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On Certain Subclasses of Multivalent Functions with Varying Arguments of Coefficients

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ABSTRACT: In this paper we introduce and study the classes $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$ and $\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$ of multivalent functions with varying arguments of coefficients. We obtain coefficients inequalities, distortion theorems and extreme points for functions in these classes. Also, we investigate several distortion inequalities involving fractional calculus. Finally, results on partial sums are considerd.

Key Words: Analytic, Multivalent functions, Hadamard product, Fractional integral and fractional derivative operators, Varying arguments.

Contents

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}(p\in\mathbb{N}=\{1,2,...\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disc $\mathbb{U}=\{z:z\in\mathbb{C}\text{ and }|z|<1\}$. We note that $\mathcal{A}(1)=\mathcal{A}$.

We recall some definitions which will be used in our paper.

Definition 1.1. *[1]. (i) A function $f(z)$ of the form (1.1) is said to be in the class of $\beta-$uniformly multivalent starlike functions, denoted by $\mathcal{UST}_p(\alpha,\beta)$, if it satisfies the following condition:*

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\beta\left|\frac{zf'(z)}{f(z)}-p\right|\quad(0\le\alpha<p;\ \beta\ge0;\ z\in\mathbb{U}).$$

(ii) A function $f(z)$ of the form (1.1) is said to be in the class of $\beta-$uniformly multivalent convex functions, denoted by $\mathcal{UCV}_p(\alpha,\beta)$, if it satisfies the following condition:

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\beta\left|1+\frac{zf''(z)}{f'(z)}-p\right|\quad(0\le\alpha<p;\ \beta\ge0;\ z\in\mathbb{U}).$$

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Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g. [2], [3], [5], [6], [7], [15], [16] and [18]). We find it to be convenient to recall here the following definitions which were used recently by Owa [8] and by Srivastava and Owa [17].

Definition 1.2. *The fractional integral of order λ is defined, for a function $f(z)$, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \ (\lambda > 0),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.3. *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \ (0 \leq \lambda < 1),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.4. *Under the hypotheses of Definition 1.3, the fractional derivative of order $n+\lambda$ is defined by*

$$D_z^{k+\lambda} f(z) = \frac{d^k}{dz^k} D_z^{\lambda} f(z) \ (0 \leq \lambda < 1;\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

In this paper, we define the following subclass of $\mathcal{A}(p)$

Definition 1.5. *A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{M}_p(\lambda, \alpha, \beta)$ if*

$$\Re\left\{\frac{\Psi_z^{(\lambda,p)}}{f(z)} - \alpha\right\} > \beta \left|\frac{\Psi_z^{(\lambda,p)}}{f(z)} - p\right| \ (0 \leq \alpha < p;\ \beta \geq 0;\ 0 \leq \lambda < 1;\ z \in \mathbb{U}), \tag{1.2}$$

where

$$\Psi_z^{(\lambda,p)} = \frac{\Gamma(p-\lambda+1)}{\Gamma(p)} z^{\lambda} D_z^{\lambda} f(z). \tag{1.3}$$

Also, a function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{N}_p(\lambda, \alpha, \beta)$ if and only if

$$\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \in \mathcal{M}_p(\lambda, \alpha, \beta).$$

We note that:
(i)$\mathcal{M}_1(\lambda, \alpha, 0) = \mathcal{S}_\lambda^*(\alpha)$ and $\mathcal{N}_1(\lambda, \alpha, 0) = \mathcal{K}_\lambda(\alpha)$ (see Owa [9]);
(ii)$\mathcal{M}_p(1, \alpha, \beta) = \mathcal{UST}_p(\alpha, \beta)$ and $\mathcal{VN}_p(1, \alpha, \beta) = \mathcal{UCV}_p(\alpha, \beta)$ (see Al-Kharsani [1]).

Also, we note that:

$$\mathcal{M}_1(\lambda, \alpha, \beta) = \mathcal{M}(\lambda, \alpha, \beta) = \left\{f(z) \in \mathcal{A} : \Re\left(\frac{\Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z)}{f(z)} - \alpha\right) > \beta \left|\frac{\Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z)}{f(z)} - 1\right|\right\};$$

and $\mathcal{N}_1(\lambda, \alpha, \beta) = \mathcal{N}(\lambda, \alpha, \beta) = \left\{f(z) \in \mathcal{A} : \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \in \mathcal{M}(\lambda, \alpha, \beta)\right\}.$

In [12], Silverman introduced and studied the univalent functions with varying arguments, as follows:

Definition 1.6. *[12]. We say that a function $f(z)$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{V}(\theta_n)$ if $f(z) \in \mathcal{S}$ (the class of analytic and univalent functions in $\mathbb{U}$) and $\arg(a_n) = \theta_n$ for all $n\,(n \geq 2)$. Further, if there exists a real number η such that*

$$\theta_n + (n-1)\eta \equiv \pi\,(mod\ 2\pi)\,,$$

then $f(z)$ is said to be in the class $\mathcal{V}(\theta_n, \eta)$. The union of $\mathcal{V}(\theta_n, \eta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers η is denoted by $\mathcal{V}$.

Silverman [12] used the concept of varying arguments of the coefficients to introduce and study the class $\mathcal{V}^*(\alpha)$, which is a subclass of $\mathcal{V}$ consisting of starlike functions of order α. For $\eta = 0$, we obtain the class $\mathcal{T}_n$ consisting of functions $f(z)$ with negative coefficients.

In [4], Aouf et al. introduced a subclass of multivalent functions with varying arguments of coefficients as follows.

Definition 1.7. *[4]. We say that a function $f(z)$ of the form (1.1) is in the class $\mathcal{V}_p(\theta_{p+n})$ if $f(z) \in \mathcal{A}(p)$ and $\arg(a_{p+n}) = \theta_{p+n}$ for all $n\,(n \geq 1)$. Further, if there exists a real number η such that*

$$\theta_{p+n} + n\eta \equiv \pi\,(mod\ 2\pi)\,,$$

then $f(z)$ is said to be in the class $\mathcal{V}_p(\theta_{p+n}, \eta)$. The union of $\mathcal{V}_p(\theta_{p+n}, \eta)$ taken over all possible sequences $\{\theta_{p+n}\}$ and all possible real numbers η is denoted by $\mathcal{V}_p$.

Let $\mathcal{VUST}_p(\alpha, \beta)$ denote the subclass of $\mathcal{V}_p$ consisting of functions $f(z) \in \mathcal{UST}_p(\alpha, \beta)$ and $\mathcal{VUCV}_p(\alpha, \beta)$ denote the subclass of $\mathcal{V}_p$ consisting of functions $f(z) \in \mathcal{UCV}_p(\alpha, \beta)$ which are the subclasses of multivalent uniformly starlike functions with varying arguments of coefficients and multivalent uniformly convex functions with varying arguments, respectively.

Using the concept of varying arguments in multivalent functions, we introduce the following subclasses.

Definition 1.8. *Let $\mathcal{VM}_{p,\eta}(\lambda, \alpha, \beta)$ denote the subclass of $\mathcal{V}_p$ consisting of functions $f(z) \in \mathcal{M}_p(\lambda, \alpha, \beta)$ and $\mathcal{VN}_{p,\eta}(\lambda, \alpha, \beta)$ denote the subclass of $\mathcal{V}_p$ consisting of functions $f(z) \in \mathcal{N}_p(\lambda, \alpha, \beta)$.*

We note that:
(i)$\mathcal{VM}_{p,\eta}(1, \alpha, 0) = \mathcal{V}_p(\alpha)$ and $\mathcal{VN}_{p,\eta}(1, \alpha, 0) = \bar{\mathcal{V}}_p(\alpha)$ (see Aouf et al. [4]);
(ii)$\mathcal{VM}_{p,0}(1, \alpha, 0) = \mathcal{T}^*(p, \alpha)$ and $\mathcal{VN}_{p,0}(1, \alpha, 0) = \mathcal{C}(p, \alpha)$ (see Owa [10]);
(iii)$\mathcal{VM}_{p,0}(1, \alpha, \beta) = \mathcal{UST}_p(\alpha, \beta)$ and $\mathcal{VN}_{p,0}(1, \alpha, \beta) = \mathcal{UCV}_p(\alpha, \beta)$ (see Al-Kharsani [1]);
(iv)$\mathcal{VM}_{1,\eta}(\lambda, \alpha, 0) = \mathcal{V}^*_\lambda(\alpha)$ and $\mathcal{VN}_{1,\eta}(\lambda, \alpha, 0) = \mathcal{W}^*_\lambda(\alpha)$ (see Owa [9]);
(v)$\mathcal{VM}_{1,\eta}(1, \alpha, 0) = \mathcal{V}^*(\alpha)$ and $\mathcal{VN}_{1,\eta}(1, \alpha, 0) = \mathcal{VK}(\alpha)$ (see Silverman [12]);
(vi)$\mathcal{VM}_{1,0}(1, \alpha, 0) = \mathcal{T}^*(\alpha)$ and $\mathcal{VM}_{1,0}(1, \alpha, 0) = \mathcal{K}(\alpha)$ (see Silverman [13]).

Also, we note that:
$\mathcal{VM}_1(\lambda, \alpha, \beta) = \mathcal{VM}(\lambda, \alpha, \beta) =$

$$\left\{ f(z) \in \mathcal{V} : \Re\left(\frac{\Gamma(2-\lambda) z^\lambda D_z^\lambda f(z)}{f(z)} - \alpha \right) > \beta \left| \frac{\Gamma(2-\lambda) z^\lambda D_z^\lambda f(z)}{f(z)} - 1 \right| \right\};$$

and $\mathcal{VN}_1(\lambda, \alpha, \beta) = \mathcal{VN}(\lambda, \alpha, \beta) = \left\{ f(z) \in \mathcal{V} : \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \in \mathcal{VM}_p(\lambda, \alpha, \beta) \right\}.$

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this paper that

$$0 \leq \alpha < p,\ \beta \geq 0,\ 0 \leq \lambda < 1,\ p \in \mathbb{N},\ z \in \mathbb{U} \text{ and } \phi_{p,n}^{\lambda} = \tfrac{\Gamma(p+n+1)\Gamma(p-\lambda+1)}{\Gamma(p+n-\lambda+1)\Gamma(p)}\ (n \in \mathbb{N}).$$

Theorem 2.1. *Let $f(z)$ be given by (1.1). Then $f(z) \in \mathcal{VM}_{p,\eta}(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} \left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta + \alpha)\right] |a_{p+n}| \leq p - \alpha. \tag{2.1}$$

Proof. Assume that the condition (2.1) holds, then it is sufficient to show the inequality (1.2) holds. Hence, it suffices to show that

$$\beta \left| \frac{\Psi_z^{(\lambda,p)}}{f(z)} - p \right| - \Re \left\{ \frac{\Psi_z^{(\lambda,p)}}{f(z)} - p \right\} < p - \alpha,$$

where $\Psi_z^{(\lambda,p)}$ is defined by (1.3). Thus

$$\beta \left| \frac{\Psi_z^{(\lambda,p)}}{f(z)} - p \right| - \Re \left\{ \frac{\Psi_z^{(\lambda,p)}}{f(z)} - p \right\} \leq (1+\beta) \left| \frac{\Psi_z^{(\lambda,p)}}{f(z)} - p \right|$$

$$\leq \frac{(1+\beta) \sum\limits_{n=1}^{\infty} \left[\phi_{p,n}^{\lambda} - p\right] |a_{p+n}|}{1 - \sum\limits_{n=1}^{\infty} |a_{p+n}|}.$$

This last expression is bounded above by $(p - \alpha)$ if (2.1) holds. Conversely, assume that

$$\Re \left\{ \frac{\Psi_z^{(\lambda,p)}}{f(z)} - \alpha \right\} > \beta \left| \frac{\Psi_z^{(\lambda,p)}}{f(z)} - p \right|,$$

or, equivalently

$$\Re \left\{ \frac{(p-\alpha) + \sum\limits_{n=1}^{\infty} \left[\phi_{p,n}^{\lambda} - \alpha\right] a_{p+n} z^n}{1 + \sum\limits_{n=1}^{\infty} a_{p+n} z^n} \right\} > \beta \left| \frac{\sum\limits_{n=1}^{\infty} \left[\phi_{p,n}^{\lambda} - p\right] a_{p+n} z^n}{1 + \sum\limits_{n=1}^{\infty} a_{p+n} z^n} \right|.$$

Since $f(z) \in \mathcal{V}_p$, then $f(z) \in \mathcal{V}_p(\theta_{p+n}, \eta)$ for some sequence $\{\theta_{p+n}\}$ and a real numbers η such that

$$\theta_{p+n} + n\eta \equiv \pi\ (mod\ 2\pi).$$

Let $z = re^{i\eta}$, we have

$$\left(\frac{(p-\alpha) + \sum\limits_{n=1}^{\infty} \left[\phi_{p,n}^{\lambda} - \alpha\right] |a_{p+n}| e^{i[\theta_{p+n} + n\eta]} r^n}{1 + \sum\limits_{n=1}^{\infty} |a_{p+n}| e^{i[\theta_{p+n} + n\eta]} r^n} \right) > \left(\frac{\beta \sum\limits_{n=1}^{\infty} \left[\phi_{p,n}^{\lambda} - p\right] |a_{p+n}| e^{i[\theta_{p+n} + n\eta]} r^n}{1 + \sum\limits_{n=1}^{\infty} |a_{p+n}| e^{i[\theta_{p+n} + n\eta]} r^n} \right).$$

Letting $r \to 1^-$, we obtain the required result and hence the proof of the inequality (2.1) is completed. Further, we consider a function $f(z)$ given by

$$f(z) = z^p + \sum_{n=1}^{\infty} \frac{(p-\alpha) e^{i\theta_{p+n}}}{n\,(n+1) \left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta + \alpha)\right]} z^{p+n}.$$

Then, writing

$$a_{p+n}=\frac{(p-\alpha)e^{i\theta_{p+n}}}{n\,(n+1)\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}\quad (n\in\mathbb{N})\,.$$

We have

$$\sum_{n=1}^{\infty}\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]|a_{p+n}|=\sum_{n=1}^{\infty}\frac{p-\alpha}{n\,(n+1)}=p-\alpha.$$

Therefore, $f(z)\in\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$ satisfies the equality in (2.1). □

Corollary 2.2. *Let $f(z)$ defined by (1.1) be in the class $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$. Then*

$$|a_{p+n}|\le\frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}.$$

The result is sharp for the function

$$f(z)=z^{p}+\frac{(p-\alpha)e^{i\theta_{p+n}}}{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}z^{p+n}.$$

Theorem 2.3. *Let $f(z)$ be given by (1.1). Then $f(z)\in\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty}\phi_{p,n}^{\lambda}\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]|a_{p+n}|\le p(p-\alpha). \tag{2.2}$$

Proof. Since $f(z)\in\mathcal{VN}_{p}(\lambda,\alpha,\beta)$ if and only if

$$\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)}z^{\lambda}D_{z}^{\lambda}f(z)\in\mathcal{VM}_{p}(\lambda,\alpha,\beta).$$

It follows that $f(z)\in\mathcal{VN}_{p}(\lambda,\alpha,\beta)$ if and only if (2.2) holds. Moreover, the equality in (2.2) holds true for

$$f(z)=z^{p}+\sum_{n=1}^{\infty}\frac{p(p-\alpha)e^{i\theta_{p+n}}}{n\,(n+1)\,\phi_{p,n}^{\lambda}\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}z^{p+n}.$$

This completes the proof of Theorem 2.3. □

Corollary 2.4. *Let $f(z)$ defined by (1.1) be in the class $\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$. Then*

$$|a_{p+n}|\le\frac{p(p-\alpha)}{\phi_{p,n}^{\lambda}\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}.$$

The result is sharp for the function

$$f(z)=z^{p}+\frac{p(p-\alpha)}{\phi_{p,n}^{\lambda}\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}e^{i\theta_{p+n}}z^{p+n}.$$

3. Distortion theorems

Theorem 3.1. *Let $f(z)$ defined by (1.1) be in the class $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$, then for $z \in \mathbb{U}$, we have*

$$|z|^p - \frac{(p-\alpha)}{\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right]}\,|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{(p-\alpha)}{\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right]}\,|z|^{p+1}. \tag{3.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{(p-\alpha)e^{i\theta_{p+1}}}{\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right]} z^{p+1}. \tag{3.2}$$

Proof. It is easy to see from Theorem 2.1 that

$$\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right] \sum_{n=1}^{\infty} |a_{p+n}| \leq \sum_{n=1}^{\infty} \left[(1+\beta)\phi^{\lambda}_{p,n} - (p\beta+\alpha)\right] |a_{p+n}| \leq p-\alpha,$$

because $\phi^{\lambda}_{p,n} < \phi^{\lambda}_{p,n+1}$ for $n \geq 1$. This gives us that

$$\sum_{n=1}^{\infty} |a_{p+n}| \leq \frac{(p-\alpha)}{\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right]}. \tag{3.3}$$

Making use of (3.3), we have

$$\begin{aligned} |f(z)| &\geq |z|^p - \sum_{n=1}^{\infty} |a_{p+n}|\,|z|^{p+1} \\ &\geq |z|^p - \frac{(p-\alpha)}{\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right]}\,|z|^{p+1}, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\leq |z|^p + \sum_{n=1}^{\infty} |a_{p+n}|\,|z|^{p+1} \\ &\leq |z|^p + \frac{(p-\alpha)}{\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right]}\,|z|^{p+1}, \end{aligned}$$

which proves the assertion (3.1). Since the equality in (3.1) is satisfied by $f(z)$ given by (3.2), the proof is thus completed. □

Using similar arguments to those in the proof of the Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let $f(z)$ defined by (1.1) be in the class $\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$, then for $z \in \mathbb{U}$, we have*

$$|z|^p - \frac{p(p-\alpha)}{\phi^{\lambda}_{p,1}\left[(1+\beta)\phi^{\lambda}_{p,1} - (p\beta+\alpha)\right]}\,|z|^{p+1} \leq |f(z)| \leq$$

$$|z|^p+\frac{p(p-\alpha)}{\phi_{p,1}^{\lambda}\left[(1+\beta)\phi_{p,1}^{\lambda}-(p\beta+\alpha)\right]}\,|z|^{p+1}.$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p+\frac{p(p-\alpha)e^{i\theta_{p+1}}}{\phi_{p,1}^{\lambda}\left[(1+\beta)\phi_{p,1}^{\lambda}-(p\beta+\alpha)\right]}z^{p+1}.$$

4. Extreme points

Theorem 4.1. *Let $f(z)$ defined by (1.1) belongs to the class $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$ with $\arg(a_{p+n})=\theta_{p+n}$ and $\theta_{p+n}+n\eta\equiv\pi\,(mod\ 2\pi)$ for all n. Also, let $f_p(z)=z^p$ and*

$$f_{p+n}(z)=z^p+\frac{(p-\alpha)\,e^{i\theta_{p+n}}}{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}z^{p+n}.$$

Then $f(z)$ is in the class $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$ if and only if can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}\mu_{p+n}f_{p+n}(z),$$

where $\mu_{p+n}\geq 0$ and $\sum_{n=0}^{\infty}\mu_{p+n}=1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\mu_{p+n}f_{p+n}(z) \\ &= z^p+\sum_{n=1}^{\infty}\frac{(p-\alpha)\,e^{i\theta_{p+n}}\mu_{p+n}}{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}z^{p+n}. \end{aligned} \tag{4.1}$$

Then it follows that

$$\sum_{n=1}^{\infty}\frac{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}{(p-\alpha)}\frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}\mu_{p+n}=\sum_{n=1}^{\infty}\mu_{p+n}=1-\mu_p\leq 1,$$

which implies that $f(z)\in\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$. Conversely, assume that the function $f(z)$ defined by (1.1) be in the class $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$. Then Corollary 2.2 gives that

$$|a_{p+n}|\leq\frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}\ (n\in\mathbb{N}).$$

Defining μ_{p+n} by

$$\mu_{p+n}=\frac{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]|a_{p+n}|}{(p-\alpha)}\ (n\in\mathbb{N}),$$

and

$$\mu_p=1-\sum_{n=1}^{\infty}\mu_{p+n},$$

we write

$$a_{p+n}=\frac{(p-\alpha)\mu_{p+n}e^{i\theta_{p+n}}}{\left[(1+\beta)\phi_{p,n}^{\lambda}-(p\beta+\alpha)\right]}\ (n\in\mathbb{N}).$$

This shows that

$$\begin{aligned} f(z) &= z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \\ &= \left(\sum_{n=0}^{\infty} \mu_{p+n} \right) z^p + \sum_{n=1}^{\infty} \frac{(p-\alpha)e^{i\theta_{p+n}}}{\left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta+\alpha)\right]} \mu_{p+n} z^{p+n} \\ &= \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z). \end{aligned}$$

This completes the proof of Theorem 4.1. □

By using similar arguments and analysis to those in the proof of Theorem 4.1, we can derive the following theorem.

Theorem 4.2. *Let $f(z)$ defined by (1.1) belongs to the class $\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$ with $\arg(a_{p+n}) = \theta_{p+n}$ and $\theta_{p+n} + n\eta \equiv \pi \,(\mathrm{mod}\ 2\pi)$ for all n. Also, let $f_p(z) = z^p$ and*

$$f_{p+n}(z) = z^p + \frac{p\,(p-\alpha)\,e^{i\theta_{p+n}}}{\phi_{p,n}^{\lambda}\left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta+\alpha)\right]} z^{p+n}.$$

Then $f(z)$ is in the class $\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$ if and only if can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z),$$

where $\mu_{p+n} \geq 0$ and $\sum_{n=0}^{\infty} \mu_{p+n} = 1$.

5. Applications of Fractional Calculus

Theorem 5.1. *Let $f(z)$ defined by (1.1) be in the class $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$. Then we have*

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left\{ 1 + \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z| \right\}, \tag{5.1}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left\{ 1 - \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z| \right\}, \tag{5.2}$$

for $\lambda > 0$ and $z \in \mathbb{U}$. Further

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} |z|^{p-\lambda} \left\{ 1 + \frac{(p+1)(p-\alpha)}{p\left[(p+1)(1+\beta) - (p\beta+\alpha)(p+1-\lambda)\right]} |z| \right\}, \tag{5.3}$$

and

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} |z|^{p-\lambda} \left\{ 1 - \frac{(p+1)(p-\alpha)}{p\left[(p+1)(1+\beta) - (p\beta+\alpha)(p+1-\lambda)\right]} |z| \right\}, \tag{5.4}$$

for $0 \leq \lambda < 1$ and $z \in \mathbb{U}$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)} a_{p+n} z^{p+n}. \end{aligned}$$

Then

$$F(z) = z^p + \sum_{n=1}^{\infty} \Omega_{p,n}^{\lambda} a_{p+n} z^{p+n},$$

where

$$\Omega_{p,n}^{\lambda} = \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)} \ (\lambda > 0).$$

We see that

$$0 < \Omega_{p,n}^{\lambda} \le \frac{p+1}{p+\lambda+1}.$$

Then

$$\begin{aligned} |F(z)| &\le \left| z^p + \sum_{n=1}^{\infty} \Omega_{p,n}^{\lambda} a_{p+n} z^{p+n} \right| \\ &\le |z|^p + \sum_{n=1}^{\infty} \Omega_{p,n}^{\lambda} |a_{p+n}| \, |z|^{p+1} \\ &\le |z|^p + \sum_{n=1}^{\infty} \Omega_{p,n}^{\lambda} \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta+\alpha)\right]} |z|^{p+n} \\ &\le |z|^p + \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z|^{p+1}, \end{aligned}$$

and

$$|F(z)| \ge |z|^p - \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z|^{p+1},$$

which proves the inequalities (5.1) and (5.2). Now, Since

$$\begin{aligned} &\left(\frac{p+1-\lambda}{p+1}\right) \frac{\Gamma(p+n+1)\Gamma(p-\lambda+1)}{\Gamma(p+n+1-\lambda)} \left[(1+\beta)\left(\frac{p+1}{p+1-\lambda}\right) - (p\beta+\alpha)\right] \\ &\le \left[(1+\beta)\frac{\Gamma(p+n+1)\Gamma(p-\lambda+1)}{\Gamma(p+n+1-\lambda)} - (p\beta+\alpha)\Gamma(p)\right], \end{aligned}$$

it follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(\frac{p+1-\lambda}{p+1}\right) \frac{\Gamma(p+n+1)\Gamma(p-\lambda+1)}{\Gamma(p+n+1-\lambda)} \left[(1+\beta)\left(\frac{p+1}{p+1-\lambda}\right) - (p\beta+\alpha)\right] |a_{p+n}| \\ &\le \sum_{n=1}^{\infty} \left[(1+\beta)\frac{\Gamma(p+n+1)\Gamma(p-\lambda+1)}{\Gamma(p+n+1-\lambda)} - (p\beta+\alpha)\Gamma(p)\right] |a_{p+n}|, \end{aligned}$$

that is,

$$\sum_{n=1}^{\infty} \phi_{p,n}^{\lambda} |a_{p+n}| \le \frac{(p+1)(p-\alpha)}{[(p+1)(1+\beta) - (p\beta+\alpha)(p+1-\lambda)]}.$$

Let

$$\begin{aligned} G(z) &= \frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(p+n-\lambda+1)} a_{p+n} z^{p+n}. \end{aligned}$$

Then

$$\begin{aligned} |G(z)| &\leq \left| z^p + \sum_{n=1}^{\infty} \frac{\phi_{p,n}^{\lambda}}{p} a_{p+n} z^{p+n} \right| \\ &\leq |z|^p + \sum_{n=1}^{\infty} \frac{\phi_{p,n}^{\lambda}}{p} \left|a_{p+n}\right| |z|^{p+1} \\ &\leq |z|^p + \frac{(p+1)(p-\alpha)}{p\left[(p+1)(1+\beta)-(p\beta+\alpha)(p+1-\lambda)\right]} |z|^{p+1}, \end{aligned}$$

and

$$|G(z)| \geq |z|^p - \frac{(p+1)(p-\alpha)}{p\left[(p+1)(1+\beta)-(p\beta+\alpha)(p+1-\lambda)\right]} |z|^{p+1}.$$

Further equalities in (5.1) and (5.2) are attained for the function

$$f(z) = z^p + \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} e^{i\theta_{p+1}} z^{p+1}, \tag{5.5}$$

and equalities in (5.3) and (5.4) are attained for the function

$$f(z) = z^p + \frac{(p+1)(p-\alpha)}{p\left[(p+1)(1+\beta)-(p\beta+\alpha)(p+1-\lambda)\right]} e^{i\theta_{p+1}} z^{p+1}.$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. *Let $f(z)$ defined by (1.1) be in the class $\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$. Then we have*

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left\{ 1 + \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z| \right\},$$

and

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left\{ 1 - \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z| \right\},$$

for $\lambda > 0$ and $z \in \mathbb{U}$. Further

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} |z|^{p-\lambda} \left\{ 1 + \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z| \right\},$$

and

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} |z|^{p-\lambda} \left\{ 1 - \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]} |z| \right\},$$

for $0 \leq \lambda < 1$ and $z \in \mathbb{U}$. The result is sharp for the functions $f(z)$ given by (5.5).

Proof. Since

$$0 < \Omega_{p,n}^{\lambda} < \Omega_{p,n}^{-\lambda} < \frac{p+1}{p-\lambda+1}.$$

and

$$\sum_{n=1}^{\infty} \phi_{p,n}^{-\lambda} \left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta+\alpha)\right] |a_{p+n}| \leq p(p-\alpha).$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{p}\phi_{p,n}^{\lambda} |a_{p+n}| \leq \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{p}\phi_{p,n}^{-\lambda} |a_{p+n}| \leq \frac{(p-\alpha)}{\left[(1+\beta)\phi_{p,1}^{\lambda} - (p\beta+\alpha)\right]}.$$

Thus, we obtain the required result. □

6. Partial sums

Following the earlier works by Silverman [11] and Silvia [14] on partial sums for univalent functions, we consider partial sums of functions in the class $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$ and obtain sharp lower bound for real part of the ratio of $f(z)$ to $f_m(z)$.

Theorem 6.1. *Let $f(z) \in \mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$ of the form (1.1) and define the partial sums of $f_p(z)$ and $f_m(z)$ by*

$$f_p(z) = z^p \text{ and } f_m(z) = z^p + \sum_{n=1}^{m} a_{p+n} z^{p+n} \ (m \in \mathbb{N}). \tag{6.1}$$

Also, let

$$\sum_{n=1}^{\infty} c_{p+n} |a_{p+n}| \leq 1,$$

where

$$c_{p+n} = \frac{\left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta+\alpha)\right]}{(p-\alpha)}. \tag{6.2}$$

Then

$$\Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{1}{c_{p+m+1}}, \tag{6.3}$$

and

$$\Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{c_{p+m+1}}{1+c_{p+m+1}}. \tag{6.4}$$

Proof. For the coefficients c_{p+n+1} defined by (6.2), it is easy to verify that

$$c_{p+n+1} > c_{p+n} > 1,$$

and so

$$\sum_{n=1}^{m} |a_{p+n}| + c_{p+m+1} \sum_{n=m+1}^{\infty} |a_{p+n}| \leq \sum_{n=1}^{\infty} c_{p+n} |a_{p+n}| \leq 1.$$

Let

$$\begin{aligned} g_1(z) &= c_{p+m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{c_{p+m+1}}\right)\right\} \\ &= 1 + \frac{c_{p+m+1}\sum\limits_{n=m+1}^{\infty} a_{p+n}z^n}{1 + \sum\limits_{n=1}^{m} a_{p+n}z^n}, \end{aligned}$$

which is analytic in $\mathbb{U}$ and $g_1(0) = 1$. To prove (6.3), it suffices to prove that $\Re\{g_1(z)\} \geq 0$, or equivalently $|g_1(z) - 1| \leq |g_1(z) + 1|$ since

$$\Re\{w\} \geq \mu \text{ if and only if } |w - 1 - \mu| \leq |w + 1 - \mu|.$$

Thus

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq \frac{c_{p+m+1}\sum\limits_{n=m+1}^{\infty}|a_{p+n}|}{2 - 2\sum\limits_{n=1}^{m}|a_{p+n}| - c_{p+m+1}\sum\limits_{n=m+1}^{\infty}|a_{p+n}|} \leq 1,$$

which readily the assertion (6.3) of Theorem 6.1. To show that

$$f(z) = z^p + \frac{z^{p+m+1}}{c_{p+m+1}}, \tag{6.5}$$

gives sharp result, we note that for $z = re^{\frac{i\pi}{n}}$,

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^{m+1}}{c_{p+m+1}} \to 1 - \frac{1}{c_{p+m+1}} \text{ as } z \to 1^-.$$

Similary, let

$$\begin{aligned} g_2(z) &= (1 + c_{p+m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{c_{p+m+1}}{1 + c_{p+m+1}}\right\} \\ &= 1 - \frac{(1 + c_{p+m+1})\sum\limits_{n=m+1}^{\infty} a_{p+n}z^n}{1 + \sum\limits_{n=1}^{\infty} a_{p+n}z^n}. \end{aligned} \tag{6.6}$$

Making use of (6.6), we have

$$\left|\frac{g_2(z) - 1}{g_2(z) + 1}\right| \leq \frac{(1 + c_{p+m+1})\sum\limits_{n=m+1}^{\infty}|a_{p+n}|}{2 - 2\sum\limits_{n=1}^{\infty}|a_{p+n}| - (c_{p+m+1} - 1)\sum\limits_{n=m+1}^{\infty}|a_{p+n}|} \leq 1,$$

which implies to the assertion (6.4) of Theorem 6.1. The bound in (6.4) is sharp for each $m \in \mathbb{N}$ with the extremal function $f(z)$ given by (6.5). □

We can obtain the following theorem by using similar arguments to those in the proof of the Theorem 6.1.

Theorem 6.2. *Let $f(z) \in \mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$ of the form (1.1) and the partial sums of $f_p(z)$ and $f_m(z)$ defined by (6.1). Also, let*

$$\sum_{n=1}^{\infty} d_{p+n} \left| a_{p+n} \right| \leq 1,$$

where

$$d_{p+n} = \frac{\phi_{p,n}^{\lambda} \left[(1+\beta)\phi_{p,n}^{\lambda} - (p\beta + \alpha) \right]}{p\,(p-\alpha)}.$$

Then

$$\Re\left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{1}{d_{p+m+1}},$$

and

$$\Re\left\{ \frac{f(z)}{f_m(z)} \right\} \geq \frac{d_{p+m+1}}{1+d_{p+m+1}}.$$

Remark 6.3. *(i) For different choices of λ, p, β in the above results, we obtain some analogous results for Aouf et al. [4], Owa [9], Silverman [11] and Silverman [12];*
(ii) For $p = 1$ in the above results, we will obtain new results fot the classes $\mathcal{VM}(\lambda,\alpha,\beta)$ and $\mathcal{VN}(\lambda,\alpha,\beta)$ mentioned in the introduction.

Conclusion

In our present investigation, we have introduced and studied the classes $\mathcal{VM}_{p,\eta}(\lambda,\alpha,\beta)$ and $\mathcal{VN}_{p,\eta}(\lambda,\alpha,\beta)$ of multivalent functions with varying arguments of coefficients. We have successfully obtained coefficients inequalities, distortion theorems and extreme points for functions in these classes. Also, we have investigated several distortion inequalities involving fractional calculus. Finally, results on partial sums have been considerd.

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