(\varepsilon,\eta).$$

Putting together all these facts lead to (4.37), while (4.36) can be obtained in the same way choosing $\beta_n(s) = B_n(s^-)$ and $\beta_n(v^\varepsilon)\varphi_+^\eta$ as test function in (4.4). □

Now let us check (4.33). For fixed $k > 0$, we choose $(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+$ as test function in (4.4), defining $\Gamma_{n,k}(s) = \int_0^s (k - T_k(r))H_n(r)\,dr$, and integrating by parts, we obtain

$$\begin{aligned}
&- \int_Q \Gamma_{n,k}(v^\varepsilon)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\,dxdt \qquad (4.38)\\
&+ \int_Q (k - T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt\\
&+ \int_Q a(x,t,\nabla u^\varepsilon)\nabla H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt\\
&- \int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt\\
&= \int_Q f^\varepsilon H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt - \int_0^T \langle \operatorname{div}(G^\varepsilon), H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\rangle\,dt\\
&+ \int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon - \int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_-^\varepsilon.
\end{aligned}$$

For $n > k$, we have

$$H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}} = a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}} \text{ a.e. in } Q,$$

then rearranging all terms of (4.38), we obtain

$$\begin{aligned}
&\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt + \int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon\\
&= - \int_Q \Gamma_{n,k}(v^\varepsilon)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\,dxdt + \frac{2k}{n}\int_{\{-2n<v\le -n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\delta^+\psi_\eta^+\,dx\,dt\\
&+ \int_Q (k - T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt\\
&- \int_Q f^\varepsilon(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt\\
&- \int_0^T \langle \operatorname{div}(G^\varepsilon), H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\rangle\,dt + \int_Q (k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,d\lambda_-^\varepsilon. \qquad (4.39)
\end{aligned}$$

Let us analyze term by term the right hand side of (4.39). Due to Proposition 4.1 we have $\Gamma_{n,k}(v^\varepsilon)$ converges to $\Gamma_{n,k}(v)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, and since $\Gamma_{n,k}(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, we deduce

$$\begin{aligned}
&\int_Q \Gamma_{n,k}(v^\varepsilon)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\,dxdt\\
&= \int_Q \Gamma_{n,k}(v)\frac{d\psi_\delta^+}{dt}\psi_\eta^+\,dxdt + \int_Q \Gamma_{n,k}(v)\frac{d\psi_\eta^+}{dt}\psi_\delta^+\,dxdt + \omega(\varepsilon) = \omega(\varepsilon,\delta).
\end{aligned}$$

Since $(k-T_k(v^\varepsilon))H_n(v^\varepsilon)$ converges to $(k-T_k(v))H_n(v)$ a.e. and $*$- weakly in $L^\infty(Q)$, thanks to Proposition 2.1, Proposition 4.1 and Lemma 4.2, we deduce

$$\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt$$

$$=\int_Q (k-T_k(v))H_n(v)\sigma_{2n}\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt+\omega(\varepsilon)=\omega(\varepsilon,\delta).$$

Moreover, $(k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+$ weakly converges to $(k-T_k(v))H_n(v)\psi_\delta^+\psi_\eta^+$ in $L^p(0,T;W_0^{1,p}(\Omega))$, and *- weakly in $L^\infty(Q)$, thanks again to Lemma 4.2, we have

$$\int_0^T \langle \mathrm{div}(G^\varepsilon),(k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\rangle\,dt=\omega(\varepsilon,\delta),$$

and

$$\int_Q f^\varepsilon(k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt=\omega(\varepsilon,\delta).$$

Using assumptions (3.2), (3.3), (3.6), Young's inequality, and since $0\le\psi_\delta^+\le 1$ we obtain

$$\left|\frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\delta^+\psi_\eta^+\,dx\,dt\right|$$

$$\le\frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}}\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\psi_\eta^+\,dx\,dt$$
$$+\frac{C}{n}\int_Q\left(|\nabla g^\varepsilon|^p+|L|^{p'}+|B|^p\right)dxdt,$$

applying Lemma 4.4 for $\varphi_+^\eta=\psi_\eta^+$, we obtain

$$\frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}}a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\eta^+\,dx\,dt=\omega(\varepsilon,n,\eta).$$

Using (4.27) in Lemma 4.2, we have

$$\left|\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_-^\varepsilon\right|\le 2k\int_Q\psi_\delta^+\psi_\eta^+d\lambda_-^\varepsilon$$

$$=2k\int_Q\psi_\delta^+\psi_\eta^+d\mu_s^-+\omega(\varepsilon)=\omega(\varepsilon,\delta).$$

Collecting all we have shown above, we get

$$\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon+\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt=\omega(\varepsilon,\delta,n,\eta).$$

Since $\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon\ge 0$, we obtain

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt\le\omega(\varepsilon,\delta,\eta).$$

On the other hand, reasoning as before with $(k+T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^-\psi_\eta^-$ as test function we can obtain

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\psi_\delta^-\psi_\eta^-\,dxdt\le\omega(\varepsilon,\delta,\eta).$$

Therefore, we obtain (4.33) which yields (4.34).

Remark 4.5. *As we have shown above we have*

$$\int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon + \int_Q \frac{\partial b(x,u^\varepsilon)}{\partial s} a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla u^\varepsilon \psi_\delta^+\psi_\eta^+\,dxdt$$

$$+ \int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla_x b(x,u^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt$$

$$- \int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla g^\varepsilon \psi_\delta^+\psi_\eta^+\,dxdt = \omega(\varepsilon,\delta,n,\eta),$$

by assumptions (3.2), (3.5), thanks to proposition 4.1 and Lemma 4.2 one obtains

$$\int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon = \omega(\varepsilon,\delta,n,\eta).$$

Analogously we obtain

$$\int_Q H_n(v^\varepsilon)(k + T_k(v^\varepsilon))\psi_\delta^-\psi_\eta^-\,d\lambda_-^\varepsilon = \omega(\varepsilon,\delta,n,\eta).$$

The two last results above show an interesting property of approximating renormalized, they expresse the fact that v^ε (and so the solution u^ε) is very large (greater than any $k > 0$) on the set where the singular measure μ_s^+ is concentrated, and small (smaller than any $k < 0$) on the set where the singular measure μ_s^- is concentrated.

⋆ **Step 3**. Far from E.
We first prove a result that will be essential to deal with the second term on the right hand side of (4.31).

Lemma 4.6. *Let $k \ge 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r) = r$ for $|r| \le k$ and supp S' is compact. Then*

$$\int_0^T\int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \ge \omega(\varepsilon,\nu).$$

Proof. The proof of the above Lemma follows the arguments in [8], Lemma 1 and we just sketch the proof of it.
Let $k \ge 0$ be fixed. Since S is increasing and $S(r) = r$ for $|r| \le k$,

$$T_k(S(v^\varepsilon)) = T_k(v^\varepsilon) \text{ and } T_k(S(v)) = T_k(v) \text{ a.e. in } Q.$$

As a consequence $T_k(S(v))_\nu = T_k(v)_\nu$ a.e. in Q, for any $\nu > 0$.
It follows that under the notation $z^\varepsilon = S(v^\varepsilon)$ and $z = S(v)$, and thanks to properties of $T_k(z)_\nu$ we have

$$\int_0^T\int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \tag{4.40}$$

$$= \int_0^T\int_0^t \langle \frac{\partial z^\varepsilon}{\partial t}, (T_k(z^\varepsilon) - T_k(z)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt$$

$$= \int_0^T\int_0^t \langle \frac{\partial (z^\varepsilon - T_k(z)_\nu)}{\partial t}, (z^\varepsilon - T_k(z)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt$$

$$- \int_0^T\int_0^t \langle \frac{\partial z^\varepsilon}{\partial t}, (z^\varepsilon - T_k(z^\varepsilon))(1-\Phi_{\delta,\eta})\rangle\,dsdt$$

$$+ \int_0^T\int_0^t \langle \frac{\partial T_k(z)_\nu}{\partial t}, (z^\varepsilon - T_k(z)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt,$$

integrating by parts we have

$$\int_0^T\int_0^t\langle\frac{\partial S(v^\varepsilon)}{\partial t},(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \tag{4.41}$$

$$=\frac{1}{2}\int_0^T\int_0^t\int_\Omega(z^\varepsilon-T_k(z)_\nu)^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdsdt-\frac{1}{2}\int_0^T\int_\Omega(z^\varepsilon-T_k(z^\varepsilon))^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdsdt$$
$$+\frac{1}{2}\int_0^T\int_\Omega(z^\varepsilon-T_k(z)_\nu)^2\,dxdt-\frac{T}{2}\int_\Omega(z^\varepsilon-T_k(z)_\nu)^2(t=0)\,dx$$
$$-\frac{1}{2}\int_0^T\int_\Omega(z^\varepsilon-T_k(z^\varepsilon))^2\,dxdt+\frac{T}{2}\int_\Omega(z^\varepsilon-T_k(z^\varepsilon))^2(t=0)\,dx$$
$$+\int_0^T\int_0^t\int_\Omega\frac{\partial T_k(z)_\nu}{\partial t}(z^\varepsilon-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\,dx\,ds\,dt,$$

since $\int_0^r(s-T_k(s))\,ds=\frac{1}{2}(r-T_k(r))^2$.

Using the definition of z^ε and z, the fact that S is bounded and v^ε converges to v a.e. on Q, we have z^ε converges to z strongly in $L^2(Q)$ and in $L^\infty(Q)$ $*-$ weakly, the strong convergence of $b(x,u_0^\varepsilon)$ to $b(x,u_0)$ in $L^1(\Omega)$ implies that $z^\varepsilon(t=0)$ converges to $S(b(x,u_0))$ strongly in $L^2(\Omega)$.

Passing to the limit as ε tends to zero in (4.41) leads to

$$\int_0^T\int_0^t\langle\frac{\partial S(v^\varepsilon)}{\partial t},(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \tag{4.42}$$

$$=\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z)_\nu)^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdt-\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z))^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdt$$
$$+\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z)_\nu)^2\,dxdt-\frac{T}{2}\int_\Omega(z-T_k(z)_\nu)^2(t=0)\,dx$$
$$-\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z))^2\,dxdt+\frac{T}{2}\int_\Omega(z-T_k(z))^2(t=0)\,dx$$
$$+\int_0^T\int_0^t\int_\Omega\frac{\partial T_k(z)_\nu}{\partial t}(z-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\,dx\,ds\,dt+\omega(\varepsilon),$$

by rewriting the definition of $T_k(u)_\nu$ in terms of $T_k(z)$ we have

$$\frac{\partial T_k(z)_\nu}{\partial t}+\nu(T_k(z)_\nu-T_k(z))=0\text{ in }D'(Q),$$

$$T_k(z)_\nu(t=0)=v_0^\nu\text{ in }\Omega.$$

By properties of $T_k(z)_\nu$ we obtain that $T_k(z)_\nu$ converges to $T_k(z)$ strongly in $L^2(Q)$ and $T_k(z)_\nu(t=0)$ converges to $T_k(S(b(x,u_0)))$ strongly in $L^2(\Omega)$ as ν tends to ∞. Passing to the limit-inf as ν tends to ∞ in (4.42) leads to

$$\varliminf_{\nu\to\infty}\lim_{\varepsilon\to0}\int_0^T\int_0^T\langle\frac{\partial S(v^\varepsilon)}{\partial t},(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt$$
$$=\nu\int_0^T\int_0^t\int_\Omega(T_k(z)-T_k(z)_\nu)(z-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\,dxdsdt.$$

Thanks to definition of $T_k(z)_\nu$ we have

$$\int_0^T\int_0^t\int_\Omega(T_k(z)-T_k(z)_\nu)(z-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\,dxdsdt$$

$$= \int_{\{|z|\le k\}} (z - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta})\; dxdsdt$$

$$+ \int_{\{z>k\}} (k - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta})\; dxdsdt$$

$$+ \int_{\{z<-k\}} (-k - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta})\; dxdsdt,$$

and the three terms are all non negatives, then

$$\int_0^T \int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\; dt \ge \omega(\varepsilon,\nu)$$

□

Now, let us multiply by $H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})$ the equation solved by u^ε and integrate to obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial v^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\Big\rangle\; dt \qquad (4.43)\\
&+ \int_Q a(x,t,\nabla u^\varepsilon).\nabla(T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\, dxdt\\
&+ \int_Q a(x,t,\nabla u^\varepsilon).\nabla H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, dxdt\\
&- \int_Q a(x,t,\nabla u^\varepsilon).\nabla \Phi_{\delta,\eta} H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)\, dxdt\\
&= \int_Q f^\varepsilon H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, dxdt\\
&- \int_0^T \langle \mathrm{div}(G^\varepsilon), H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\, dt\\
&+ \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, d\lambda_+^\varepsilon\\
&- \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, d\lambda_-^\varepsilon.
\end{aligned}$$

Let us analyze term by term the identity (4.43), by Lemma 4.6 we have

$$\int_0^T \Big\langle \frac{\partial v^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\Big\rangle\; dt \ge \omega(\varepsilon,\nu).$$

The almost everywhere and *-weak convergence of $H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)$ to $H_n(v)(T_k(v) - T_k(v)_\nu)$ in $L^\infty(Q)$, the properties of $T_k(v)_\nu$ and thanks to Propositions 2.1 and 4.1 we have

$$\int_Q a(x,t,\nabla u^\varepsilon).\nabla \Phi_{\delta,\eta} H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)\, dxdt = \omega(\varepsilon,\nu).$$

Due the strong convergence of $\mathrm{div}(G^\varepsilon)$ to $\mathrm{div}(G)$ in $L^{p'}(0,T,W^{-1,p'}(\Omega))$, Proposition 4.1 and the properties of $T_k(v)_\nu$ one obtains

$$\int_0^T \langle \mathrm{div}(G^\varepsilon), H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\, dt = \omega(\varepsilon,\nu).$$

The weak convergence of f^ε to f in $L^1(Q)$, the almost everywhere and $*-$ weak convergence of $H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)$ to $H_n(v)(T_k(v)-T_k(v)_\nu)$ in $L^\infty(Q)$, Propositions 2.1, the properties of $T_k(v)_\nu$ and the Lebesgue's dominated convergence theorem leads to

$$\int_Q f^\varepsilon H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dxdt=\omega(\varepsilon,\nu).$$

By Lemma 4.2 and the fact that $|H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)|\le 2k$ we obtain

$$\begin{aligned}&\Big|\int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,d\lambda_+^\varepsilon\Big|\\&\le 2k\int_Q(1-\psi_\delta^+\psi_\eta^+)d\lambda_+^\varepsilon+2k\int_Q\psi_\delta^-\psi_\eta^-\,d\lambda_+^\varepsilon,\end{aligned}$$

and

$$\int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,d\lambda_+^\varepsilon=\omega(\varepsilon,\delta,\eta),$$

and similarly we get

$$\int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,d\lambda_-^\varepsilon=\omega(\varepsilon,\delta,\eta).$$

It remains to prove that

$$\int_Q a(x,t,\nabla u^\varepsilon).\nabla H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dxdt=\omega(\varepsilon,n,\delta,\eta).$$

We have

$$\begin{aligned}&\Big|\frac{1}{n}\int_{\{n\le|v^\varepsilon|<2n\}}a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dx\,dt\Big|\\&\le\frac{2k}{n}\int_{\{n\le|v^\varepsilon|<2n\}}\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\Phi_{\delta,\eta})\,dx\,dt\\&\quad+\frac{C}{n}\int_Q\Big(|\nabla g^\varepsilon|^p+|L|^{p'}+|B|^p\Big)\,dxdt=I_1+I_2,\end{aligned}$$

we have $I_2=\omega(n)$, and we rewrite I_1 as follows:

$$\begin{aligned}I_1=&\frac{2k}{n}\int_{\{n\le v^\varepsilon<2n\}}\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\psi_\delta^+\psi_\eta^+)\,dx\,dt,\\&-\frac{2k}{n}\int_{\{n\le v^\varepsilon<2n\}}\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\psi_\delta^-\psi_\eta^-\,dx\,dt,\\&+\frac{2k}{n}\int_{\{-2n<v^\varepsilon\le-n\}}\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\psi_\delta^-\psi_\eta^-)\,dx\,dt,\\&-\frac{2k}{n}\int_{\{-2n<v^\varepsilon\le-n\}}\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\psi_\delta^+\psi_\eta^+\,dx\,dt.\end{aligned}$$

We can apply Lemma 4.4 for every term above. Indeed, if we define $\varphi_-^{\delta,\eta}=1-\psi_\delta^+\psi_\eta^+$, we have by Lemma 4.2,

$$\int_Q\varphi_-^{\delta,\eta}\,d\mu_s^+\le\eta+\delta,$$

then $\varphi_-^{\delta,\eta}$ satisfies (4.33), thanks to Lemma 4.4 we obtain

$$\begin{aligned}\frac{2k}{n}\int_{\{n\le v^\varepsilon<2n\}}\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\psi_\delta^+\psi_\eta^+)\,dx\,dt &\le \omega(\varepsilon,n)+\delta+\eta\\ &= \omega(\varepsilon,n,\delta,\eta).\end{aligned}$$

In analogous way we obtain the same result for the others terms. Therefore, we obtain our estimate far from E

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon)-T_k(v)_\nu)H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\,dxdt \le \omega(\varepsilon,\nu,n,\delta,\eta). \tag{4.44}$$

□

⋆ **Step 4**. Strong convergence of truncates.
Collecting together (4.31), (4.34) and (4.44), we have by taking again $n>k$,

$$\overline{\lim_{\varepsilon\to 0}}\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,dxdt \le \int_Q \sigma_k\nabla T_k(v)\,dxdt. \tag{4.45}$$

Now, we prove that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_Q \frac{\partial b(x,u^\varepsilon)}{\partial s}\Big[a(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}})-a(x,t,\nabla u\chi_{\{|v|\le k\}})\Big]\\ \times\Big[\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}-\nabla u\chi_{\{|v|\le k\}}\Big]\,dx\,dt=0.\end{aligned} \tag{4.46}$$

We set

$$\begin{aligned}A^\varepsilon=\int_Q\frac{\partial b(x,u^\varepsilon)}{\partial s}\Big[a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}-a(x,t,\nabla u)\chi_{\{|v|\le k\}}\Big]\\ \times[\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}-\nabla u\chi_{\{|v|\le k\}}]\,dxdt.\end{aligned}$$

We split (4.38), into $A^\varepsilon=A_1^\varepsilon+A_2^\varepsilon+A_3^\varepsilon$, where

$$A_1^\varepsilon=\int_Q\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}\,dxdt,$$

$$A_2^\varepsilon=-\int_Q\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u\chi_{\{|v^\varepsilon|\le k\}}\chi_{\{|v|\le k\}}\,dxdt,$$

$$A_3^\varepsilon=-\int_Q\frac{\partial b(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u)(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}})-\nabla u\chi_{\{|v|\le k\}}))\,dxdt.$$

We pass to the limit as ε tends to 0 in A_1^ε, A_2^ε and A_3^ε. Let us remark that we have $\frac{\partial b(x,u^\varepsilon)}{\partial s}\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}=\nabla T_k(v^\varepsilon)-\nabla_x b(x,u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}+\nabla g^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}$ a.e in Q, and we have also $\chi_{\{|v^\varepsilon|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ in Q (see [14]), we obtain:

$$\begin{aligned}&\lim_{\varepsilon\to 0}A_1^\varepsilon\\ &=\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,dxdt+\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla g^\varepsilon\,dx\,dt\\ &\le\int_Q\sigma_k\nabla T_k(v)\,dx\,dt-\int_Q\sigma_k\nabla_x b(x,u)\chi_{\{|v|\le k\}}\,dx\,dt+\int_Q\sigma_k\nabla g\chi_{\{|v|\le k\}}\,dx\,dt.\end{aligned} \tag{4.47}$$

As a consequence of Proposition 4.1, we deduce that

$$\lim_{\varepsilon\to 0} A_2^\varepsilon = -\int_Q \sigma_k(\nabla T_k(v) - \nabla_x b(x,u) + \nabla g)\,dx\,dt, \tag{4.48}$$

and

$$\lim_{\varepsilon\to 0} A_3^\varepsilon = -\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u)\Big(\nabla T_k(v^\varepsilon) - (\nabla_x b(x,u^\varepsilon) + \nabla g^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}} \tag{4.49}$$

$$-\frac{\partial b(x,u^\varepsilon)}{\partial s}\Big(\frac{\partial b(x,u)}{\partial s}\Big)^{-1}\Big(\nabla T_k(v) - (\nabla_x b(x,u) + \nabla g)\chi_{\{|v|\le k\}}\Big)\Big)\,dx\,dt = 0.$$

Therefore collecting (4.47), (4.48) and (4.49) yield (4.46). Using (4.45), (4.46) and the usual Minty's argument we deduce that, $\sigma_k = a(x,t,\nabla u)\chi_{\{|v|\le k\}}$. Through the monotonicity argument which relies on (3.7) (see [17], Lemma 5), we can deduce from (4.46) that

$$\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}} \to \nabla u \chi_{\{|v|\le k\}} \text{ a.e. in } Q,$$

and since $a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}}$ converges to $a(x,t,\nabla u)\nabla u \chi_{\{|v|\le k\}}$ weakly in $L^1(Q)$, by coercivity argument we have that $|\nabla u^\varepsilon|^p \chi_{\{|v^\varepsilon|\le k\}}$ is equi-integrable, as a consequence of Vitali's theorem and since g^ε strongly converges in $L^p(0,T;W_0^{1,p}(\Omega))$ yields

$$T_k(v^\varepsilon) \to T_k(v) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)),$$

the proof of Theorem 4.3, is complete. □

Proof. (*Proof of Theorem 4.1*). Now we are able to prove that Problem (1.1)-(1.3) has a renormalized solutions.
Let S in $W^{2,\infty}(\mathbb{R})$, such that S' has a compact support as in Definition 3.1, and let $\varphi \in C_c^\infty(Q)$, then the approximating solutions u^ε (and v^ε) satisfy

$$-\int_0^T \langle \varphi_t, S(v^\varepsilon)\rangle\,dt + \int_Q S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla\varphi\,dxdt + \int_Q S''(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\varphi\,dxdt \tag{4.50}$$

$$= \int_Q f^\varepsilon S'(v^\varepsilon)\varphi\,dxdt + \int_Q G^\varepsilon S'(v^\varepsilon)\nabla\varphi\,dxdt + \int_Q S''(v^\varepsilon)G^\varepsilon\nabla v^\varepsilon\varphi\,dxdt$$

$$+\int_Q S'(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon - \int_Q S'(v^\varepsilon)\varphi\,d\lambda_-^\varepsilon.$$

Thanks to Theorem 4.3, all terms in (4.50) easily pass to the limit on ε except the last two terms that give some problem. We can write following the arguments in [33] we have

$$\int_Q S'(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon = \int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\,d\lambda_+^\varepsilon + \int_Q S'(v^\varepsilon)\varphi(1-\psi_\delta^+)\,d\lambda_+^\varepsilon. \tag{4.51}$$

Let ψ_δ^+ be defined as in Lemma 4.2, then we have

$$\Big|\int_Q S'(v^\varepsilon)\varphi(1-\psi_\delta^+)\,d\lambda_+^\varepsilon\Big| \le C\int_Q (1-\psi_\delta^+)\,d\lambda_+^\varepsilon = \omega(\varepsilon,\delta),$$

while choosing $S'(v^\varepsilon)\varphi\psi_\delta^+$ in (4.4) one gets,

$$\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\,d\lambda_+^\varepsilon = -\int_Q f^\varepsilon S'(v^\varepsilon)\varphi\psi_\delta^+\,dxdt - \int_Q G^\varepsilon S'(v^\varepsilon)\nabla(\varphi\psi_\delta^+)\,dxdt \tag{4.52}$$

$$-\int_Q G^\varepsilon S''(v^\varepsilon)\nabla v^\varepsilon\varphi\psi_\delta^+\,dxdt + \int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\,d\lambda_-^\varepsilon - \int_Q S(v^\varepsilon)(\varphi\psi_\delta^+)_t\,dxdt$$

$$+\int_Q S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\varphi)\,dxdt+\int_Q S''(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\delta^+\varphi\,dxdt.$$

Now, thanks to Proposition 4.1 and the properties of ψ_δ^+, we have

$$\int_Q f^\varepsilon S'(v^\varepsilon)\varphi\psi_\delta^+\,dxdt=\omega(\varepsilon,\delta)\text{ and }\int_Q G^\varepsilon S'(v^\varepsilon)\nabla(\varphi\psi_\delta^+)\,dxdt=\omega(\varepsilon,\delta).$$

By Lemma 4.2, we deduce

$$\Big|\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\,d\lambda_-^\varepsilon\Big|\leq C\int_Q\psi_\delta^+\,d\lambda_-^\varepsilon=\omega(\varepsilon,\delta).$$

Again by Lemma 4.2, and since $S(v)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$,

$$\int_Q S(v^\varepsilon)(\varphi\psi_\delta^+)_t\,dxdt=\omega(\varepsilon,\delta).$$

By Theorem 4.3 and Lemma 4.2, we have

$$\int_Q S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\varphi)\,dxdt=\omega(\varepsilon,\delta),$$

and

$$\int_Q S''(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\delta^+\varphi\,dxdt=\omega(\varepsilon,\delta).$$

Therefore, from (4.52) we deduce

$$\int_Q S'(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon=\omega(\varepsilon).\tag{4.53}$$

Similarly, we can prove that

$$\int_Q S'(v^\varepsilon)\varphi\,d\lambda_-^\varepsilon=\omega(\varepsilon).\tag{4.54}$$

As a consequence of the above convergence results, we are in a position to pass to the limit as ε tends to 0 in (4.50) and to conclude that u satisfies (3.11).

It remains to show that $S(v)$ satisfies the initial condition (3.12). To this end, firstly remark that $S(v^\varepsilon)$ being bounded in $L^\infty(Q)$, secondly, (4.50) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial S(v^\varepsilon)}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))$.

As a consequence, an Aubin's type lemma (see e.g., [42], Corollary 4) implies that $S(v^\varepsilon)$ lies in a compact set of $\mathcal{C}([0,T];W^{-1,s}(\Omega))$ for any $s<\inf(p',\frac{N}{N-1})$. It follows that, on one hand, $S(v^\varepsilon)(t=0)$ converges to $S(v)(t=0)$ strongly in $W^{-1,s}(\Omega)$, On the other hand, the smoothness of S imply that $S(v^\varepsilon)(t=0)$ converges to $S(b(x,u))(t=0)$ strongly in $L^q(\Omega)$ for all $q<\infty$. Due to (4.2), we conclude that $S(v^\varepsilon)(t=0)=S(b(x,u_0^\varepsilon))$ converges to $S(b(x,u)(t=0)$ strongly in $L^q(\Omega)$. Then v satisfies (3.12).

Now choosing $\beta_n(v^\varepsilon)$ as test function in (4.4) where $\varphi\in C_c^\infty(Q)$, we obtain

$$\begin{aligned}-\int_0^T\langle\varphi_t,\overline{\beta_n}(v^\varepsilon)\rangle\,dt+\int_Q\beta_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla\varphi\,dxdt\\ +\frac{1}{n}\int_{\{n\leq v^\varepsilon<2n\}}a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\varphi\,dxdt\\ =\int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi\,dxdt-\int_0^T\langle\operatorname{div}(G^\varepsilon),\beta_n(v^\varepsilon)\varphi\rangle\,dxdt\\ +\int_Q\beta_n(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon-\int_Q\beta_n(v^\varepsilon)\varphi\,d\lambda_-^\varepsilon.\end{aligned}\tag{4.55}$$

Reasoning as before (in particular as in the proof of Lemma 4.4) we obtain

$$\int_0^T \langle \varphi_t, \overline{\beta_n}(v^\varepsilon)\rangle \, dt = \omega(\varepsilon, n), \qquad \int_Q \beta_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon).\nabla\varphi \, dxdt = \omega(\varepsilon, n),$$

$$\int_Q f^\varepsilon \beta_n(v^\varepsilon)\varphi \, dxdt = \omega(\varepsilon, n), \qquad \int_0^T \langle \operatorname{div}(G^\varepsilon), \beta_n(v^\varepsilon)\varphi\rangle \, dxdt = \omega(\varepsilon, n).$$

Thanks to Theorem 4.3 we have

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon \varphi \, dxdt = \frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v \; \varphi \, dxdt + \omega(\varepsilon).$$

Now we deal with the two last terms in the right hand side of (4.55) we can write

$$\int_Q \beta_n(v^\varepsilon)\varphi \, d\lambda_+^\varepsilon = -\int_Q h_n(v^\varepsilon)\varphi \, d\lambda_+^\varepsilon + \int_Q \varphi \, d\lambda_+^\varepsilon,$$

where $h_n(s) = H_n(s^+)$. By construction of λ_+^ε we have

$$\int_Q \varphi \, d\lambda_+^\varepsilon = \int_Q \varphi \, d\mu_s^+ + \omega(\varepsilon).$$

Following the same argument as in (4.50) and (4.51) by taking $h_n(v^\varepsilon) = S'(v^\varepsilon)$ we obtain

$$\int_Q h_n(v^\varepsilon)\varphi \, d\lambda_+^\varepsilon = \omega(\varepsilon).$$

If we prove that

$$\int_Q \beta_n(v^\varepsilon)\varphi \, d\lambda_-^\varepsilon = \omega(\varepsilon), \tag{4.56}$$

then, we obtain for every $\varphi \in C_c^\infty(Q)$

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v \; \varphi \, dxdt = \int_Q \varphi \, d\mu_s^+ \tag{4.57}$$

We can write

$$\int_Q \beta_n(v^\varepsilon)\varphi \, d\lambda_-^\varepsilon = \int_Q \beta_n(v^\varepsilon)\varphi\psi_\delta^- \, d\lambda_-^\varepsilon + \int_Q \beta_n(v^\varepsilon)\varphi(1-\psi_\delta^-) \, d\lambda_-^\varepsilon,$$

by Lemma 4.2, we obtain

$$\int_Q \beta_n(v^\varepsilon)\varphi(1-\psi_\delta^-) \, d\lambda_-^\varepsilon = \omega(\varepsilon, \delta).$$

Choosing $\beta_n(v^\varepsilon)\varphi\psi_\delta^-$ as a test function in the formulation of u^ε

$$\begin{aligned}
\int_Q \beta_n(v^\varepsilon)\varphi\psi_\delta^- \, d\lambda_-^\varepsilon &= \int_0^T \langle (\varphi\psi_\delta^-)_t, \overline{\beta_n}(v^\varepsilon)\rangle \, dt - \int_Q \beta_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon)\nabla(\varphi\psi_\delta^-) \, dxdt \\
&\quad - \frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon \varphi\psi_\delta^- \, dxdt + \int_Q f^\varepsilon \beta_n(v^\varepsilon)\varphi\psi_\delta^- \, dxdt \\
&\quad + \int_Q G^\varepsilon \beta_n(v^\varepsilon)\nabla(\varphi\psi_\delta^-) \, dxdt + \frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} G^\varepsilon \nabla v^\varepsilon \varphi\psi_\delta^- \, dxdt + \int_Q \beta_n(v^\varepsilon)\varphi\psi_\delta^- \, d\lambda_+^\varepsilon.
\end{aligned}$$

Using again Proposition 2.1, Proposition 4.1, Lemma 4.2 and Lemma 4.4 yields (4.56), and therefore we obtain (4.57) for every $\varphi \in C_c^\infty(Q)$. Now if $\varphi \in C^\infty(\overline{Q})$, we can split

$$\frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v \; \varphi \, dxdt = \frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v \; \varphi\psi_\delta^+ \, dxdt \tag{4.58}$$

$$+\frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi(1-\psi_\delta^+)\,dxdt,.$$

Thanks to (4.57), we have

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi\psi_\delta^+\,dxdt=\int_Q \varphi\,d\mu_s^+ +\omega(\delta),$$

By Lemma 4.4, we obtain

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\ \varphi(1-\psi_\delta^+)\,dxdt=\omega(\varepsilon,n,\delta).$$

Thanks to Theorem 4.3, we deduce

$$\frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi(1-\psi_\delta^+)\,dxdt=\omega(n,\delta).$$

Putting together all these facts above, we get (3.13) for every $\varphi\in C^\infty(\overline{Q})$, and by density argument (3.13) holds for every $\varphi\in C(\overline{Q})$. To obtain (3.14) we can reason as before using ψ_δ^+ in the place of ψ_δ^- and viceversa, and this conclude the proof of Theorem 4.1. □

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