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Optimal Correction of Infeasible Equations System as $Ax + B|x| = b$ Using ℓ_p-norm Regularization

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ABSTRACT: Optimal correction of an infeasible equations system as $Ax + B|x| = b$, leads into a non-convex fractional problem. In this paper, a regularization method(ℓ_p- norm, $0 < p < 1$), is presented to solve mentioned fractional problem. In this method, the obtained problem can be formulated as a non-convex and non-smooth optimization problem which is not Lipschitz. The objective function of this problem can be decomposed as a difference of convex functions (DC). For this reason, we use a special smoothing technique based on DC programming. The numerical results obtained for generated problem show high performance and effectiveness of the proposed method.

Key Words: Infeasiblity linear equalities, Infeasible absolute value equalities, ℓ_p- norm regularization method, Difference of convex optimization, Non-convex and non-smooth optimization.

Contents

1. Introduction

We consider the system of equations as below:

$$Ax + B|x| = b, \tag{1.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are given and $|.|$ denotes absolute value. System(1.1) was introduced in [23] and investigated in a more general context in [19]. This is infeasible for a number of reasons such as, error in data, error in formula, etc (for the more details see [1,2]). As stated in [1,2,15,16,17,18,20,7], remodeling and finding error in formula is not economical. Hence, optimal correction of this system is considered by minimal changes in data which the below fractional problem has been obtained

$$\min_{x\in\mathbb{R}^n} \frac{\|Ax + B|x| - b\|^2}{1 + \|x\|^2}. \tag{1.2}$$

In problem (1.2) by considering special forms of the matrix B, we investigate two cases:

1) If $B = 0$, then the optimal correction obtained from the infeasible system of linear equations ($Ax = b$) is as follows

$$\min_{x\in\mathbb{R}^n} \frac{\|Ax - b\|^2}{1 + \|x\|^2}. \tag{1.3}$$

Problem (1.3) is non-convex. To solve it, a lot of algorithms were devloped without considering the non-convexity property of the objective function [2,17,18,24,5].

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2) If $B = -I$, then the optimal correction achieved from infeasible system absolute value equations $(Ax - |x| = b)$ is

$$\min_{x\in\mathbb{R}^n} \frac{\|Ax - |x| - b\|^2}{1 + \|x\|^2}. \tag{1.4}$$

Problem (1.4) is non-convex and non-smooth. To solve it, a few algorithms were applied without considering non-convexity and unsmoothness properties of the objective function [15,16,20].

In addition, it should be noted, since the solutions of problems (1.3) and (1.4) are with a large norm, to control solutions were proposed the Tikhonov regularization methods which are inexact (see [20,24,5,6]). In addition to focousing on non-convexity and unsmoothness properties of the objective function, the present study proposes the exact regularization method (ℓ_p- norm, $0 < p < 1$) with the following general form

$$\min_{x\in\mathbb{R}^n} \frac{\|Ax + B|x| - b\|^2}{1 + \|x\|^2} + \lambda\|x\|_p^p, \tag{1.5}$$

where $\|x\|_p^p = \sum_{i=1}^n |x_i|^p$ (for more information on the exactness of this method, see [4,8,9,10,11]).
Problem (1.5) is a non-Lipschitz optimization problem that its the objective function is non-convex and non-smooth. Therefore, to find a solution for the problem, a second order classic algorithm is presented based on diference of convex algorithm(DCA) in [3].
This paper, has been divided into four sections: Section 2 describes a regularization method (ℓ_p- norm, $0 < p < 1$). An algorithm based on DC programming is discussed in Section 3. In Section 4 the numerical results are demonstrated. Finally, conclusion is given in Section 5.

We use the following notations in this paper: all vectors will be column. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. $\mathbb{R}^n$ is an n-dimensional Euclidean space. For the two vectors x and y in the n-dimensional real space, $x^\top y := \sum_{i=1}^n x_i y_i$ is an inner product in $\mathbb{R}^n$ which in $x^\top$ denotes transpose of a vector. For $x \in \mathbb{R}^n$, $\|x\|$ denotes 2-norm and ℓ_p-norm is defined as $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ and $\|x\|_\infty = \max |x_i|$. we will use ∇f and $\partial^2 f$ to denote the gradient and the generalized second-order derivative of function f.

2. ℓ_p-Norm regularization method

Consider the below fractional problem

$$\min_{x\in\mathbb{R}^n} f(x) = \frac{\|Ax + B|x| - b\|^2}{1 + \|x\|^2}.$$

As previously mentioned, above problem has an optimal solution with a large norm and it is meaningless from a practical point of view. In order to control the optimal solution, we apply the regularization method(ℓ_p-norm, $0 < p < 1$) as following

$$\min_{x\in\mathbb{R}^n} g(x) = \frac{\|Ax + B|x| - b\|^2}{1 + \|x\|^2} + \lambda\|x\|_p^p.$$

Above problem is a non-convex and non-smooth optimization problem. Therefore, to solve it, first we smooth this problem by using smoothing method obtained in what comes next.
In the rest of this paper the following lemmas from Chen [8] play a crucial role for smoothing problem (1.5).

Lemma 2.1. *The function $s_\mu(x) : \mathbb{R} \longrightarrow \mathbb{R}_+$, as the following is a smooth approximation for $|x|$*

$$s_\mu(x) = \begin{cases} |x|, & if \quad |x| \geq \mu, \\ (\frac{x^2}{2\mu} + \frac{\mu}{2}), & if \quad |x| < \mu, \end{cases}$$

where $\mu > 0$ is a smoothing parameter.

Lemma 2.2. *The function $(s_\mu(x))^p : \mathbb{R} \longrightarrow \mathbb{R}_+$, is a smooth approximation for $|x|^p$ in $\mathbb{R}$. Moreover, the the following items hold:*

1) *Let $k_0 = (\frac{k}{2})^p$ where $k := \int_{-\infty}^{\infty} |s|\rho(s)ds$. Then for any $x \in \mathbb{R}$,*

$$0 \leq (s_\mu(x))^p - |x|^p \leq 2k_0\mu^p.$$

2) *for any fixed $\mu > 0$, $(s_\mu(x))^p$ is a Lipschitz continuous function in $\mathbb{R}$. In particular, its gradients is bounded by $2p(s_\mu(0))^{p-1}$.*

3) *Assume there is $\rho_0 > 0$ such that $|\rho(s)| \leq \rho_0$. Let*

$$\nu_\mu = p(1-p)(s_\mu(0))^{p-2} + \frac{1}{\mu}p(s_\mu(0))^{p-1}\rho_0$$

Then for any fixed $\mu > 0$, the gradient of $(s_\mu(x))^p$ is Lipschitz continuous with Lipschitz constant $2\nu_\mu$.

Therefore, according to Lemmas 2.1 and 2.2 problem (1.5) a is converted to the following smoothed problem

$$\min_{x\in\mathbb{R}^n} g_\mu(x) = \frac{\|Ax + B\psi_\mu(x) - b\|^2}{(1 + \|x\|^2)} + \lambda\|\psi_\mu(x)\|_p^p, \tag{2.1}$$

where $\psi_\mu(x) = (s_\mu(x_1); s_\mu(x_2); \ldots; s_\mu(x_n))$ and $\|\psi_\mu(x)\|_p^p = \sum_{i=1}^{n}(s_\mu(x_i))^p$.
Futhemore, the smoothed problem (2.1) is nonconvex. Thus, using Dinkel bach theorems [12], we reformulate this problem to form a univariate equation. That is,

$$F(t) = \min_{x\in\mathbb{R}^n}\{\|Ax + B\psi_\mu(x) - b\|^2 - t(1 + \|x\|^2) + \lambda\|\psi_\mu(x)\|_p^p(1 + \|x\|^2)\} = 0. \tag{2.2}$$

Now, in problems (2.1) and (2.2), we set:

$$\begin{aligned} &P(x) = \|Ax + B\psi_\mu(x) - b\|^2 + \lambda\|\psi_\mu(x)\|_p^p(1 + \|x\|^2), \\ &Q(x) = 1 + \|x\|^2. \end{aligned}$$

Then, we consider problems (2.3) and (2.4) as below:

$$\min_{x\in\mathbb{R}^n} \frac{P(x)}{Q(x)} \tag{2.3}$$

$$\min_{x\in\mathbb{R}^n}\{P(x) - tQ(x)\}, t \in \mathbb{R} \tag{2.4}$$

Note that, the objective functions of the problems (2.3) and (2.4) are coercive. Therefore, the objective function takes on its minimum.

Theorem 2.1. *Suppose that $x^\star$ is a solution of problem (2.3). Then, $t^\star = \frac{P(x^\star)}{Q(x^\star)} = \min_{x\in\mathbb{R}^n} \frac{P(x)}{Q(x)}$ if, and only if*

$$\min_{x\in\mathbb{R}^n}\{P(x) - t^\star Q(x)\} = 0.$$

Proof. ($\Rightarrow$). Let $x^\star$ be a solution of problem (2.3). We have

$$t^\star = \frac{P(x^\star)}{Q(x^\star)} \leq \frac{P(x)}{Q(x)} \quad \text{for all } x \in \mathbb{R}^n.$$

Hence

$$P(x) - t^\star Q(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}^n, \tag{2.5}$$

$$P(x^\star) - t^\star Q(x^\star) = 0. \tag{2.6}$$

From (2.5) and (2.6) we have $\min_{x\in\mathbb{R}^n}\{P(x)t^\star Q(x)\} = 0$ and we see that the minimum is taken on, at $x^\star$. Thus, the first part of the proof is finished.

($\Leftarrow$). Let $x^\star$ be a solution of problem $\min_{x\in\mathbb{R}^n}\{P(x) - t^\star Q(x)\} = 0$ that means $P(x^\star) - t^\star Q(x^\star) = 0$. On the other hand, we have

$$P(x) - t^\star Q(x) \geq P(x^\star) - t^\star Q(x^\star) = 0 \text{ for all } x \in \mathbb{R}^n.$$

Hence, we have $t^\star \leq \dfrac{P(x)}{Q(x)}$ (since $Q(x) > 0$) for all $x \in \mathbb{R}^n$, that is $t^\star$ is the minimum of problem (2.3). Then, we have $t^\star = \dfrac{P(x^\star)}{Q(x^\star)}$, that is $x^\star$ is a solution vector of problem (2.3). □

As it is seen in Theorem 2.1, the root of equation (2.2) is equivalent to the global optimal solution of problem (2.1). Therefore, instead of solving problem (2.1), we find a solution for equation (2.2). Considering the properties of inner problem of Eq.2.2, to solve it, the generalized Newtown method is applied. For implementing this method, we need to compute the generalized gradient value. On the other hand, to compute the mentioned generalized gradient, a solution for a non-convex and non-smooth optimization subproblem is required.

Before solving this subproblem, we first state some of the theorems and lemmas in the following.

Lemma 2.3. *$F(t)$ is a strictly decreasing and concave function and it has a unique root in $[0, \|b\|^2]$.*

Proof. It is similar to Lemma 2.1, in [5]. □

Now, the function $F(t)$ is written as following

$$G(t) = -F(t) = t - \min_{x\in\mathbb{R}^n}\{\|Ax + B\psi_\mu(x) - b\|^2 - t(\|x\|^2) + \lambda\|\psi_\mu(x)\|_p^p(1 + \|x\|^2)\}.$$

Theorem 2.2. *$G(t)$ is a subdifferentiable, convex and strictly increasing function and $1 + \|x_t\|^2$ is a subgradient of this fuction in point t, where*

$$x_t = \min_{x\in\mathbb{R}^n}\{\|Ax + B\psi_\mu(x) - b\|^2 - t(1 + \|x\|^2) + \lambda\|\psi_\mu(x)\|_p^p(1 + \|x\|^2)\}.$$

Proof. It is analogous to Theorem 2.2 in [5]. □

As you can see below, function $G(t)$ contains an unconstrained optimization problem which its objective function is a non-convex and smoothed

$$\min_{x\in\mathbb{R}^n} g_\mu(x) = \{\|Ax + B\psi_\mu(x) - b\|^2 - t(\|x\|^2) + \lambda\|\psi_\mu(x)\|_p^p(1 + \|x\|^2)\}. \tag{2.7}$$

For solving problem (2.7), we reformulate it as a DC optimization problem. In order to achieve this purpose, we bring some fundemental theorems and lemmas from [14] in following without proof.

Lemma 2.4. *Let $\gamma_1 \geq \dfrac{p(1-p)}{\mu^{2-p}}$ then $g : \mathbb{R} \longrightarrow \mathbb{R}$*

$$g(x) = (s_\mu(x))^p + \frac{\gamma_1}{2}x^2,$$

where $0 < p < 1$ and $\mu > 0$, is a convex function.

Lemma 2.5. *Let $\gamma_2 \geq p(\frac{\mu}{2})^{p-2}$ then $k : \mathbb{R}^2 \longrightarrow \mathbb{R}$*

$$k(x, y) = s^p_\mu(x)y^2 + \frac{\gamma_2}{2}(x^2 + y^2)^2,$$

where $p, \mu,$ values are mentioned in the lemma 2.4, is a convex function.

Theorem 2.3. *Let $\gamma \geq \lambda p(\frac{\mu}{2})^{p-2}$ then $f : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is convx, as follows*

$$f(x) = \lambda\|\psi_\mu(x)\|^p_p(1 + \|x\|^2) + \frac{\gamma}{2}\|x\|^2(1 + \|x\|^2),$$

where $\lambda > 0$ is a regularization parameter and p, μ are given values in the Lemmas 2.4 and 2.5.

In addition to Lemmas 2.4, 2.5 and Theorem 2.3, for converting problem (2.7) into a DC optimization problem, the following proposition needs to be stated.

Proposition 2.1. *There exists $\beta > 0$ such that the function $g(x) = \|Ax + B\psi_\mu(x) - b\|^2 + \frac{\beta}{2}\|x\|^2$ is a convex function.*

It follows that the function $f(x) = \|Ax + B\psi_\mu(x) - b\|^2$ can be represented as $g(x) - \frac{\beta}{2}\|x\|^2$, where $g(x) = f(x) + \frac{\beta}{2}\|x\|^2$ is convex.

The following theorem shows the transformation of problem (2.7) into a DC optimization problem.

Theorem 2.4. *Suppose that assumptions of Theorem 2.3 and Proposition 2.1 hold, then the below problem is a DC optimization problem*

$$\min_{x\in\mathbb{R}^n}\{\|Ax + B\psi_\mu(x) - b\|^2 + \frac{\beta}{2}\|x\|^2 + \lambda\|\psi_\mu(x)\|^p_p(1 + \|x\|^2) + \frac{\gamma}{2}\|x\|^2$$
$$(1 + \|x\|^2) - \big((t + \frac{\beta}{2} + \frac{\gamma}{2})\|x\|^2 + \frac{\gamma}{2}\|x\|^4\big)\}. \tag{2.8}$$

Proof. In problem (2.8), we set

$$g_1(x) = \|Ax + B\psi_\mu(x) - b\|^2 + \frac{\beta}{2}\|x\|^2 + \lambda\|\psi_\mu(x)\|^p_p(1 + \|x\|^2) + \frac{\gamma}{2}\|x\|^2(1 + \|x\|^2),$$
$$g_2(x) = \big((t + \frac{\beta}{2} + \frac{\gamma}{2})\|x\|^2 + \frac{\gamma}{2}\|x\|^4\big).$$

According to the stated theorem and the proposition in assumption, functions $g_1(x)$ and $g_2(x)$ are covex. consequently, the proof is complete. □

The objective function of problem (2.8) is smooth and according to Lemmas 2.1 and 2.2, its gradient is continuously Lipschitz. Therefore, its Hessian is exist almost everywhere [21]. Thus, in order to deal with it, in the next section a classic algorithm of second order type will be presented.

3. Algorithms and their convergence

Consider the following problem

$$\min_{x\in\mathbb{R}^n} g_\mu(x) = g_1(x) - g_2(x), \tag{3.1}$$

where g_1 and g_2 are convex functions (as they were introduced in the previous section).
To find a solution for problem (3.1), by considering $x_k \in dom f_1$, in each iteration, the following relaxation problem will be solved

$$\min_{x\in\mathbb{R}^n} g_\mu(x) = g_1(x) - \big(\nabla g_2(x_k)\big)^\top(x - x_k). \tag{3.2}$$

Since the objective function of problem (3.2) is convex and continuously differentiable and its Hessian is available almost every where, to solve it, a second order classic algorithm (i.e., smoothing difference of convex) is proposed here:

Algorithm 1 Smoothing difference of convex algorithm

1. Choose the preliminary parameters $\delta \in (0,1)$, $\lambda > 0$, $\mu_0 > 0$, $\theta_1, \theta_2 > 0$, $\gamma, \beta > 0$, $p \in (0,1)$ and the initial points $x_0, y_0 \in \mathbb{R}^n$ and set $i = 0$, $\epsilon = 1e-6$, $c_i = \|x_i - y_i\|_\infty$.
2. While $c_i > \epsilon$ do.
   A) Set $x_i = y_i$. Thus, calculate the following value $y_{i+1} = y_i - \alpha(M_i + \delta I)^{-1}\nabla f(y_i)$, where α is determined by the Armijo line search and $M_i \in \partial^2 g_\mu(y_i)$.
   B) Update γ, β by using Theorem 2.4.
   C) Update the value of μ as following:
   If $\|\nabla g_\mu(y_{i+1})\| > \theta_1 \mu_i$, then set $\mu_{i+1} = \mu_i$; otherwise, choose$\mu_{i+1} = \theta_2 \mu_i$.
   D) Set $y_i = y_{i+1}$ and then, compute $c_i = \|x_i - y_i\|_\infty$.
3. End

Theorem 3.1. *If $\{x_i\}$ be a generated sequence by Algorithm 1 then it is convergent to a local optimal solution for prblem (3.1).*

Proof. Since the objective function problem (3.1) is coercive and also, $\{g_1(x_i) - g_2(x_i)\}$ is a decreasing sequence, therefore, convergence is clearly implied (for more details see [3] and the references given there). □

Now, applying the results obtained from Algorithm 1, the below generalized Newtown algorithm is suggested to determine the root of Equation $G(t) = 0$.

Algorithm 2 Generalized Newton method

1. Choose the values $A, B \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, accuracy parameter $\epsilon' = 1e-6$ and a starting point $t_k \in \left[0, \|b\|^2\right]$ and set $k = 0$.
2. While $|t_{k+1} - t_k| \geq \epsilon'$
   Solve problem (3.3) using Alorithm 1
   $$\min_{x\in\mathbb{R}^n} g(x) = \|Ax + B|x| - b\|^2 + \lambda\|\psi_\mu(x)\|_p^p(1 + \|x\|^2) - t\|x\|^2. \tag{3.3}$$
   Calculate
   $$G(t_k) = t_k - g(x(t_k)),$$
   where $f(x(t))$ is the value of objective function in problem (3.3). Set $t_{k+1} = t_k - \frac{G(t_k)}{1+\|x(t_k)\|^2}$.
3. If the condition of step 2 is held, continue; otherwise, the algorithm will be stopped.

Remark 3.2. *The convergence of Algorithm2 is similar to the convergence of Newtown method which is used for finding the roots of a nolinear equation (see [22]).*

4. The numerical results

In this section, the numerical results for Algorithm 2 are reported. The results were derived by a personal computer (CPU: Core i7, 12GB RAM, OS: Windows 8). The test problems contain six classes of consistent and inconsistent systems as $Ax = b$, $Ax - |x| = b$ and $Ax + B|x| = b$. These systems were generated randomly by the following MATLAB codes.

Code 1: A randomly generated consistent system of equalities as $Ax + B|x| = b$

```
m=input('Enterm:');
n=input('Entern:');
d=input('Enterdin(0,1]:');
A=sprand(m,n,d);
A=100*(A-.5*spones(A));
x=spdiags(rand(n,1),0,n,n)*(rand(n,1)-rand(n,1));
x=spdiags(ones(n,1)-sign(x),0,n,n)*10*(rand(n,1)-rand(n,1));
B=spdiags(x,0,n,n);
b=A*x+B*abs(x);
```

Code 2: A randomly generated inconsistent system of equalities as $Ax + B|x| = b$

```
u=10*(rand(n,1)-rand(n,1));
u=u/norm(u);
k =null(u');
k=[k,zeros(n,1)];
x=spdiags(rand(n,1),0,n,n)*(rand(n,1)-rand(n,1));
x=spdiags(ones(n,1)-sign(x),0,n,n)*10*(rand(n,1)-rand(n,1));
B=spdiags(x,0,n,n);
A=k+B;b=u;
```

Remark 4.1. *In this MATLAB codes, by considering above mentioned systems in special case, the matrix B is $B = zeros(n, n)$ or $B = -eye(n)$.*

In the below tables, each column is defined as following :
The values of n denote the dimensions of A, B and d denotes the density of these matrices. In column 2, $p \in (0, 1)$ indicates the size of regularization norm and other columns are defined as the following:

$$Me = \frac{1}{20}\Big(\sum_{j=1}^{20}(\|Ax^\star + B|x^\star| - b\|)_j\Big),\ M\|x^\star\| = \frac{1}{20}\Big(\sum_{j=1}^{20}(\|x^\star\|)_j\Big),$$

$$Mee = \frac{1}{20}\Big(\sum_{j=1}^{20}(\|(A + E^\star)x^\star + B|x^\star| - (b + r^\star)\|)_j\Big),\ Mi = \frac{1}{20}\Big(\sum_{j=1}^{20}(k)_j\Big),$$

$$Mf = \frac{1}{20}\Big(\sum_{j=1}^{20}\big(\frac{\|(Ax^\star + B|x^\star| - b)\|}{1 + \|x^\star\|^2}\big)_j\Big),\ Merror = \frac{1}{20}\Big(\sum_{j=1}^{20}(|t_{k+1} - t_k|)_j\Big),$$

$$Mtime = \frac{1}{20}\Big(\sum_{j=1}^{20}(time)_j\Big).$$

In the formulas above, number 20 shows the frequency in which every random system with identical dimension is generated; $x^\star$ is a solution for problem(1.5); **time** denotes the running time of Algorithm 2; k is the iteration number of generalized Newton algorithm. Finally, $E^\star$ and $r^\star$ are values that are computed in [17,20,24,5] as below

$$E^\star = -\big(\frac{Ax^\star + B|x^\star| - b}{1 + \|x^\star\|^2}\big)x^{\star\top},\ r^\star = \frac{Ax^\star + B|x^\star| - b}{1 + \|x^\star\|^2}.$$

Table 1: Numerical results derived from solving the consistent system of absolute value equations with $d = 1$

n	p	Me	$M\|x^\star\|$	Mi	Mf	$Merror$	$Mtime$
	0.1	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.06
	0.2	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
	0.3	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
	0.4	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
100	0.5	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
	0.6	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
	0.7	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
	0.8	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
	0.9	6.94e-11	2.39	2	1.91e-20	5.87e-13	0.04
	0.1	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.2	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.3	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.4	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
500	0.5	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.6	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.7	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.8	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.9	2.29e-07	5.33	2	7.36e-15	1.15e-12	1.6
	0.1	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.9
	0.2	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.9
	0.3	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.9
	0.4	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.9
1000	0.5	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.9
	0.6	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.8
	0.7	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.7
	0.8	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.6
	0.9	7.37e-08	7.53	2	4.54e-16	1.68e-12	8.6
	0.1	2.38e-06	10.4	2	2.71e-13	2.13e-12	52.9
	0.2	2.38e-06	10.4	2	2.71e-13	2.13e-12	52.9
	0.3	2.38e-06	10.4	2	2.71e-13	2.13e-12	53
	0.4	2.38e-06	10.4	2	2.71e-13	2.13e-12	52.9
2000	0.5	2.38e-06	10.4	2	2.71e-13	2.13e-12	52.6
	0.6	2.38e-06	10.4	2	2.71e-13	2.13e-12	52.5
	0.7	2.38e-06	10.4	2	2.71e-13	2.13e-12	52
	0.8	2.38e-06	10.4	2	2.71e-13	2.13e-12	50.9
	0.9	2.38e-06	10.4	2	2.71e-13	2.13e-12	50.2
	0.1	7.47e-07	13.09	2	1.62e-14	1.39e-12	111.9
	0.2	7.47e-07	13.09	2	1.62e-14	1.39e-12	112.3
	0.3	7.47e-07	13.09	2	1.62e-14	1.39e-12	112.4
	0.4	7.47e-07	13.09	2	1.62e-14	1.39e-12	111.9
3000	0.5	7.47e-07	13.09	2	1.62e-14	1.39e-12	111.7
	0.6	7.47e-07	13.09	2	1.62e-14	1.39e-12	111.3
	0.7	7.47e-07	13.09	2	1.62e-14	1.39e-12	110.8
	0.8	7.47e-07	13.09	2	1.62e-14	1.39e-12	109.7
	0.9	7.47e-07	13.09	2	1.62e-14	1.39e-12	108.5
	0.1	4.85e-06	16.64	2	1.32e-13	2.32e-13	636.6
	0.2	4.85e-06	16.64	2	1.32e-13	2.32e-13	632.5
	0.3	4.85e-06	16.64	2	1.32e-13	2.32e-13	647
	0.4	4.85e-06	16.64	2	1.32e-13	2.32e-13	648
5000	0.5	4.85e-06	16.64	2	1.32e-13	2.32e-13	668.2
	0.6	4.85e-06	16.64	2	1.32e-13	2.32e-13	673.1
	0.7	4.85e-06	16.64	2	1.32e-13	2.32e-13	668
	0.8	4.85e-06	16.64	2	1.32e-13	2.32e-13	658.5
	0.9	4.85e-06	16.64	2	1.32e-13	2.32e-13	634.1

Table 2: Numerical results derived from correcting the inconsistent system of absolute value equations

n	p	Mee	$M\|x^\star\|$	Mi	Mf	$Merror$	$Mtime$
	0.1	8.66e-17	2.27e-06	2	1	1.01e-09	0.1
	0.2	8.28e-17	2.64e-06	2	1	8.21e-10	0.1
	0.3	9.12e-17	2.82e-06	2	9.9999e-01	2.58e-10	0.1
	0.4	1.06e-16	2.87e-06	2	9.9999e-01	1.09e-10	0.1
100	0.5	9.24e-17	6.56e-06	2	9.9999e-01	2.24e-10	0.1
	0.6	8.06e-17	4.66e-06	2	9.9999e-01	6.02e-11	0.2
	0.7	7.68e-17	4.64e-06	2	9.9999e-01	3.03e-12	0.21
	0.8	1.08e-16	8.98e-06	2.4	9.9998e-01	1.34e-07	0.3
	0.9	9.66e-17	3.33e-02	3.2	9.5209e-01	3.37e-08	0.5
	0.1	8.78e-17	7.14e-06	2.6	9.9999e-01	6.96e-09	6
	0.2	8.92e-17	3.13e-06	2	9.9999e-01	6.42e-10	3.8
	0.3	8.72e-17	4.41e-06	2	9.9999e-01	1.31e-10	4
	0.4	1.02e-16	7.76e-06	2	9.9998e-01	3.74e-11	4.3
500	0.5	9.49e-17	6.98e-06	2	9.9999e-01	3.29e-11	6.1
	0.6	8.99e-17	1.21e-05	2	9.9997e-01	1.71e-10	6.7
	0.7	9.74e-17	1.15e-05	2	9.9998e-01	2.13e-11	6.9
	0.8	8.55e-17	2.85e-05	2	9.9994e-01	2.09e-09	7.9
	0.9	8.87e-17	6.86e-05	2.8	9.9985e-01	1.67e-08	12.7
	0.1	8.58e-17	7.13e-06	2	9.9999e-01	8.36e-08	18.6
	0.2	1.02e-16	3.13e-06	2	9.9999e-01	1.52e-09	18.7
	0.3	9.44e-17	4.39e-06	2	9.9999e-01	3.71e-10	20.1
	0.4	9.93e-17	7.75e-06	2	9.9999e-01	6.89e-10	22.1
1000	0.5	8.12e-17	1.50e-05	2	9.9997e-01	4.17e-09	24.1
	0.6	8.30e-17	1.21e-05	2	9.9998e-01	6.00e-11	28.4
	0.7	9.16e-17	2.75e-05	2	9.9995e-01	2.99e-09	32.2
	0.8	9.95e-17	2.84e-05	2	9.9994e-01	7.52e-11	39.2
	0.9	9.43e-17	7.35e-05	2	9.9985e-01	1.18e-08	44.8
	0.1	9.61e-17	7.14e-06	2	9.9999e-01	1.06e-09	97.3
	0.2	9.88e-17	8.20e-06	2	9.9998e-01	6.33e-08	96
	0.3	9.56e-17	4.40e-06	2	9.9999e-01	4.08e-10	108.6
	0.4	8.44e-17	7.76e-06	2	9.9998e-01	1.01e-10	118.7
2000	0.5	8.72e-17	1.50e-05	2	9.9997e-01	7.12e-09	130.8
	0.6	8.62e-17	1.21e-05	2	9.9997e-01	2.30e-11	157.5
	0.7	9.07e-17	2.76e-05	2	9.9994e-01	3.22e-09	181.4
	0.8	8.15e-17	6.53e-05	2	9.9986e-01	1.26e-08	197.1
	0.9	8.86e-17	7.36e-05	2	9.9984e-01	1.17e-08	238.5
	0.1	9.28e-17	7.14e-06	2	9.9999e-01	3.29e-09	273
	0.2	1.08e-16	8.19e-06	2	9.9998e-01	8.63e-08	259.8
	0.3	9.62e-17	4.40e-06	2	9.9999e-01	8.23e-10	290.3
	0.4	7.99e-17	7.76e-06	2	9.9998e-01	3.11e-10	324.4
3000	0.5	8.28e-17	1.50e-05	2	9.9997e-01	9.06e-11	365.3
	0.6	8.42e-17	3.14e-05	2	9.9994e-01	1.59e-08	410.5
	0.7	8.50e-17	2.76e-05	2	9.9994e-01	4.26e-09	600.9
	0.8	8.67e-17	6.52e-05	2	9.9987e-01	1.41e-08	639.1
	0.9	9.48e-17	7.33e-05	2	9.9985e-01	1.23e-08	675.1
	0.1	9.22e-17	7.14e-06	2	9.9999e-01	6.07e-09	1147.3
	0.2	1.08e-16	8.20e-06	3	9.9998e-01	2.36e-08	1657.3
	0.3	8.42e-17	1.27e-05	2	9.9997e-01	1.25e-07	1114.5
	0.4	8.13e-17	7.76e-06	2	9.9998e-01	6.59e-10	1524
5000	0.5	9.17e-17	1.50e-05	2	9.9997e-01	2.87e-10	1458.3
	0.6	7.28e-17	3.14e-05	2	9.9994e-01	2.45e-08	1631.6
	0.7	9.09e-17	2.76e-05	2	9.9995e-01	1.24e-10	1990.4
	0.8	8.38e-17	6.52e-05	2	9.9987e-01	1.64e-08	2193.7
	0.9	8.24e-17	7.32e-05	2	9.9985e-01	6.58e-10	2735

Table 3: Numerical results derived from solving the consistent system of linear equations with $d = 1$

n	p	Me	$M\|x^\star\|$	Mi	Mf	$Merror$	$Mtime$
	0.1	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.01
	0.2	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
	0.3	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
	0.4	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
100	0.5	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
	0.6	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
	0.7	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
	0.8	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
	0.9	5.52e-11	2.29	2	1.11e-20	1.22e-12	0.007
	0.1	1.28e-12	5.46	2.2	6.09e-24	9.57e-13	0.3
	0.2	1.24e-12	5.46	2.2	6.04e-24	9.57e-13	0.3
	0.3	1.29e-12	5.46	2.2	6.08e-24	9.57e-13	0.3
	0.4	1.25e-12	5.46	2.2	5.98e-24	9.57e-13	0.3
500	0.5	1.23e-12	5.46	2.2	6.11e-24	9.57e-13	0.3
	0.6	1.25e-12	5.46	2.2	6.06e-24	9.57e-13	0.3
	0.7	1.28e-12	5.46	2.2	6.05e-24	9.57e-13	0.3
	0.8	1.25e-12	5.46	2.2	6.05e-24	9.57e-13	0.3
	0.9	1.23e-12	5.46	2.2	6.15e-24	9.57e-13	0.3
	0.1	4.30e-12	7.40	2	5.88e-23	3.05e-12	1.2
	0.2	4.38e-12	7.40	2	5.88e-23	3.05e-12	1.2
	0.3	4.34e-12	7.40	2	5.88e-23	3.05e-12	1.2
	0.4	4.37e-12	7.40	2	5.88e-23	3.05e-12	1.2
1000	0.5	4.35e-12	7.40	2	5.88e-23	3.05e-12	1.2
	0.6	4.48e-12	7.40	2	5.87e-23	3.05e-12	1.2
	0.7	4.34e-12	7.40	2	5.88e-23	3.05e-12	1.2
	0.8	4.30e-12	7.40	2	5.88e-23	3.05e-12	1.2
	0.9	4.52e-12	7.40	2	5.89e-23	3.05e-12	1.2
	0.1	3.52e-12	10.51	2	9.53e-24	3.57e-13	6.4
	0.2	3.04e-12	10.51	2	9.52e-24	3.57e-13	6.5
	0.3	3.36e-12	10.51	2	9.61e-24	3.57e-13	6.5
	0.4	3.24e-12	10.51	2	9.54e-24	3.57e-13	6.4
2000	0.5	3.22e-12	10.51	2	9.65e-24	3.57e-13	6.4
	0.6	3.34e-12	10.51	2	9.54e-24	3.57e-13	6.4
	0.7	3.22e-12	10.51	2	9.52e-24	3.57e-13	6.3
	0.8	3.50e-12	10.51	2	9.62e-24	3.57e-13	6.4
	0.9	3.10e-12	10.51	2	9.49e-24	3.57e-13	6.5
	0.1	5.22e-12	13.08	2.2	2.03e-23	1.08e-12	42.9
	0.2	4.57e-12	13.08	2.2	2.02e-23	1.08e-12	42.5
	0.3	4.91e-12	13.08	2.2	2.04e-23	1.08e-12	42.5
	0.4	4.86e-12	13.08	2.2	2.03e-23	1.08e-12	44.5
3000	0.5	5.01e-12	13.08	2.2	2.03e-23	1.08e-12	44.2
	0.6	4.93e-12	13.08	2.2	2.01e-23	1.08e-12	42.9
	0.7	4.98e-12	13.08	2.2	1.99e-23	1.08e-12	42.3
	0.8	4.77e-12	13.08	2.2	2.01e-23	1.08e-12	42.3
	0.9	4.52e-12	13.08	2.2	2.00e-23	1.08e-12	42.6
	0.1	7.75e-12	16.59	2	3.56e-23	1.16e-12	71.2
	0.2	7.79e-12	16.59	2	3.58e-23	1.16e-12	71.1
	0.3	7.68e-12	16.59	2	3.57e-23	1.16e-12	72.6
	0.4	7.64e-12	16.59	2	3.52e-23	1.16e-12	73.9
5000	0.5	7.09e-12	16.59	2	3.59e-23	1.16e-12	71.9
	0.6	7.64e-12	16.59	2	3.56e-23	1.16e-12	71.9
	0.7	8.23e-12	16.59	2	3.58e-23	1.16e-12	70.5
	0.8	9.23e-12	16.59	2	3.64e-23	1.16e-12	70.2
	0.9	7.50e-12	16.59	2	3.59e-23	1.16e-12	70.7

Table 4: Numerical results derived from correcting the inconsistent system of linear equations

n	p	Mee	$M\|x^\star\|$	Mi	Mf	$Merror$	$Mtime$
100	0.1	5.37e-15	12.35	3	9.9007e-01	2.26e-15	0.03
	0.2	5.77e-15	12.35	3	9.9007e-01	9.10e-15	0.02
	0.3	5.37e-15	12.35	3	9.9007e-01	2.10e-14	0.02
	0.4	5.12e-15	12.35	3	9.9007e-01	3.99e-14	0.02
	0.5	5.15e-15	12.35	3	9.9007e-01	6.50e-14	0.02
	0.6	5.25e-15	12.35	3	9.9007e-01	9.72e-14	0.02
	0.7	5.38e-15	12.35	3	9.9007e-01	1.39e-13	0.02
	0.8	5.43e-15	12.35	3	9.9007e-01	1.89e-13	0.02
	0.9	5.31e-15	12.35	3	9.9007e-01	2.49e-13	0.02
500	0.1	2.63e-14	27.59	3	9.9800e-01	9.32e-16	0.6
	0.2	2.51e-14	27.59	3	9.9800e-01	4.04e-15	0.6
	0.3	2.56e-14	27.59	3	9.9800e-01	1.87e-14	0.6
	0.4	2.49e-14	27.59	3	9.9800e-01	4.32e-14	0.6
	0.5	2.58e-14	27.59	3	9.9800e-01	5.90e-14	0.6
	0.6	2.59e-14	27.59	3	9.9800e-01	9.45e-14	0.6
	0.7	2.55e-14	27.59	3	9.9800e-01	1.34e-13	0.6
	0.8	2.61e-14	27.59	3	9.9800e-01	1.77e-13	0.6
	0.9	2.56e-14	27.59	3	9.9800e-01	2.39e-13	0.6
1000	0.1	5.12e-14	39	3	9.9900e-01	7.10e-16	3.5
	0.2	5.07e-14	39	3	9.9900e-01	6.66e-15	3.6
	0.3	5.12e-14	39	3	9.9900e-01	1.37e-14	3.5
	0.4	5.13e-14	39	3	9.9900e-01	1.29e-14	3.6
	0.5	5.13e-14	39	3	9.9900e-01	4.98e-14	3.6
	0.6	5.10e-14	39	3	9.9900e-01	9.11e-14	3.6
	0.7	5.18e-14	39	3	9.9900e-01	1.30e-13	3.6
	0.8	5.10e-14	39	3	9.9900e-01	1.65e-13	3.6
	0.9	5.11e-14	39	3	9.9900e-01	2.14e-13	3.5
2000	0.1	1.03e-13	55.16	3	9.9950e-01	4.44e-16	20.4
	0.2	1.03e-13	55.16	3	9.9950e-01	4.08e-15	20.6
	0.3	1.01e-13	55.16	3	9.9950e-01	1.82e-14	20.5
	0.4	1.02e-13	55.16	3	9.9950e-01	2.29e-14	20.6
	0.5	1.03e-13	55.16	3	9.9950e-01	5.06e-14	20.5
	0.6	1.02e-13	55.16	3	9.9950e-01	1.62e-13	20.9
	0.7	1.02e-13	55.16	3	9.9950e-01	9.99e-14	20.9
	0.8	1.03e-13	55.16	3	9.9950e-01	1.63e-13	20.9
	0.9	1.02e-13	55.16	3	9.9950e-01	1.23e-13	20.4
3000	0.1	1.54e-13	67.56	3	9.9967e-01	1.77e-15	56.4
	0.2	1.55e-13	67.56	3	9.9967e-01	2.66e-15	56.5
	0.3	1.53e-13	67.56	3	9.9967e-01	5.15e-15	56
	0.4	1.53e-13	67.56	3	9.9967e-01	7.28e-15	56.3
	0.5	1.54e-13	67.56	3	9.9967e-01	3.02e-14	56.2
	0.6	1.52e-13	67.56	3	9.9967e-01	2.31e-14	56.2
	0.7	1.53e-13	67.56	3	9.9967e-01	6.91e-14	56.4
	0.8	1.54e-13	67.56	3	9.9967e-01	1.33e-13	55.7
	0.9	1.53e-13	67.56	3	9.9967e-01	2.22e-13	55.7
5000	0.1	2.55e-13	87.22	3	9.9980e-01	0	215.7
	0.2	2.54e-13	87.22	3	9.9980e-01	1.75e-14	216.8
	0.3	2.55e-13	87.22	3	9.9980e-01	4.08e-15	216.9
	0.4	2.57e-13	87.22	3	9.9980e-01	5.33e-15	218.5
	0.5	2.56e-13	87.22	3	9.9980e-01	1.01e-14	219.2
	0.6	2.56e-13	87.22	3	9.9980e-01	1.95e-14	217.4
	0.7	2.58e-13	87.22	3	9.9980e-01	3.16e-14	219.2
	0.8	2.58e-13	87.22	3	9.9980e-01	3.60e-14	217.4
	0.9	2.59e-13	87.22	3	9.9980e-01	7.42e-14	222.8

Table 5: Numerical results derived from solving the consistent system of general absolute value equations with $d = 1$

n	p	Me	$M\|x^\star\|$	Mi	Mf	$Merror$	$Mtime$
100	0.1	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.08
	0.2	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.06
	0.3	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.06
	0.4	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.06
	0.5	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.05
	0.6	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.05
	0.7	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.05
	0.8	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.05
	0.9	1.78e-06	9.66	2	3.52e-13	6.13e-09	0.05
500	0.1	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.2	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.3	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.4	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.5	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.6	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.7	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.8	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
	0.9	2.80e-09	14.50	2	4.69e-18	6.21e-11	1.3
1000	0.1	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.2
	0.2	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.3
	0.3	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.2
	0.4	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.3
	0.5	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.2
	0.6	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.2
	0.7	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.2
	0.8	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.2
	0.9	5.21e-11	18.39	2	1.04e-21	3.82e-11	8.3
2000	0.1	3.83e-06	29.06	2	6.71e-12	2.86e-08	67.6
	0.2	3.83e-06	29.06	2	6.71e-12	2.86e-08	69.8
	0.3	3.83e-06	29.06	2	6.71e-12	2.86e-08	63.9
	0.4	3.83e-06	29.06	2	6.71e-12	2.86e-08	60.9
	0.5	3.83e-06	29.06	2	6.71e-12	2.86e-08	59.7
	0.6	3.83e-06	29.06	2	6.71e-12	2.86e-08	60.2
	0.7	3.83e-06	29.06	2	6.71e-12	2.86e-08	59.6
	0.8	3.83e-06	29.06	2	6.71e-12	2.86e-08	59.3
	0.9	3.83e-06	29.06	2	6.71e-12	2.86e-08	58.9
3000	0.1	4.78e-07	31.64	2	2.07e-16	2.90e-11	174.4
	0.2	4.78e-07	31.64	2	2.08e-16	2.90e-11	173.9
	0.3	4.78e-07	31.64	2	2.07e-16	2.90e-11	174.1
	0.4	4.77e-07	31.64	2	2.07e-16	2.89e-11	177.5
	0.5	4.77e-07	31.64	2	2.06e-16	2.90e-11	173.6
	0.6	4.77e-07	31.64	2	2.06e-16	2.90e-11	174.9
	0.7	4.76e-07	31.64	2	2.06e-16	2.90e-11	174
	0.8	4.76e-07	31.64	2	2.06e-16	2.90e-11	174.7
	0.9	4.76e-07	31.64	2	2.06e-16	2.90e-11	152.7
5000	0.1	6.72e-09	41.06	2	3.93e-17	8.20e-10	765.7
	0.2	7.17e-09	41.06	2	4.51e-17	8.20e-10	689.1
	0.3	7.43e-09	41.06	2	4.88e-17	8.20e-10	617.9
	0.4	7.54e-09	41.06	2	5.02e-17	8.20e-10	591.2
	0.5	7.58e-09	41.06	2	5.08e-17	8.20e-10	576.2
	0.6	7.59e-09	41.06	2	5.10e-17	8.20e-10	572.2
	0.7	7.59e-09	41.06	2	5.10e-17	8.20e-10	557
	0.8	7.59e-09	41.06	2	5.10e-17	8.20e-10	560.9
	0.9	7.59e-09	41.06	2	5.10e-17	8.20e-10	562.8

Table 6: Numerical results derived from correcting the inconsistent system of general absolute value equations

n	p	Mee	$M\|x^\star\|$	Mi	Mf	$Merror$	$Mtime$
100	0.1	9.24e-17	1.22e-06	2	1	9.58e-10	0.1
	0.2	8.96e-17	1.18e-06	2	1	2.63e-10	0.1
	0.3	7.38e-17	1.27e-06	2	1	8.24e-11	0.1
	0.4	9.60e-17	1.17e-06	2	1	3.36e-11	0.2
	0.5	8.73e-17	1.30e-06	2	1	6.28e-12	0.2
	0.6	1.01e-16	1.15e-06	2	1	5.30e-13	0.2
	0.7	8.70e-17	2.83e-06	2	1	2.01e-11	0.2
	0.8	1.03e-16	2.04e-03	2.8	9.97e-01	2.49e-11	0.4
	0.9	8.63e-17	2.15e-02	3.2	9.73e-01	3.05e-09	0.5
500	0.1	9.26e-17	2.15e-06	2	1	3.92e-09	2.5
	0.2	1.04e-16	2.12e-06	2	1	1.02e-09	2.3
	0.3	8.49e-17	2.77e-06	2	1	6.17e-10	2.4
	0.4	1.02e-16	4.16e-06	2	1	8.53e-10	2.7
	0.5	7.69e-17	3.31e-06	2	1	1.85e-11	3
	0.6	7.76e-17	5.98e-06	2	9.9999e-01	2.10e-10	3.5
	0.7	8.99e-17	5.58e-06	2	9.9999e-01	1.25e-10	4.1
	0.8	9.17e-17	6.98e-06	2	9.9999e-01	1.57e-10	4.9
	0.9	8.72e-17	1.34e-05	2.2	9.9998e-01	2.14e-07	6.1
1000	0.1	8.21e-17	3.96e-06	2	1	4.04e-08	13.7
	0.2	9.31e-17	2.75e-06	2	1	3.90e-09	12.6
	0.3	9.31e-17	3.02e-06	2	1	4.12e-10	13.7
	0.4	9.56e-17	4.74e-06	2	9.9999e-01	3.47e-10	14.8
	0.5	9.05e-17	5.87e-06	2	9.9999e-01	6.79e-10	16.7
	0.6	8.20e-17	7.07e-06	2	9.9999e-01	2.28e-10	19.8
	0.7	9.95e-17	8.45e-06	2	9.9999e-01	1.97e-10	22.8
	0.8	9.71e-17	1.64e-05	2	9.9998e-01	7.21e-10	26.8
	0.9	1.04e-16	1.79e-05	2	9.9998e-01	1.48e-07	31.8
2000	0.1	9.01e-17	4.43e-06	2.25	9.9999e-01	3.12e-09	82.5
	0.2	8.86e-17	4.85e-06	2	9.9999e-01	2.92e-08	63.8
	0.3	8.03e-17	6.52e-06	2.5	9.9999e-01	1.30e-08	83.8
	0.4	9.35e-17	4.80e-06	2	9.9999e-01	2.43e-10	76.1
	0.5	1.09e-16	8.76e-06	2	9.9999e-01	2.77e-09	83
	0.6	9.30e-17	7.22e-06	2	9.9999e-01	2.23e-11	100.2
	0.7	9.83e-17	1.60e-05	2	9.9998e-01	3.47e-10	112.2
	0.8	8.40e-17	1.64e-05	2	9.9998e-01	5.28e-11	136.1
	0.9	1.10e-16	4.24e-05	2	9.9995e-01	3.93e-09	145.2
3000	0.1	9.46e-17	4.43e-06	2	9.9999e-01	6.02e-09	200
	0.2	8.37e-17	4.85e-06	2.2	9.9999e-01	3.68e-08	194.9
	0.3	8.60e-17	7.63e-06	2	9.9999e-01	3.76e-08	183.3
	0.4	9.52e-17	4.80e-06	2	9.9999e-01	3.80e-10	213.6
	0.5	9.12e-17	8.75e-06	2	9.9999e-01	3.94e-09	236.7
	0.6	9.23e-17	1.60e-05	2.2	9.9998e-01	3.46e-09	301.4
	0.7	1.15e-16	1.60e-05	2	9.9998e-01	4.10e-11	325.7
	0.8	8.39e-17	1.65e-05	2	9.9998e-01	3.37e-11	381.8
	0.9	9.99e-17	4.23e-05	2	9.9995e-01	4.12e-10	415.8
5000	0.1	9.76e-17	4.44e-06	2	1	8.6358e-09	706
	0.2	9.77e-17	5.24e-06	2	9.9999e-01	5.3467e-09	635
	0.3	8.60e-17	7.67e-06	2.2	9.9999e-01	1.3235e-08	723
	0.4	7.72e-17	4.82e-06	2	9.9999e-01	7.63e-10	750
	0.5	8.77e-17	8.90e-06	2	9.9999e-01	2.64e-10	834
	0.6	9.97e-17	1.82e-05	2.2	9.9998e-01	7.74e-09	1020
	0.7	9.31e-17	1.61e-05	2	9.9998e-01	3.03e-11	1139
	0.8	8.77e-17	3.76e-05	2	9.9996e-01	5.54e-09	1273
	0.9	9.21e-17	4.23e-05	2	9.9995e-01	2.58e-10	1505

Tables 1, 3 and 5 have been presented to show the correctness of Algorithm 2. As we can see, in each three tables the values obtained from columns 3 and 7, confirm the correctness of the algorithm. On the other hand, Tables 2, 4 and 6 report the results acheived from the optimal correction of infeasible systems. The data attained from the third column of Tables 2, 4 and 6, prove the high precision of Agorithm 2 in correcting these infeasible systems.
Furthermore, in all tables, the seventh column displays convergence of Algorithm 2 to high accuracy. When the data in Tables 1 and 2 are compared with those in Tables 3 and 4, it can be concluded that the running time during which system $Ax = b$ is corrected by Algorithm 2 is less than the time system $Ax - |x| = b$ is corrected by the same algorithm. The reason of this difference is smoothing the term $\|Ax - |x| - b\|^2$.
By comparing the reported results for different p in $(0, 1)$, we can claim that the solutions are almost identical. The only difference is observed in the running times. Using Dolan-Moré method in [13], we can present this time difference in the following graph.

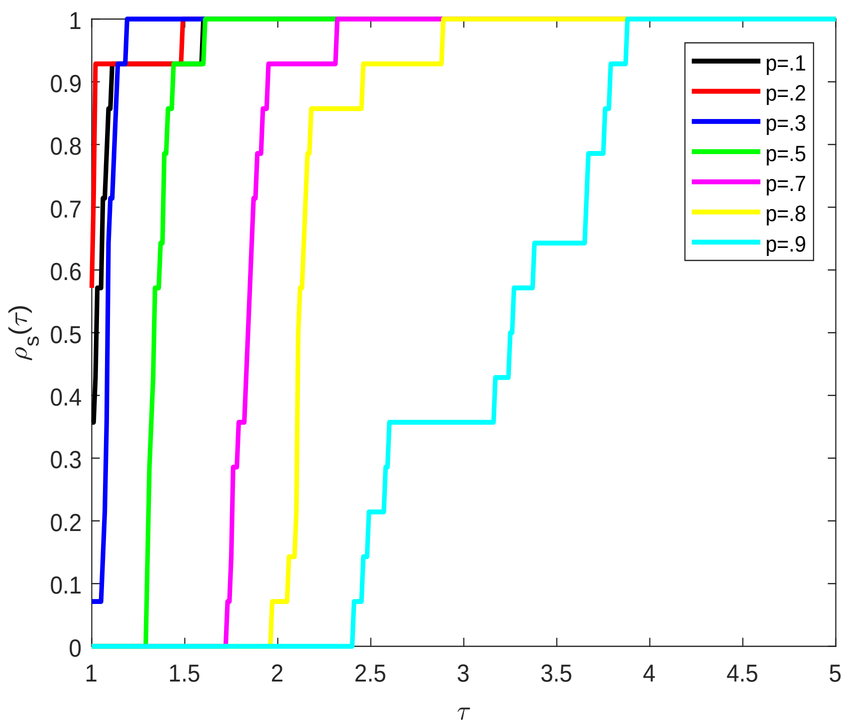

Figure 1: Performance profile of computing time of Algorithm 2 for $p = 0.1, p = 0.2, \ldots, p = 0.9$

Figure 1 shows the performance time during which Algorithm 2 corrects the infeasible system of absolue value equations ($Ax - |x| = b$). This procedure has been performed on systems with $100 - 5000$ dimensions (note that the number of systems examined in this algorithm is 1000). In cases where $p \leq .5$ the percentag of systems corrected by this algorithm is greater and it has been corrected in less time. In contrast, in cases where $p > .5$, it has taken more time for the system to be corrected. In general, this algorithm has been successful in correcting all these systems.

5. Conclusion

In this paper, we proposed a regularization method (ℓ_p-norm, $0 < p < 1$), to solve fractional problem derived form the optimal correction of the system $Ax + B|x| = b$. This regularization method is convergent for a sufficiently small value of $\lambda > 0$, but this value in Tikhonov regularization method must

be sufficiently large. When λ is large then the conditional number of Hessian matrix, obtained from the objective function, may be very large. Subsequently, the solutions would be inexact. For this reason, an exact regularization method (ℓ_p-norm, $0 < p < 1$) was suggested. Since this regularized problem leads to a non-covex and non-smooth optimization problem, to solve it, a generalized Newtown algorithm based on smooth generalized Newtown-DC algorithm was presented.
The numerical results observed in Tables 2, 4 and 6 show that our algorithm has corrected the infeasible systems in the least iteration and high performance[1].

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[1] The MATLAB code of programs is available and would be submitted upon request.

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