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Design and Analysis of a Faster King-Werner-type Derivative Free Method

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ABSTRACT: We introduce a new faster King-Werner-type derivative-free method for solving nonlinear equations. The local as well as semi-local convergence analysis is presented under weak center Lipschitz and Lipschitz conditions. The convergence order as well as the convergence radii are also provided. The radii are compared to the corresponding ones from similar methods. Numerical examples further validate the theoretical results.

Key Words: King-Werner method, local convergence, Semi-local convergence, Banach space, Euclidean space.

Contents

1. Introduction

The study of many physical phenomena often leads to finding a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subset X \to Y$ is Fréchet differentiable operator, X and Y are Banach spaces, and D is a nonempty subset of X. One prefers the solutions to be found in closed form but this can be achieved only in special cases. That is why most solution methods for such equations are iterative. There is a plethora of iterative methods for generating a sequence $\{x_n\}$ approximating x^*, see, for example [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18] and references therein. High convergence order methods that do not involve the usage of the Fréchet derivative are of particular importance, since many equations contain a non differentiable term.

In the present study, based on the King-Werner-type method (KWTM) defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1} F(x_n) \\ y_{n+1} &= x_{n+1} - A_n^{-1} F(x_{n+1}), \end{aligned} \tag{1.2}$$

where $x_0,\, y_0 \in D$ are initial points and $A_n = [x_n, y_n; F]$ is a divided difference of order one on D [1], we shall derive a modified King-Werner-type method. The order of convergence of KWTM is $1 + \sqrt{2}$. KWTM has been studied extensively in [19,20,21,22,23]. We shall show that modified method is of convergence order $3 > 1 + \sqrt{2}$ under the same hypotheses of convergence. Then, we compare the radius of convergence with the corresponding ones of Secant and Newton's method under common hypotheses.

The study of convergence of iterative methods is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence

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balls. Local results are important since they provide the degree of difficulty in choosing initial points. Moreover, we present the largest radius of convergence among these methods. Furthermore, the semi-local convergence of method (2.10) is presented. Finally, the paper is concluded with numerical examples.

2. Derivation and local convergence

The motivation for new modified method is taken from the real case $X = Y = \mathbb{R}$ and then rewritten in Banach space form as it is usually the case. Consider the method defined on $\mathbb{R}$ for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} x_{n+1} &= x_n - [x_n, y_n; f]^{-1} f(x_n) \\ y_{n+1} &= x_{n+1} - P'(x_{n+1})^{-1} f(x_{n+1}), \end{aligned} \tag{2.1}$$

where polynomial P is defined by $P(t) = at^2 + bt + c$ for solving the equation $f(t) = 0$. We shall impose the conditions

$$P(x_n) = f(x_n),\ P(y_n) = f(y_n) \quad \text{and} \quad P(x_{n+1}) = f(x_{n+1}). \tag{2.2}$$

Then, we have that

$$\begin{aligned} ax_n^2 + bx_n + c &= f(x_n) \\ ay_n^2 + by_n + c &= f(y_n) \\ ax_{n+1}^2 + bx_{n+1} + c &= f(x_{n+1}), \end{aligned}$$

so

$$\begin{aligned} a(x_n^2 - y_n^2) + b(x_n - y_n) &= f(x_n) - f(y_n) \\ a(x_{n+1}^2 - x_n^2) + b(x_{n+1} - x_n) &= f(x_{n+1}) - f(x_n), \end{aligned}$$

or

$$\begin{aligned} a(x_n + y_n) + b &= [x_n, y_n; f] \\ a(x_{n+1} + x_n) + b &= [x_{n+1}, x_n; f] \end{aligned} \tag{2.3}$$

and

$$a(x_{n+1} - y_n) = [x_{n+1}, x_n; f] - [x_n, y_n; f]. \tag{2.4}$$

Estimate (2.4) motivates us to choose

$$a = [x_{n+1}, x_n, y_n; f]. \tag{2.5}$$

Then, by substituting (2.5) in (2.3), we get

$$b = [x_n, y_n; f] - [x_{n+1}, x_n, y_n; f](x_n + y_n) \tag{2.6}$$

so

$$\begin{aligned} P'(x_{n+1}) &= 2ax_{n+1} + b \\ &= [x_n, y_n; f] + [x_{n+1}, y_n; f] - [x_n, y_n; f] \\ &\quad + [x_{n+1}, x_n; f] - [y_n, x_n; f]. \end{aligned} \tag{2.7}$$

Therefore, in view of (2.7) method (2.1) can be written as

$$x_{n+1} = x_n - [x_n, y_n; f]^{-1} f(x_n) \tag{2.8}$$

and

$$y_{n+1} = x_{n+1} - ([x_{n+1}, y_n; f] + [x_{n+1}, x_n; f] - [y_n, x_n; f])^{-1} f(x_{n+1}). \tag{2.9}$$

In a Banach space setting the method (2.9) can be expressed as

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1}F(x_n) \\ y_{n+1} &= x_{n+1} - B_{n+1}^{-1}F(x_{n+1}), \end{aligned} \tag{2.10}$$

where $B_{n+1} = [x_{n+1}, y_n;\ F] + [x_{n+1}, x_n;\ F] - [y_n, x_n;\ F]$. Here onwards we denote this method by MKWTM. The local convergence analysis that follows uses some parameters and scalars functions. Let l_0, l and l_1 be given parameters. Define parameters r_0 and r_A by

$$r_0 = \frac{1}{l_0 + l + 2l_1} \quad \text{and} \quad r_A = \frac{1}{2l_0 + l}.$$

Notice that $r_0 < r_A$ for $l_0 \le l$. Define functions ϕ and ψ on the interval $[0, r_0)$ by

$$\phi(t) = \frac{\left(l_1 + l_2 + \frac{l_2 t}{1-(l_0+l)t}\right)t}{1 - (l_0 + l + 2l_1)t}$$

and

$$\psi(t) = \phi(t) - 1.$$

We have $\psi(0) = -1 < 0$ and $\psi(t) \to +\infty$ as $t \to r_0^-$. It follows from the intermediate value theorem that equation $\psi(t) = 0$ has solution in the interval $(0, r_0)$. Denote by r the smallest such solution. Then, we have that for each $t \in [0, r)$

$$(l_0 + l + 2l_1)t < 1$$

and

$$0 \le \phi(t) < 1.$$

Notice that the convergence radius r can be given in closed form by the positive solution of the quadratic equation

$$q(t) = 0,$$

where

$$q(t) = \lambda_2 t^2 + \lambda_1 t + 1,$$

since $q(t) = 0$ is equivalent to $\psi(t) = 0$ provided that $(l_0 + l)t \neq 1$ and $(l_0 + l + 2l_1)t \neq 1$. Here, $\lambda_2 = (l_0 + l)(l_0 + l + 3l_1 + l_2) - l_2$ and $\lambda_1 = -(2l_0 + 2l + 3l_1 + l_2)$.

Let $B(x, \varrho)$ and $\bar{B}(x, \varrho)$ stand, respectively for the open and closed balls in X with center $x \in X$ and of radius $\varrho > 0$.

Next, we present the local convergence analysis of method (2.10) using the preceding notation and conditions whereas and X, Y are Banach spaces until otherwise specified.

Theorem 2.1. *Let $F : D \subset X \to Y$ be a Fréchet differentiable operator and $[\cdot, \cdot; F] : D^2 \to \mathcal{L}(X, Y)$ be a divided difference of order one. Suppose : there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in \mathcal{L}(Y, X)$, there exist $l_0, l > 0$ such that for each $x, y \in D$*

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \le l_0\|x - x^*\| + l\|y - x^*\|. \tag{2.11}$$

Let $D_0 = D \cap B(x^*, \frac{1}{l_0+l})$. There exist $l_1, l_2 > 0$ such that for each $x, y, u, v \in D_0$

$$\|F'(x^*)^{-1}([x, y; F] - [u, v; F])\| \le l_1\|x - u\| + l_2\|y - v\|. \tag{2.12}$$

and

$$\bar{B}(x^*, r) \subseteq D, \tag{2.13}$$

where the radius of convergence is defined previously. Then, sequences $\{x_n\}$, $\{y_n\}$ generated by method (2.10) for $x_0, y_0 \in B(x^*, r) - \{x^*\}$ are well defined in $B(x^*, r)$, remain in $B(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converge to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|x_{n+1} - x^*\| \le \frac{l_2\|y_n - x^*\|\|x_n - x^*\|}{1 - (l_0\|x_n - x^*\| + l\|y_n - x^*\|)} \le \|y_n - x^*\| < r, \tag{2.14}$$

$$\begin{aligned}\|y_{n+1}-x^*\| &\leq \frac{\big((l_1+l_2)\|y_n-x^*\|+l_2\|x_{n+1}-x^*\|\big)\|x_{n+1}-x^*\|}{1-\big((l_0+l_1)\|x_{n+1}-x^*\|+(l+l_1)\|y_n-x^*\|\big)}\\ &\leq \|x_{n+1}-x^*\|<r \end{aligned}\tag{2.15}$$

and

$$e_{n+4}\leq C_1e_{n+3}^2e_{n+2}e_{n+1}+C_2e_{n+3}^2e_{n+2}^2+C_3e_{n+3}^3,\tag{2.16}$$

where $e_n=\|x_n-x^*\|$, $C_1>0$, $C_2>0$, *and* $C_3>0$. *Furthermore, for* $R\in[r,\frac{1}{l_0})$ *the vector* x^* *is the only solution of equation* $F(x)=0$ *in* $D_1=D\cap B(x^*,R)$.

Proof: We shall show estimates (2.14) and (2.15) using induction on k. By hypotheses $x_0,y_0\in B(x^*,r)-\{x^*\}$, the definition of r and (2.11), we have in turn that

$$\|F'(x^*)^{-1}([x_k,y_k;F]-F'(x^*))\|\leq(l_0\|x_k-x^*\|+l\|y_k-x^*\|)<(l_0+l)r<1.\tag{2.17}$$

It follows from (2.17) and the Banach lemma on invertible operators [2] that $[x_k,y_k;F]^{-1}\in\mathcal{L}(Y,X)$ and

$$\begin{aligned}\|[x_k,y_k;F]^{-1}F'(x^*)\| \leq&\frac{1}{1-(l_0\|x_k-x^*\|+l\|y_k-x^*\|)}\\ <&\frac{1}{1-(l_0+l)r}.\end{aligned}\tag{2.18}$$

Then, x_{k+1} is well defined by the first substep of method (2.10) and we can write

$$\begin{aligned}x_{k+1}-x^* &= x_k-x^*-[x_k,y_k;F]^{-1}[x_k,x^*;F](x_k-x^*)\\ &=[x_k,y_k;F]^{-1}([x_k,y_k;F]-[x_k,x^*;F])(x_k-x^*).\end{aligned}\tag{2.19}$$

In view of (2.12), (2.18) and (2.19), we get in turn that

$$\begin{aligned}\|x_{k+1}-x^*\| \leq&\frac{l_2\|y_k-x^*\|\|x_k-x^*\|}{1-(l_0\|x_k-x^*\|+l\|y_k-x^*\|)}\\ \leq&\|y_k-x^*\|<r,\end{aligned}\tag{2.20}$$

which shows (2.14) for $n=k$ and $x_{k+1}\in B(x^*,r)-\{x^*\}$. By the second sub-step of method (2.10) we can write

$$\begin{aligned}y_{k+1}-x^* &= x_{k+1}-x^*-B_{k+1}^{-1}F(x_{k+1})\\ &=B_{k+1}^{-1}\big(B_{k+1}-[x_{k+1},x^*;F]\big)(x_{k+1}-x^*).\end{aligned}\tag{2.21}$$

Using (2.11), (2.12) and (2.21), we obtain in turn that

$$\begin{aligned}\|F'(x^*)^{-1}(B_{k+1}-F'(x^*))\| \leq&\|F'(x^*)^{-1}([x^*,x^*;F]-[x_{k+1},y_k;F])\|\\ &+\|F'(x^*)^{-1}([y_k,x_k;F]-[x_{k+1},x_k;F])\|\\ \leq&l_0\|x_{k+1}-x^*\|+l\|y_k-x^*\|\\ &+l_1\|x_{k+1}-x^*+x^*-y_k\|\\ \leq&(l_0+l_1)\|x_{k+1}-x^*\|+(l+l_1)\|y_k-x^*\|\\ \leq&(l_0+l+2l_1)r<1,\end{aligned}\tag{2.22}$$

so $B_{k+1}^{-1}\in\mathcal{L}(Y,X)$ and

$$\begin{aligned}\|B_{k+1}^{-1}F'(x^*)\| \leq&\frac{1}{1-((l_0+l_1)\|x_{k+1}-x^*\|+(l+l_1)\|y_k-x^*\|)}\\ \leq&\frac{1}{1-(l_0+l+2l_1)r}.\end{aligned}\tag{2.23}$$

In view of (2.12), (2.21), (2.23) and the definition of r, we get in turn that

$$\begin{aligned}\|y_{k+1}-x^*\| \leq & \frac{(l_1\|y_k-x^*\|+l_2\|x_{k+1}-y_k\|)\|x_{k+1}-x^*\|}{1-((l_0+l_1)\|x_{k+1}-x^*\|+(l+l_1)\|y_k-x^*\|)} \\ \leq & \frac{((l_1+l_2)\|y_k-x^*\|+l_2\|x_{k+1}-x^*\|)\|x_{k+1}-x^*\|}{1-(l_0+l+2l_1)r} \\ \leq & \phi(r)\|x_{k+1}-x^*\| \leq \|x_{k+1}-x^*\| < r, \end{aligned} \tag{2.24}$$

which shows (2.15) for $n=k$ and $y_{k+1}\in B(x^*,r)$. The induction for (2.14) and (2.15) is completed. By (2.20) and (2.24), we get the estimate

$$\|y_{k+1}-x^*\| \leq c\|y_k-x^*\| < r,$$

where

$$c=\frac{l_2\|x_0-x^*\|}{1-(l_0\|x_0-x^*\|+l\|y_0-x^*\|)}\in[0,1),$$

so $\lim_{k\to\infty} y_k = x^*$ and consequently by (2.20) $\lim_{k\to\infty} x_k = x^*$. We can write by (2.14) and (2.15) in turn that

$$\begin{aligned}\|x_{k+2}-x^*\| \leq & \frac{l_2\|y_{k+1}-x^*\|\|x_{k+1}-x^*\|}{1-(l_0+l)r}, \\ \|x_{k+3}-x^*\| \leq & \frac{l_2\|y_{k+2}-x^*\|\|x_{k+2}-x^*\|}{1-(l_0+l)r} \\ \leq & \frac{l_2((l_1+l_2)\|y_{k+1}-x^*\|+l_2\|x_{k+2}-x^*\|)\|x_{k+2}-x^*\|^2}{(1-(l_0+l)r)(1-(l_0+l+2l_1)r)}, \end{aligned}$$

so

$$\begin{aligned} &\|x_{k+4}-x^*\| \\ &\leq \frac{l_2\big((l_1+l_2)\|y_{k+2}-x^*\|+l_2\|x_{k+3}-x^*\|\big)\|x_{k+3}-x^*\|^2}{(1-(l_0+l)r)(1-(l_0+l+2l_1)r)} \\ &\leq \frac{l_2\Big(\frac{(l_1+l_2)\big((l_1+l_2)\|x_{k+1}-x^*\|+l_2\|x_{k+2}-x^*\|\big)\|x_{k+2}-x^*\|}{1-(l_0+l+2l_1)r}+l_2\|x_{k+3}-x^*\|\Big)\|x_{k+3}-x^*\|^2}{(1-(l_0+l+2l_1)r)(1-(l_0+l)r)}, \end{aligned}$$

which shows (2.16) for

$$C_1 = 4a_1a_2^2 > 0$$
$$C_2 = 2a_1a_2^2 > 0$$

and

$$C_3 = a_1a_2,$$

where $a_1=\frac{l}{1-2l_0\varrho}$ and $a_2=\frac{l}{1-2(l_0+l)r}$. To show the uniqueness part, let $Q=[x^*,y^*;F]$, where $y^*\in D_1$ with $F(y^*)=0$. By the definition of R and (2.11), we obtain that

$$\|F'(x^*)^{-1}(Q-F'(x^*))\| \leq l_0\|x^*-y^*\| \leq l_0R < 1,$$

so $Q^{-1}\in\mathcal{L}(Y,X)$. Then, from the identity

$$0=F(x^*)-F(y^*)=[x^*,y^*;F](x^*-y^*)=Q(x^*-y^*),$$

we conclude that $x^*=y^*$. □

Next, we shall present the local convergence for Secant methods

$$x_{n+1}=x_n-[x_n,x_{n-1};F]^{-1}F(x_n) \tag{2.25}$$

and

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \tag{2.26}$$

under conditions (2.11) and (2.12), so we can compare the radii with r. Using the approximations

$$x_{n+1} - x^* = [x_n, x_{n-1}; F]^{-1}\big([x_n, x_{n-1}; F] - [x_n, x^*; F]\big)(x_n - x^*) \tag{2.27}$$

and

$$x_{n+1} - x^* = [x_{n-1}, x_n; F]^{-1}\big([x_{n-1}, x_n; F] - [x_n, x^*; F]\big)(x_n - x^*), \tag{2.28}$$

respectively as in the proof of Theorem 2.1, we arrive at:

Proposition 2.2. *Suppose that the hypotheses of Theorem 2.1 until (2.12) and*

$$\bar{B}(x^*, r^*) \subseteq D \tag{2.29}$$

hold, where

$$r^* = \frac{1}{l_0 + l + l_2} \tag{2.30}$$

hold. Then, sequence $\{x_n\}$ generated by Secant method (2.25) for $x_{-1}, x_0 \in B(x^*, r^*)$-$\{x^*\}$ is well defined in $B(x^*, r^*)$, remains in $B(x^*, r^*)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|x_{n+1} - x^*\| \le \frac{l_2 \|x_{n-1} - x^*\| \|x_n - x^*\|}{1 - (l_0 \|x_n - x^*\| + l \|x_n - x^*\|)}. \tag{2.31}$$

Furthermore, for $R \in [r^*, \frac{1}{l_0})$ the vector x^* is the only solution of equation $F(x) = 0$ in $D_1 = D \cap B(x^*, R)$.

Remark 2.3. *Condition (2.12) can be replaced for Secant method (2.25) by*

$$\|F'(x^*)^{-1}([x, y; F] - [x, z; F])\| \le l_3 \|y - z\|, \tag{2.32}$$

for some $l_3 > 0$. Then, r^*, l_2, (2.31) can be replaced by r_1^*, l_3, (2.34), respectively in Proposition (2.1)

$$r_1^* = \frac{1}{l_0 + l + l_3}, \tag{2.33}$$

$$\|x_{n+1} - x^*\| \le \frac{l_3 \|x_{n-1} - x^*\| \|x_n - x^*\|}{1 - (l_0 \|x_n - x^*\| + l \|x_n - x^*\|)}. \tag{2.34}$$

Notice that

$$l_3 \le l_2, \tag{2.35}$$

so

$$r^* \le r_1^* \tag{2.36}$$

and (2.35) is a more precise estimate than (2.31).

Proposition 2.4. *Suppose that the hypotheses of Theorem 2.1 and*

$$\bar{B}(x^*, r^{**}) \subseteq D \tag{2.37}$$

hold, where

$$r^{**} = \frac{1}{l_0 + l + 2l_1 + l_2}. \tag{2.38}$$

Then, sequence $\{x_n\}$ generated by Secant method (2.26) for $x_{-1}, x_0 \in B(x^*, r^{**}) - \{x^*\}$ is well defined in $B(x^*, r^{**})$, remains in $B(x^*, r^{**})$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|x_{n+1} - x^*\| \le \frac{l_1 \|x_{n-1} - x^*\| + (l_1 + l_2)\|x_n - x^*\|}{1 - (l_0 \|x_{n-1} - x^*\| + l \|x_n - x^*\|)} \|x_n - x^*\|. \tag{2.39}$$

Furthermore, for $R \in [r^{**}, \frac{1}{l_0})$ the vector x^* is the only solution of equation $F(x) = 0$ in $D_1 = D \cap B(x^*, R)$.

Proposition 2.5. *It was shown in [1,2] that under conditions (2.11) and (2.12) the radius of convergence for Newton's method*

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{2.40}$$

is given by

$$r_A^* = \frac{1}{2(l_0+l)+l_1+l_2}. \tag{2.41}$$

Then, it follows from the preceding definitions of the convergence radii that r_1^* is the largest convergence radius among methods (2.10), (2.25), (2.26), and (2.40).

We present the local convergence analysis of KWTM based on scalar parameters and functions. Let $\alpha \geq 0$, $\beta \geq 0$ and $b > 0$ with $\alpha + \beta \neq 0$. Define parameters ϱ_0, ϱ_1 and functions f and h_f on interval $[0, \varrho_0)$ by

$$\varrho_0 = \frac{1}{\alpha+\beta}, \quad \varrho_1 = \frac{1}{\alpha+\beta+b},$$

$$f(t) = \big(b + \frac{\alpha b t}{1-(\alpha+\beta)t} + \beta\big)t$$

and

$$h_f(t) = f(t) - 1.$$

We have that $h_f(0) = -1$ and $h_f(t) \to +\infty$ as $t \to \varrho_0^-$. The intermediate value theorem assures that equation $h_f(t) = 0$ has solutions on the interval $(0, \varrho_0)$. Denote by ϱ^* the smallest such solution. Notice that $h_f(\varrho_1) = 0$, so $\varrho^* \leq \varrho_1$. Then, we have for each $t \in [0, \varrho^*)$

$$0 \leq \frac{bt}{1-(\alpha+\beta)t} < 1.$$

and

$$0 \leq f(t) < 1.$$

The local convergence analysis of KWTM is based on the hypotheses (H):

(h1) $F : D \subseteq X \to Y$ is a continuously Fréchet-differentiable operator and $[\cdot,\cdot;F] : D \times D \to \mathcal{L}(X,Y)$ is a divided difference operator of order one.

(h2) There exists parameters $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta \neq 0$, $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in \mathcal{L}(Y,X)$ and for each $x, y \in D$

$$\|F'(x^*)^{-1}([x,y;F] - F'(x^*))\| \leq \alpha\|x-x^*\| + \beta\|y-x^*\|.$$

Set: $D_0 = D \cap \bar{U}(x^*, \varrho_0)$, where ϱ_0 is defined previously.

(h3) There exists $b > 0$ such that for each $x, y \in D_0$

$$\|F'(x^*)^{-1}([x,y;F] - [x,x^*;F])\| \leq b\|y-x^*\|.$$

(h4) $\bar{U}(x^*, \varrho^*) \subseteq D$, where ϱ^* is defined previously.

(h5) There exists $R^* \geq \varrho^*$ such that

$$R^* < \frac{1}{\beta}, \quad \beta \neq 0.$$

Set $D_1 = D \cap \bar{U}(x^*, R^*)$.

Theorem 2.6. *Suppose that the hypotheses (H) hold. Then, sequences $\{x_n\}$, $\{y_n\}$ starting from x_0, $y_0 \in U(x^*, \varrho^*) - \{x^*\}$ and generated by KWTM are well defined in $U(x^*, \varrho^*)$ for each $n = 0, 1, 2, \dots$ and remain in $U(x^*, \varrho^*)$ converge to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \dots$*

$$\|x_{n+1} - x^*\| \leq \frac{b\|y_n - x^*\|}{1 - (\alpha\|x_n - x^*\| + b\|y_n - x^*\|)}\|x_n - x^*\| \leq \|x_n - x^*\| < \varrho^* \tag{2.42}$$

and

$$\|y_{n+1} - x^*\| \leq \frac{b\|y_n - x^*\|}{1 - (\alpha\|x_{n+1} - x^*\| + b\|y_n - x^*\|)}\|x_{n+1} - x^*\|. \tag{2.43}$$

Furthermore, the limit point x^* is the only solution of equation $F(x) = 0$ in D_1, where D_1 is defined in (h_5).

Proof: Let $x, y \in U(x^*, \varrho^*)$. Using (h$_2$), we have in turn that

$$\begin{aligned}\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| &\leq \alpha\|x - x^*\| + \beta\|y - x^*\| \\ &< (\alpha + \beta)\varrho^* < 1.\end{aligned} \tag{2.44}$$

In view of (2.44) and the Banach lemma on invertible operators [24] $[x, y; F]^{-1} \in \mathcal{L}(Y, X)$ and

$$\|[x, y; F]^{-1}F'(x^*)\| \leq \frac{1}{1 - (\alpha\|x - x^*\| + \beta\|y - x^*\|)}. \tag{2.45}$$

In particular $[x_0, y_0; F]^{-1} \in \mathcal{L}(Y, X)$, since $x_0, y_0 \in U(x^*, \varrho^*)$. We can write by the first substep of method KWTM

$$\begin{aligned}x_1 - x^* &= x_0 - x^* - [x_0, y_0; F]^{-1}F(x_0) \\ &= [x_0, y_0; F]^{-1}\big([x_0, y_0; F] - [x_0, x^*; F]\big)(x_0 - x^*)\end{aligned} \tag{2.46}$$

By (h$_3$), (2.45) for $x = x_0$, $y = y_0$ and (2.46), we get in turn

$$\begin{aligned}\|x_1 - x^*\| &= \|[x_0, y_0; F]^{-1}F'(x^*)\|\|F'(x^*)^{-1}\big([x_0, y_0; F] - [x_0, x^*; F]\big)(x_0 - x^*)\| \\ &\leq \frac{b\|y_0 - x^*\|}{1 - (\alpha\|x_0 - x^*\| + \beta\|y_0 - x^*\|)}\|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < \varrho^*,\end{aligned} \tag{2.47}$$

so (2.42) holds for $n = 0$ and $x_1 \in U(x^*, \varrho^*)$ and $[x_1, y_0; F]^{-1} \in \mathcal{L}(Y, X)$. We also have by (2.45) that

$$\|[x_1, y_0; F]^{-1}F'(x^*)\| \leq \frac{1}{1 - (\alpha\|x_1 - x^*\| + \beta\|y_0 - x^*\|)}. \tag{2.48}$$

Moreover, we can write by the second substep of KWTM that

$$\begin{aligned}y_1 - x^* &= x_1 - x^* - [x_1, y_0; F]^{-1}F(x_1) \\ &= [x_1, y_0; F]^{-1}\big([x_1, y_0; F] - [x_1, x^*; F]\big)(x_1 - x^*),\end{aligned} \tag{2.49}$$

so

$$\begin{aligned}\|y_1 - x^*\| &\leq \frac{b\|y_0 - x^*\|\|x_1 - x^*\|}{1 - (\alpha\|x_1 - x^*\| + \beta\|y_0 - x^*\|)} \\ &\leq \frac{b\varrho^*}{1 - (\alpha + \beta\varrho^*)\varrho^*}\|x_1 - x^*\| < \varrho^*,\end{aligned}$$

which shows (2.43) for $n = 0$ and $y_1 \in U(x^*, \varrho^*)$. The induction for (2.42) and (2.43) is completed analysis if x_0, y_0, x_1, y_1 are replaced by x_m, y_m, x_{m+1}, y_{m+1} in the preceding estimates, respectively. Then, from the estimates

$$\|x_{m+1} - x^*\| \leq \mu_1 \|x_m - x^*\| < \varrho^*$$

and

$$\|y_{m+1} - x^*\| \leq \mu_2 \|x_{m+1} - x^*\| < \varrho^*,$$

where $\mu_1 = \frac{b\varrho^*}{1-(\alpha+\beta)\varrho^*} \in [0,1)$, and $\mu_2 = f(\varrho^*) \in [0,1)$, we deduce that $\lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = x^*$, $x_{m+1} \in U(x^*, \varrho^*)$ and $y_{m+1} \in U(x^*, \varrho^*)$. The uniqueness part is shown by letting $T = [x^*, y^*; F]$ for some $y^* \in D_1$ with $F(y^*) = 0$. Using (h_2) and (h_5), we obtain in turn that

$$\|F'(x^*)^{-1}([x^*, y^*; F] - F'(x^*))\| \leq \beta \|y^* - x^*\| \leq \beta R < 1,$$

so $T^{-1} \in \mathcal{L}(Y, X)$. Finally, from the identity

$$0 = F(x^*) - F(y^*) = [x^*, y^*; F](x^* - y^*),$$

we conclude that $x^* = y^*$. □

Remark 2.7. *Define the computational order of convergence (COC) [25] by*

$$COC = log\left\|\frac{w_{n+2} - x^*}{w_{n+1} - x^*}\right\| \Big/ log\left\|\frac{w_{n+1} - x^*}{w_n - x^*}\right\|, \quad \text{for each } n = 1, 2, \ldots. \tag{2.50}$$

and the approximate computational order of convergence (ACOC) [26], by

$$ACOC = log\left\|\frac{w_{n+2} - w_{n+1}}{w_{n+1} - w_n}\right\| \Big/ log\left\|\frac{w_{n+1} - w_n}{w_n - w_{n-1}}\right\|, \quad \text{for each } n = 1, 2, \ldots. \tag{2.51}$$

This way we obtain a practical order of convergence.

3. Semi-local convergence analysis

The semi-local convergence analysis of method (2.10) is based on some scalars sequences and parameters. Let $L_0 > 0$, $L > 0$, $L_1 > 0$, $L_2 > 0$, $\eta_0 \geq 0$ and $\eta \geq 0$ be given parameters. Define sequences $\{t_n\}$, $\{s_n\}$ for each $n = 0, 1, 2, \ldots$ by $t_0 = 0$, $s_0 = \eta_0$, $t_1 = \eta$, $s_1 = \big(1 + \frac{L_0 t_1 + L s_0}{1-((L_0+L_1)t_1 + L_1 s_0)}\big)t_1$, $t_2 = (1 + \frac{L_0 t_1 + L s_0}{1-(L_0 t_1 + L(s_1+s_0))})t_1$,

$$s_{n+1} = t_{n+1} + \frac{\big(L_1(t_{n+1} - t_n) + L_2(s_n - t_n)\big)(t_{n+1} - t_n)}{1 - \big(L_0 t_{n+1} + L(s_n + s_0) + L_1((t_{n+1} - t_n) + (s_n - t_n))\big)}, \tag{3.1}$$

$$t_{n+2} = t_{n+1} + \frac{\big(L_1(t_{n+1} - t_n) + L_2(s_n - t_n)\big)(t_{n+1} - t_n)}{1 - \big(L_0 t_{n+1} + L(s_{n+1} + s_0)\big)}.$$

Moreover, define polynomials g and h by

$$g(t) = (L_0 + L + 2L_1)t^2 + (L_2 - L_1)t - (L_1 + L_2)$$

and

$$h(t) = Lt^3 + L_0 t^2 + (L_1 + L_2)t - (L_1 + L_2).$$

We have that $g(0) = h(0) = -(L_1 + L_2) < 0$ and $g(1) = h(1) = L_0 + L_1 > 0$. Denote by α_g and α_h the unique solutions (by Descartes rule of sign) of equations $g(t) = 0$ and $h(t) = 0$, respectively.

We have that

$$h(\alpha_g) = L\alpha_g^3 + L_0\alpha_g^2 + (L_1 + L_2)\alpha_g - (L_1 + L_2) - g(\alpha_g),$$

since $g(\alpha_g) = 0$.

$$\text{Case}(I) : L\alpha_g \geq 2L_1 \Rightarrow h(\alpha_g) \leq 0 \Rightarrow \alpha_g \leq \alpha_h,$$

$$\text{Case}(II) : L\alpha_g \le 2L_1 \Rightarrow h(\alpha_g) \ge 0 \Rightarrow \alpha_h \le \alpha_g.$$

Then, define parameters α by

$$\alpha = \begin{cases} \alpha_h, & L\alpha_g \ge 2L_1, \\ \alpha_g, & L\alpha_g \le 2L_1. \end{cases} \tag{3.2}$$

Next, we present a convergence result for sequences $\{t_n\}$, $\{s_n\}$ using the preceding notation.

Lemma 3.1. *Let parameter α be defined as in (3.2) if $t_0 \le s_0$, $L_0\eta + L(s_1+\eta_0) < 1$, $(L_0+L)\eta + L_1\eta_0 < 1$,*

$$(L_0t_1 + Ls_0)max\Big\{\frac{1}{1-(L_0\eta + L(s_1+\eta_0))}, \frac{1}{1-((L_0+L_1)\eta + L_1\eta_0)}\Big\} \le \alpha \tag{3.3}$$

and

$$\alpha \le 1 - \frac{(L_0+L)\eta}{1-L\eta_0},$$

then the sequences $\{t_n\}$, $\{s_n\}$ are non-decreasing, bounded from above by $t^{**} = \frac{t_1}{1-\alpha}$ and converge to the unique least upper bound t^* satisfying $t_1 \le t^* \le t^{**}$. Moreover, for each $n = 1, 2, \dots,$ we have the estimates

$$0 \le s_{n+1} - t_{n+1} \le \alpha(t_{n+1} - t_n), \tag{3.4}$$

$$0 \le t_{n+2} - t_{n+1} \le \alpha(t_{n+1} - t_n)$$

and

$$0 \le t_n \le s_n.$$

Proof: Estimates (3.4) shall be shown using mathematical induction. If $t_1 = 0$, $t_k = s_k = 0$ follows from (3.1) and (3.4) holds for each $k = 1, 2, \dots$. For other values of $t_1 = \eta > 0$, (3.4) is satisfied for each k provided that

$$0 \le \frac{L_1(t_{k+1} - t_k) + L_2(s_k - t_k)}{1-(L_0t_{k+1} + L(s_k + s_0) + L_1(t_{k+1} - t_k) + L_1(s_k - t_k))} \le \alpha, \tag{3.5}$$

$$0 \le \frac{L_1(t_{k+1} - y_k) + L_2(s_k - t_k)}{1-(L_0t_{k+1} + L(s_{k+1} + s_0))} \le \alpha \tag{3.6}$$

and

$$0 \le t_k \le s_k. \tag{3.7}$$

By (3.1), (3.5)–(3.7), we have that

$$0 \le s_k - t_k \le \alpha^k(t_1 - t_0), \tag{3.8}$$

$$0 \le t_{k+1} - t_k \le \alpha^k(t_1 - t_0), \tag{3.9}$$

so

$$\begin{aligned} s_k \le& t_k + \alpha^k(t_1 - t_0) \le t_{k-1} + \alpha^{k-1}(t_1 - t_0) + \alpha^k(t_1 - t_0) \\ \le& t_1 + \alpha(t_1 - t_0) + \cdots + \alpha^k(t_1 - t_0) = \frac{1-\alpha^{k+1}}{1-\alpha}(t_1 - t_0) < t^{**} \end{aligned} \tag{3.10}$$

and analogously

$$t_{k+1} \le \frac{1-\alpha^{k+1}}{1-\alpha}(t_1 - t_0) < t^{**}. \tag{3.11}$$

Evidently (3.5)–(3.7) hold by the definition of t_0, s_0, $t_0 \le s_0$ and the left hand side inequality in (3.3). In view of (3.8)–(3.11), estimate (3.5) holds, if

$$(L_1 + L_2)\alpha^k t_1 + \alpha L_0 \frac{1-\alpha^{k+1}}{1-\alpha} t_1 + \alpha L \frac{1-\alpha^{k+1}}{1-\alpha} t_1 + 2\alpha L_1 \alpha^k t_1 - \alpha + \alpha s_0 \le 0. \tag{3.12}$$

Estimate (3.12) motivates us to define recurrent functions on the interval $[0,1)$ by

$$\phi_k(t) = \Big((L_1+L_2)t^{k-1} + (L_0+L_1)(1+t+\cdots+t^k) + 2L_1t^k\Big)t_1 + Ls_0 - 1. \tag{3.13}$$

A relationship is needed between two consecutive function ϕ_k and ϕ_{k+1}. We have in turn by (3.3)

$$\begin{aligned}\phi_{k+1}(t) =&(L_1+L_2)t^kt_1 + (L_0+L_1)(1+t+\cdots+t^{k+1})t_1 + 2L_1t^{k+1}t_1 + Ls_0 - 1\\ &- (L_1+L_2)t^{k-1}t_1 - (L_0+L)(1+t+\cdots+t^k)t_1 - 2L_1t^kt_1\\ &- Ls_0 + 1 + \phi_k(t)\\ =&\,\phi_k(t) + g(t)t^{k-1}t_1.\end{aligned} \tag{3.14}$$

In particular by (3.13), estimate (3.12) holds, if

$$\phi_k(\alpha) \le 0 \quad \text{for each} \quad k = 0,1,2,\ldots. \tag{3.15}$$

Define function ϕ_∞ on $[0,1)$ by

$$\phi_\infty(t) = \lim_{k\to\infty} \phi_k(t) = \frac{L_0+L}{1-t}t_1 + L_0s - 1. \tag{3.16}$$

Using (3.2), (3.14) and (3.16) we get that

$$\phi_k(\alpha) \le \phi_{k+1}(\alpha) \le \ldots \phi_\infty(\alpha), \tag{3.17}$$

so (3.15) holds, if

$$\phi_\infty(\alpha) \le 0, \tag{3.18}$$

which is true by right hand-side inequality in condition (3.3). Hence, the induction for (3.5) is completed. Similarly, to show (3.6), it suffices to have

$$(L_1+L_2)\alpha^kt_1 + \alpha L_0\frac{1-\alpha^{k+1}}{1-\alpha}t_1 + \alpha L\frac{1-\alpha^{k+2}}{1-\alpha}t_1 + \alpha Ls_0 - \alpha \le 0. \tag{3.19}$$

Then, again define recurrent functions ψ_k on $[0,1)$ by

$$\psi_k(t) = \Big((L_1+L_2)t^{k-1} + L_0(1+t+\cdots+t^k) + L(1+t+\cdots+t^{k+1})\Big)t_1 + Ls_0 - 1.$$

Then, we get in turn that

$$\begin{aligned}\phi_{k+1}(t) =& (L_1+L_2)t^kt_1 + L_0(1+t+\cdots+t^{k+1})t_1 + L(1+t+\cdots+t^{k+2})t_1 + Ls_0\\ &- 1 - (L_1+L_2)t^{k-1}t_1 - L_0(1+t+\cdots+t^k)t_1\\ &- L(1+t+\cdots+t^{k+1})t_1\\ &- Ls_0 + 1 + \psi_k(t)\\ =&\,\psi_k(t) + h(t)t^{k-1}t_1.\end{aligned} \tag{3.20}$$

Then, (3.19) is satisfied, if

$$\psi_k(\alpha) \le 0, \quad \text{for each} \quad k = 1,2,\ldots. \tag{3.21}$$

Define function ψ_∞ on $[0,1)$ by

$$\psi_\infty(t) = \lim_{k\to\infty} \psi_k(t) = \Big(\frac{L_0+L}{1-t}\Big)t_1 + Ls_0 - 1. \tag{3.22}$$

Using (3.2), (3.20) and (3.22) we obtain that

$$\psi_k(\alpha) \le \psi_{k+1}(\alpha) \le \ldots \psi_\infty(\alpha), \tag{3.23}$$

so (3.21) holds, if

$$\psi_\infty(\alpha) \le 0, \tag{3.24}$$

which is true by the right hand side inequality in condition (3.3). Hence, the induction for (3.6) is completed. Then, the induction for (3.7) is completed by (3.1), (3.5) and (3.6). Moreover, estimates (3.4) hold. Hence, sequences $\{t_k\}$, $\{s_k\}$ are nondecreasing, bounded above by t^{**} and such they converge to their unique least upper bound t^*. □

The following definition is useful for the semi-local convergence analysis of method (3.3) that follows.

Definition 3.2. *Let $L_0 > 0$, $L > 0$, $L_1 > 0$, $L_2 > 0$, $\eta_0 \ge 0$ and $\eta \ge 0$ be given parameters. The triplet (F, x_0, y_0) belongs to the $\tau(L_0, L, L_1, L_2, \eta_0, \eta)$, if*

$$A_0^{-1} \in \mathcal{L}(Y, X) \quad \textit{for some} \quad x_0, y_0 \in D, \tag{3.25}$$

$$\|A_0^{-1}F(x_0)\| \le \eta, \ \|x_0 - y_0\| \le \eta_0 \quad \textit{for some} \quad \eta \ge 0, \eta_0 \ge 0, \tag{3.26}$$

for each $x, y \in D$

$$\|A_0^{-1}([x, y; F] - [x_0, y_0; F])\| \le L_0\|x - x_0\| + L\|y - y_0\|. \tag{3.27}$$

Set $D_0 = D \cap B(x_0, \frac{1}{L_0+L})$. For each $x, y, z, w, \in D_0$

$$\|A_0^{-1}([x, y; F] - [z, w; F])\| \le L_1\|x - z\| + L_2\|y - w\|, \tag{3.28}$$

$$L_0 t^* + L(r + \eta_0) < 1, \quad \textit{for some} \quad R \ge t^*, \tag{3.29}$$

$$B(x_0, t^*) \subset D \tag{3.30}$$

and conditions of Lemma 3.1 hold, where t^* is given in Lemma 3.1.

Next, we present the semi-local convergence analysis of method (2.10).

Theorem 3.3. *Let $F \in \tau$. Then, sequences $\{x_n\}$, $\{y_n\}$ starting from some $x_0, y_0 \in D$ and generated by method (2.10) are well defined in $B(x_0, t^*)$, remain in $B(x_0, t^*)$ for each $n = 0, 1, 2, \dots$ and converge to a unique solution x^* of equation $F(x) = 0$ in $B(x_0, t^*)$. Moreover, the following estimates holds for each $n = 0, 1, 2, \dots$*

$$\|y_{n+1} - x_{n+1}\| \le s_{n+1} - t_{n+1}, \tag{3.31}$$

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \tag{3.32}$$

and

$$\|x_n - x^*\| \le t^* - t_n. \tag{3.33}$$

Furthermore, x^* is the only solution of equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x_0, R)$.

Proof: The result is shown using mathematical induction. If $k = 0$, estimates (3.31) and (3.32) hold by the definition of τ and $y_0, x_1 \in \bar{B}(x_0, t^*)$. For $k = 1$, using (2.10), (3.1), (3.3), (3.27) and (3.28) we have in turn that

$$\begin{aligned}\|y_1 - x_1\| = \|A_0^{-1}F(x_1)\| &= \|A_0^{-1}\big([x_1, x_0; F](x_1 - x_0) - [x_0, y_0; F](x_1 - x_0)\big)\| \\ &= \|A_0^{-1}\big([x_1, x_0; F] - [x_0, y_0; F]\big)(x_1 - x_0)\| \\ &\le \big(L_0\|x_1 - x_0\| + L\|x_0 - y_0\|\big)\|x_1 - x_0\| \\ &\le (L_0\eta + L\eta_0)\eta = s_1 - t_1 \end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\|A_0^{-1}(A_1 - A_0)\| &\le \big(L_0\|x_1 - x_0\| + L\|y_1 - y_0\|\big) \\ &\le L_0\|x_1 - x_0\| + L(\|y_1 - x_1\| + \|x_1 - x_0\| + \|x_0 - y_0\|) \\ &\le L_0\eta + L(s_1 - t_1 + t_1 - t_0 + s_0 - t_0) \\ &= L_0\eta + L(s_1 + \eta_0) < 1. \end{aligned} \tag{3.35}$$

It follows from (3.34) that (3.31) holds for $k = 1$ and by (3.35) and the Banach lemma on invertible operators $A_1^{-1} \in \mathcal{L}(Y, X)$ and

$$\|A_1^{-1}A_0\| \leq \frac{1}{1-(L_0t_1+L(s_1+s_0))} \tag{3.36}$$

By first substep of method (2.10) for $k = 1$, (3.1), (3.27) and (3.36), we get in turn that

$$\begin{aligned}\|x_2-x_1\| \leq& \|A_1^{-1}A_0\|\|A_0^{-1}(F(x_1)-F(x_0)+F(x_0))\| \\ \leq& \frac{\|A_0^{-1}([x_1,x_0;F]-[x_0,y_0;F])(x_1-x_0)\|}{1-(L_0t_1+L(s_1+s_0))} \\ \leq& \frac{L_0t_1+Ls_0}{1-(L_0t_1+L(s_1+s_0))}t_1 = t_2-t_1,\end{aligned}$$

which shows (3.32) for $k = 1$. Similarly for $k = 2, 3, \ldots$

$$\|A_k^{-1}A_0\| \leq \frac{1}{1-(L_0t_k+L(s_k+s_0))}$$

and

$$\begin{aligned}\|x_{k+1}-x_k\| =& \|A_k^{-1}F(x_k)\| \\ \leq& \|A_k^{-1}A_0\|\|A_0^{-1}(F(x_k)-F(x_{k-1})+F(x_{k-1}))\| \\ \leq& \frac{\|A_0^{-1}([x_k,x_{k-1};F]-[x_{k-1},y_{k-1};F])(x_k-x_{k-1})\|}{1-(L_0t_k+L(s_k+s_0))} \\ \leq& \frac{L_1(t_k-t_{k-1})+L_2(s_{k-1}-t_{k-1})}{1-(L_0t_k+L(s_k+s_0))}(t_k-t_{k-1}) = t_{k+1}-t_k,\end{aligned}$$

which shows (3.32) for $n = k$. Similarly, we have

$$\begin{aligned}\|A_0^{-1}(B_1-A_0)\| =& \|A_0^{-1}\big(([x_1,y_0;F]-[x_0,y_0;F])+([x_1,x_0;F]-[y_0,x_0;F])\big)\| \\ \leq& L_0\|x_1-x_0\|+L\|y_0-y_0\|+L_1\|x_1-y_0\| \\ \leq& L_0\|x_1-x_0\|+L_1(\|x_1-x_0\|+\|x_0-y_0\|) \\ \leq& (L_0+L_1)\|x_1-x_0\|+L_1\|x_0-y_0\| \\ \leq& (L_0+L_1)t_1+L_1s_0 < 1,\end{aligned}$$

then, $B_1^{-1} \in \mathcal{L}(Y, X)$ and

$$\|B_0^{-1}A_0\| \leq \frac{1}{1-((L_0+L_1)t_1+L_1s_0)},$$

so

$$\begin{aligned}\|y_1-x_1\| =& \|B_1^{-1}F(x_1)\| \\ \leq& \|B_1^{-1}A_0\|\|A_0^{-1}F(x_1)\| \\ \leq& \frac{\|A_0^{-1}(F(x_1)-F(x_0)+F(x_0)\|}{1-((L_0+L_1)t_1+L_1s_0)} \\ \leq& \frac{\|A_0^{-1}([x_1,x_0;F]-[x_0,y_0;F])(x_1-x_0)\|}{1-((L_0+L_1)t_1+L_1s_0)} \\ \leq& \frac{L_1(t_1-t_0)+L_2(s_0-t_0)}{1-((L_0+L_1)t_1+L_1s_0)}(t_1-t_0) = s_1-t_1,\end{aligned}$$

which shows (3.31) for $n = 1$. Similarly, we have for $k = 1, 2, \dots$

$$\begin{aligned}
&\|A_0^{-1}(B_{k+1} - A_0)\| \\
&\le \|A_0^{-1}\big(([x_{k+1}, y_k; F] - [x_0, y_0; F]\big)\| + \|A_0^{-1}\big(([x_{k+1}, x_k; F] - [y_k, x_k; F])\big)\| \\
&\le L_0\|x_{k+1} - x_0\| + L\|y_k - y_0\| + L_1\|x_{k+1} - y_k\| \\
&\le L_0(t_{k+1} - t_0) + L(\|y_k - x_0\| + \|x_0 - y_0\|) + L_1(\|x_{k+1} - x_k\| + \|y_k - x_k\|) \\
&\le L_0 t_{k+1} + L(s_k + \eta_0) + L_1(t_{k+1} - t_k + s_k - t_k) < 1,
\end{aligned}$$

so $B_{k+1}^{-1} \in \mathcal{L}(Y, X)$ and

$$\|B_{k+1}^{-1}A_0\| \le \frac{1}{1 - (L_0 t_{k+1} + L(s_k + \eta_0) + L_1(t_{k+1} - t_k + s_k - t_k))}$$

and

$$\begin{aligned}
\|y_{k+1} - x_{k+1}\| &\le \|B_{k+1}^{-1}A_0\|\|A_0^{-1}([x_{k+1}, x_k; F] - [x_k, y_k; F])(x_{k+1} - x_k)\| \\
&\le \frac{(L_1(t_{k+1} - t_k) + L_2(s_k - t_k))(t_{k+1} - t_k)}{1 - (L_0 t_{k+1} + L(s_k + \eta_0) + L_1(t_{k+1} - t_k + s_k - t_k))} \\
&= s_{k+1} - t_{k+1},
\end{aligned}$$

which shows (3.31), where we have also used $x_k, y_k \in U(x_0, t^*)$. Then,

$$\begin{aligned}
\|x_{k+1} - x_0\| &\le \|x_{k+1} - x_k\| + \cdots + \|x_1 - x_0\| \\
&\le t_{k+1} - t_k + \cdots + t_1 - t_0 \\
&= t_{k+1} < t^*
\end{aligned}$$

and

$$\begin{aligned}
\|y_{k+1} - x_0\| &\le \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_k\| + \cdots + \|x_1 - x_0\| \\
&\le s_{k+1} - t_{k+1} + t_{k+1} - t_k + \cdots + t_1 - t_0 \\
&= t_{k+1} < t^*,
\end{aligned}$$

so $x_{k+1}, y_{k+1} \in U(x_0, t^*)$.

We have

$$\|A_0^{-1}F(x_{k+1})\| \le (L_1(t_{k+1} - t_k) + L_2(s_k - t_k))(t_{k+1} - t_k),$$

so by the continuity of F, we get $F(x^*) = 0$.

Finally, to show the uniqueness part, let $y^* \in D_1$ such that $F(y^*)$. Set $H = [x^*, y^*; F]$. Then, using (3.27) and (3.29), we obtain in turn

$$\begin{aligned}
\|A_0^{-1}(H - A_0)\| &\le L_0\|x^* - x_0\| + L\|y^* - y_0\| \\
&\le L_0\|x^* - x_0\| + L(\|y^* - x_0\| + \|x_0 - y_0\|) \\
&\le L_0 t^* + LR + Ls_0 < 1,
\end{aligned}$$

so $H^{-1} \in \mathcal{L}(Y, X)$. In view of the identity

$$0 = F(x^*) - F(y^*) = H(x^* - y^*),$$

so $x^* = y^*$.

□

4. Numerical examples

We present some numerical examples in this section.

Example 1. Let $X = Y = \mathbb{R}$, $D = (-1, 1)$ and define F on D by

$$F(x) = e^x - 1.$$

Then, $x^* = 0$ is a solution of (1.1) and $F'(x^*) = 1$. Note that for any $x, y, u, v \in D$, we have

$$\begin{aligned}
&|F'(x^*)^{-1}\big([x, y; F] - [u, v; F]\big)| \\
=&\Big|\int_0^1 \big(F'(tx + (1-t)y) - F'(tu + (1-t)v)\big)dt\Big| \\
=&\Big|\int_0^1\int_0^1 \big(F''(\theta(tx + (1-t)y) + (1-\theta)(tu + (1-t)v)\big) \\
&\times (tx + (1-t)y - (tu + (1-t)v)\big)d\theta dt\Big| \\
=&\Big|\int_0^1\int_0^1 e^{(\theta(tx+(1-t)y)+(1-\theta)(tu+(1-t)v)}(tx + (1-t)y - (tu + (1-t)v)d\theta dt\Big| \\
\le& \int_0^1 e|t(x-u) + (1-t)(y-v)|dt \\
\le& \frac{e}{2}(|x-u| + |y-v|)
\end{aligned}$$

and

$$\begin{aligned}
&|F'(x^*)^{-1}\big([x, y; F] - F'(x^*)\big)| \\
=&\Big|\int_0^1 F'(tx + (1-t)y)dt - F'(x^*))\Big| \\
=&\Big|\int_0^1 (e^{tx+(1-t)y} - 1)dt\Big| \\
=&\Big|\int_0^1 (tx + (1-t)y)\big(1 + \frac{tx + (1-t)y}{2!} + \frac{(tx + (1-t)y)^2}{3!} + \cdots\big)dt\Big| \\
\le&\Big|\int_0^1 (tx + (1-t)y)\big(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\big)dt\Big| \\
\le& \frac{e-1}{2}(|x - x^*| + |y - x^*|).
\end{aligned}$$

That is to say, the center Lipschitz condition (2.11) and Lipschitz condition (2.12) are true for $l_0 = l = \frac{e-1}{2}$ and $l_1 = l_2 = \frac{e}{2}$, respectively. The parameter values are given as $r_0 = 0.2254$, $r_A = 0.387985$, $r = 0.135239$, $r^* = r_1^* = 0.324948$, $r^{**} = 0.172542$ and $r_A^* = 0.0630374$.

Example 2. Let $X = Y = C[0, 1]$, the space of continuous functions defined on the interval $[0, 1]$, equipped with the max norm and $D = \bar{U}(0, 1)$. Define function F on D, given by

$$F(x)(s) = x(s) - 5\int_0^1 stx^3(t)dt.$$

and divided difference of F is defined by

$$[x, y; F] = \int_0^1 F'(tx + (1-t)y)dt.$$

Then, we have

$$[F'(x)y](s) = y(s) - 15\int_0^1 stx^2(t)y(t)dt, \quad \text{for each} \quad y \in D.$$

We have $x^* = 0$ for all $s \in [0,1]$, $l_0 = l = 3.75$ and $l_1 = l_2 = 7.5$. The parameter values are given as $r_0 = 0.04444$, $r_A = 0.08888$, $r = 0.026491$, $r^* = r_1^* = 0.06666$, $r^{**} = 0.03333$ and $r_A^* = 0.00296296$.

Example 3. Let $X = Y = \mathcal{C}[0,1]$ equipped with max norm and $D = U(0,r)$ for some $r > 1$. Define F on D by

$$F(x)(s) = x(s) - y(s) - \mu \int_0^1 G(s,t)x^3(t)dt, \quad x, y \in \mathcal{C}[0,1], \quad s \in [0,1],$$

where μ is a real parameter and the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ (1-t)s, & s \le t. \end{cases}$$

Then, the Fréchet derivative of F is defined by

$$(F'(x)(w))(s) = w(s) - 3\mu \int_0^1 G(s,t)x^2(t)w(t)dt, \quad w \in \mathcal{C}[0,1], \quad s \in [0,1].$$

Let us choose $x_0(s) = y_0(s) = y(s) = 1$ and $|\mu| < \frac{8}{3}$. Then, we have that

$$\|I - A_0\| \le \frac{3}{8}\mu, \quad A_0^{-1} \in \mathcal{L}(Y,X),$$

$$\|A_0^{-1}\| \le \frac{8}{8-3|\mu|}, s_0 = 0, t_1 = \frac{|\mu|}{8-3|\mu|}, L_0 = L = \frac{3(1+r)|\mu|}{2(8-3|\mu|)},$$

$$L_1 = L_2 = \frac{3r|\mu|}{(8-3|\mu|)}.$$

Let us choose $r = 3$ and $\mu = \frac{1}{2}$. Then, we have that

$$t_1 = 0.07692307, \quad L_0 = L = 0.461538462, \quad L_1 = L_2 = 0.692307692.$$

and

$$(L_0 t_1 + L s_0) \max\{\frac{1}{1-(L_0\eta + L(s_1+\eta_0))}, \frac{1}{1-((L_0+L)\eta + L_1\eta_0)}\} \approx 0.038961,$$

$$x^* \approx 0.711345739, \qquad 1 - \frac{(L_0+L)\eta}{1-L\eta_0} \approx 0.928994.$$

That is, condition (3.3) is satisfied and Theorem 3.1 applies.

Example 4. This example is intended to verify the third order of convergence of the MKWTM. Consider a system of two equations

$$\begin{aligned} x_1 + e^{x_2} - \cos x_2 &= 0, \\ 3x_1 - \sin x_1 - x_2 &= 0. \end{aligned}$$

With the initial approximations $x_0 = \{\frac{1}{2}, \frac{3}{10}\}^T$ and $y_0 = \{\frac{1}{4}, 1\}^T$, we obtain the zero $x^* = \{0,0\}^T$. The number of iterations (n) needed to converge to the solution using the stopping criterion $(\|x_{n+1} - x_n\| + \|F(x_n)\|) < 10^{-200}$ is 6. Then, using the last three approximations x_{n+1}, x_n, x_{n-1} in (2.51), we get ACOC = 3.0000.

Example 5. Here also we confirm order of convergence by considering a system of three equations [16]:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4, \\ 2x_2 + x_3 + x_1 &= 4, \\ x_1 x_2 x_3 &= 1. \end{aligned}$$

The zero $x^* = \{1,1,1\}^T$ is obtained by assuming the initial approximations $x_0 = \{1,2,3\}^T$ and $y_0 = \{\frac{1}{10}, \frac{1}{2}, \frac{1}{2}\}^T$, The number of iterations (n) needed to converge to the solution applying the stopping criterion $(\|x_{n+1} - x_n\| + \|F(x_n)\|) < 10^{-200}$ is 10. Then, using the last three approximations x_{n+1}, x_n, x_{n-1} in (2.51), we get ACOC = 3.0000.

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